FINA556 – Structured Products and Exotic Options

Topic 2 — Exotic swaps

2.1 Implied forward rates

2.2 Asset swaps

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2.1 Implied forward rates

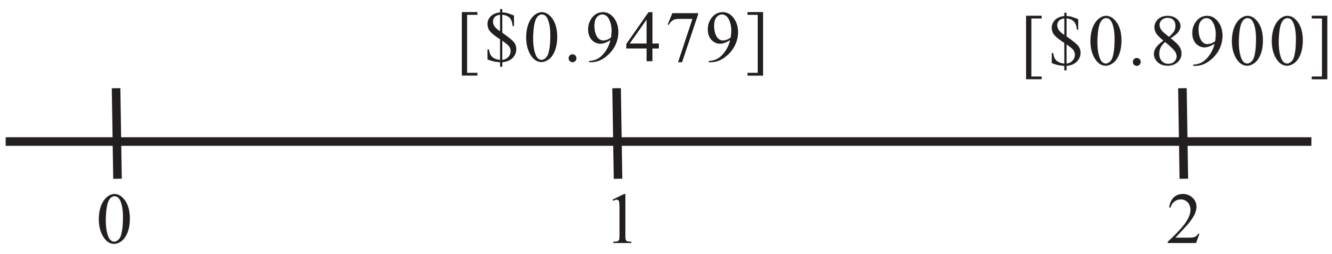
$B_t(T)$ = time-t price of T-maturity *unit par discount bond*

The *discount factor* over the period $[t, T]$ is implied by $B_t(T)$. Suppose the market prices of unit par zero coupon bonds with maturity T_1 and T_2 are known, what is the interest rate applied over the future period $[T_1, T_2]$?

Forward interest rate, $R(t; T_1, T_2)$ is the interest rate determined at the current time t which is applied over the future period $[T_1, T_2]$.

Example

Suppose $B_0(1) = 0.9479, B_0(2) = 0.8900$; what is the **implied** forward interest rate over Year One to Year Two?



Calculation formula:-

$$\underbrace{B_t(T_1)}_{\text{discount factor over } [t,T]} \underbrace{\frac{1}{1 + R(t;T_1,T_2)(T_2 - T_1)}}_{\text{discount factor over } [T_1,T_2]} = \underbrace{B_t(T_2)}_{\text{discount factor over } [t,T_2]}$$

$$R(t;T_1,T_2) = \frac{1}{T_2 - T_1}\left[\frac{B_t(T_1)}{B_t(T_2)} - 1\right].$$

In our numerical example,

$$R(0;1,2) = \frac{1}{2-1}\left(\frac{0.9479}{0.8900} - 1\right) = 0.065.$$

Calculation of forward rates from zero rates

Year	Zero rate for n-year investment (% per annum)	Forward rate for n^{th} year (% per annum)
1	3.0	
2	4.0	5.0
3	4.6	5.8
4	5.0	6.2
5	5.3	6.5

(i) $e^{0.03\times 1} \cdot e^{R_{12}\times 1} = e^{0.04\times 2}$

$e^{R_{12}} = e^{0.08}/e^{0.03} = e^{0.05}$; so $R_{12} = 0.05$

The calculation is based on continuous compounding.

(ii) $e^{0.03\times 1}e^{0.05\times 1}e^{R_{23}\times 1} = e^{0.046\times 3}$

$$e^{R_{23}} = e^{0.138}/e^{0.08} = e^{0.058};\ \text{so}\ R_{23} = 0.058$$

(iii) $e^{0.03\times 1}e^{0.05\times 1}e^{0.058\times 1}e^{R_{34}\times 1} = e^{0.05\times 4}$

$$e^{R_{34}} = e^{0.2}/e^{0.138} = e^{0.62};\ \text{so}\ R_{34} = 0.062$$

(iv) $e^{0.03\times 1}e^{0.05\times 1}e^{0.058\times 1}e^{0.062\times 1}e^{R_{45}\times 1} = e^{0.053\times 5}$

$$e^{R_{45}} = e^{0.265}/e^{0.2} = e^{0.065};\ \text{so}\ R_{45} = 0.065.$$

Instead of using discount factors over successive time periods, here we use growth factors over successive time periods.

Forward rate agreement (FRA)

The FRA is an agreement between two counterparties to exchange floating and fixed interest payments on the future settlement date T_2.

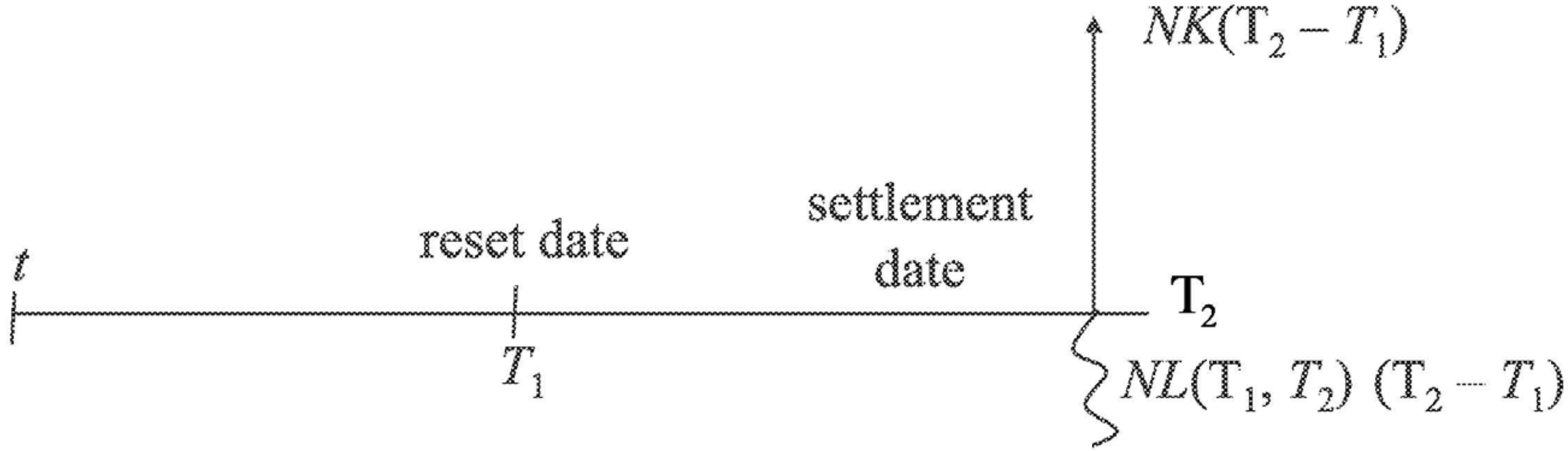
- The floating rate will be the LIBOR (London InterBank Offered Rate) $L[T_1, T_2]$ as observed on the future reset date T_1.

Recall that the implied forward rate over the future period $[T_1, T_2]$ has been fixed by the current market prices of discount bonds maturing at T_1 and T_2.

The fixed rate is expected to be equal to the implied forward rate over the same period as observed today.

Determination of the forward price of LIBOR

$$\begin{aligned} L[T_1, T_2] &= \text{LIBOR rate observed at future time } T_1 \\ &\quad \text{for the accrual period } [T_1, T_2] \\ K &= \text{fixed rate} \\ N &= \text{notional of the FRA} \end{aligned}$$



Cash flow of the fixed rate receiver

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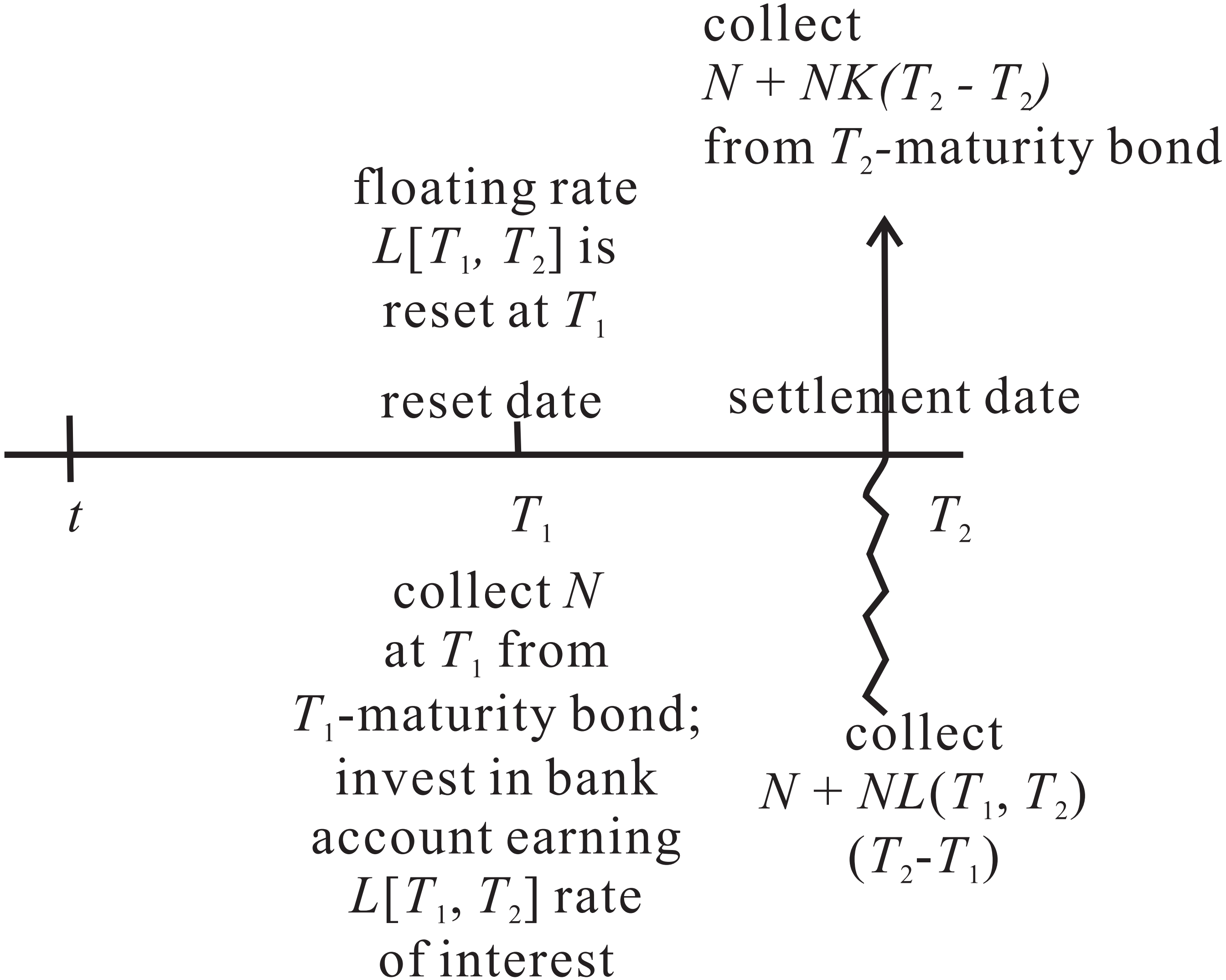


Valuation principle

Apparently, the cash flow at T_2 is uncertain since LIBOR $L[T_1, T_2]$ is set (or known) at T_1. Can we construct portfolio of discount bonds that replicate the cash flow?

- For convenience of presenting the argument, we add N to both floating and fixed rate payments.

The cash flows of the fixed rate payer can be replicated by

(i) long holding of the T_2-maturity zero coupon bond with par $N[1 + K(T_2 - T_1)]$.

(ii) short holding of the T_1-maturity zero coupon bond with par N.

The N dollars collected from the T_1-maturity bond at T_1 is invested in bank account earning interest rate of $L[T_1, T_2]$ over $[T_1, T_2]$.

By *no-arbitrage principle*, the value of the FRA is the same as that of the replicating portfolio. The fixed rate is determined so that the FRA is *entered at zero cost* to both parties. Now,

$$\begin{aligned} & \text{Value of the replicating portfolio at the current time} \\ = \; & N\{[1 + K(T_2 - T_1)]B_t(T_2) - B_t(T_1)\}. \end{aligned}$$

We find K such that the above value is zero. This gives

$$K = \underbrace{\frac{1}{T_2 - T_1}\left[\frac{B_t(T_1)}{B_t(T_2)} - 1\right]}_{\text{implied forward rate over } [T_1, T_2]}.$$

K is seen to be the forward price of $L[T_1, T_2]$ over $[T_1, T_2]$. This is the same as the forward interest rate implied from the discount bond prices.

Consider a FRA that exchanges floating rate $L[1,2]$ at the end of Year Two for some fixed rate K. Suppose

$$B_0(1) = 0.9479 \quad \text{and} \quad B_0(2) = 0.8900.$$

The implied forward rate applied from Year One to Year Two:

$$\frac{1}{2-1}\left(\frac{0.9479}{0.8900} - 1\right) = 0.065.$$

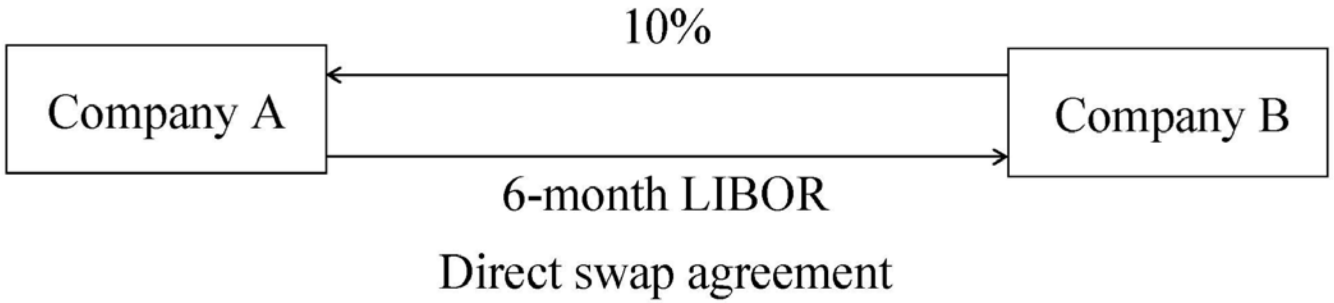
The fixed rate set for the FRA at time 0 should be 0.065 so that the value of the FRA is zero at time 0.

Suppose notional = \$1 million and $L[1,2]$ turns out to be 7% at Year One, then the fixed rate payer receives

$$(7\% - 6.5\%) \times 1 \text{ million} = \$5,000$$

at the settlement date (end of Year Two).

Interest rate swaps



In an interest swap, the two parties agree to exchange periodic interest payments.

- The interest payments exchanged are calculated based on some predetermined dollar principal, called the notional amount.

- One party is the fixed-rate payer and the other party is the floating-rate payer. The floating interest rate is based on some reference rate (the most common index is the LIBOR).

Example

Notional amount = \$50 million
fixed rate = 10%
floating rate = 6-month LIBOR

Tenor = 3 years, semi-annual payments

6-month period	Cash flows		
	Net (float-fix)	floating rate bond	fixed rate bond
0	0	-50	50
1	$\text{LIBOR}_1/2 \times 50 - 2.5$	$\text{LIBOR}_1/2 \times 50$	-2.5
2	$\text{LIBOR}_2/2 \times 50 - 2.5$	$\text{LIBOR}_2/2 \times 50$	-2.5
3	$\text{LIBOR}_3/2 \times 50 - 2.5$	$\text{LIBOR}_3/2 \times 50$	-2.5
4	$\text{LIBOR}_4/2 \times 50 - 2.5$	$\text{LIBOR}_4/2 \times 50$	-2.5
5	$\text{LIBOR}_5/2 \times 50 - 2.5$	$\text{LIBOR}_5/2 \times 50$	-2.5
6	$\text{LIBOR}_6/2 \times 50 - 2.5$	$\text{LIBOR}_6/2 \times 50$	-2.5

A swap can be interpreted as a package of cash market instruments – a portfolio of forward rate agreements.

- Buy \$50 million par of a 3-year floating rate bond that pays 6-month LIBOR semi-annually.

- Finance the purchase by borrowing \$50 million for 3 years at 10% interest rate paid semi-annually.

The fixed-rate payer has a cash market position equivalent to a long position in a floating-rate bond and a short position in a fixed rate bond (borrowing through issuance of a fixed rate bond).

Valuation of interest rate swaps

- When a swap is entered into, it typically has zero value.

- Valuation involves finding the fixed swap rate K such that the fixed and floating legs have equal value at inception.

- Consider a swap with payment dates $T_1, T_2, \cdots, T_n$ (tenor structure) set in the term of the swap. L_{i-1} is the LIBOR observed at T_{i-1} but payment is made at T_i. Write δ_i as the accrual period in year fraction over $[T_{i-1}, T_i]$ according to some day count convention. We expect $\delta_i \approx T_i - T_{i-1}$.

- The fixed payment at T_i is $KN\delta_i$ while the floating payment at T_i is $L_{i-1}N\delta_i, i = 1, 2, \cdots n$. Here, N is the notional.

Day count convention

For the 30/360 day count convention of the time period $(D_1, D_2]$ with D_1 excluded but D_2 included, the year fraction is

$$\frac{\max(30 - d_1, 0) + \min(d_2, 30) + 360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1)}{360}$$

where d_i, m_i and y_i represent the day, month and year of date $D_i, i = 1, 2$.

For example, the year fraction between *Feb 27, 2006* and *July 31, 2008*

$$= \frac{30 - 27 + 30 + 360 \times (2008 - 2006) + 30 \times (7 - 2 - 1)}{360}$$

$$= \frac{33}{360} + 2 + \frac{4}{12}.$$

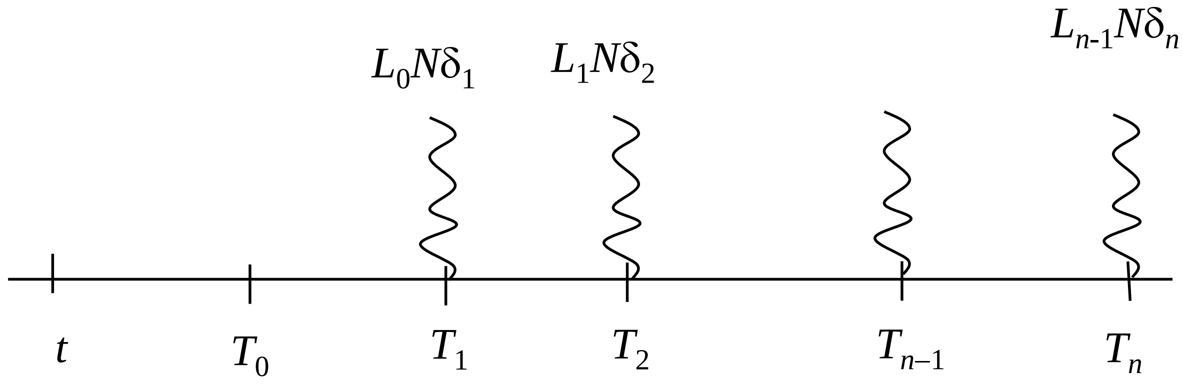
Replication of cash flows

- The fixed payment at T_i is $KN\delta_i$. The fixed payments are packages of bonds with par $KN\delta_i$ at maturity date $T_i, i = 1, 2, \cdots, n$.

- To replicate the floating leg payments at $t, t < T_0$, we long T_0-maturity bond with par N and short T_n-maturity bond with par N. The N dollars collected at T_0 can generate the floating leg payment $L_{i-1}N\delta_i$ at all $T_i, i = 1, 2, \cdots, n$. The remaining N dollars at T_n is used to pay the par of the T_n-maturity bond.

- Let $B(t, T)$ be the time-t price of the discount bond with maturity T. These bond prices represent market view on the discount factors.

Follow the strategy that consists of exchanging the notional principal at the beginning and the end of the swap, and investing it at a floating rate in between.



$$
\begin{aligned}
& \text{Present value of the floating leg payment } L_i N\delta_i \\
= \; & N[B(t,T_{i-1}) - B(t,T_i)], \quad i = 1, 2, \cdots, n. \\
& \text{Sum of the present value of the floating leg payments} \\
= \; & N \sum_{i=1}^{n} [B(t,T_{i-1}) - B(t,T_i)] = N[B(t,T_0) - B(t,T_N)].
\end{aligned}
$$

- Sum of present value of fixed leg payments

$$= NK \sum_{i=1}^{n} \delta_i B(t, T_i).$$

- The value of the interest rate swap is set to be zero at initiation. We set K such that the present value of the floating leg payments equals that of the fixed leg payment. Therefore

$$K = \frac{B(t, T_0) - B(t, T_n)}{\sum_{i=1}^{n} \delta_i B(t, T_i)}.$$

Pricing a plain interest rate swap

Notional = \$10 million, 5-year swap

Period	*Zero-rate (%)*	*discount factor*	*forward rate (%)*
1	5.50	0.9479	5.50
2	6.00	0.8900	6.50
3	6.25	0.8337	6.75
4	6.50	0.7773	7.25
5	7.00	0.7130	9.02
		sum = 4.1619	

$\star$ Discount factor over the 5-year period $= \frac{1}{(1.07)^5} = 0.7130$

Forward rate between Year Two and Year Three

$= \frac{0.8900}{0.8337} - 1 = 0.0675.$

$$K = \frac{B(T_0,T_0) - B(T_0,T_n)}{\sum_{i=1}^{n} \delta_i B(T_0,T_i)} = \frac{1 - 0.7130}{4.1619} = 6.90\%$$

$$\begin{aligned} \text{PV (floating leg payments)} &= 10,000,000 \times 1 - 10,000,000 \times 0.7130 \\ &= N[B(T_0,T_0) - B(T_0,T_n)] = 2,870,137. \end{aligned}$$

Period	*fixed payment*	*floating payment**	*PV fixed*	*PV floating*
1	689,625	550,000	653,673	521,327
2	689,625	650,000	613,764	578,709
3	689,625	675,000	574,945	562,899
4	689,625	725,000	536,061	563,834
5	689,625	902,000	491,693	643,369

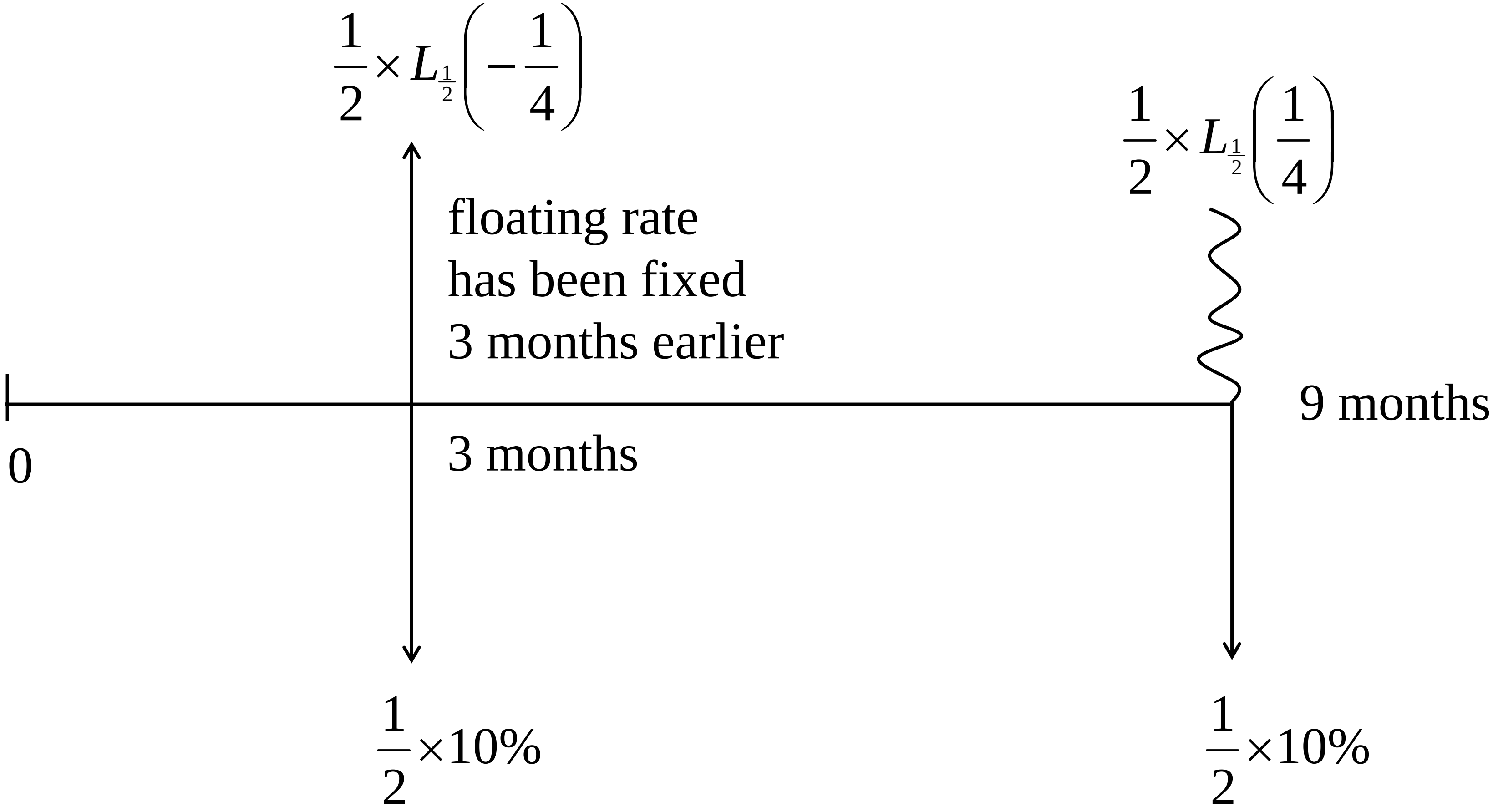
⋆ Calculated based on the assumption that the LIBOR will equal the forward rates.

Example (Valuation of an in-progress interest rate swap)

- An interest rate swap with notional $=$ \$1 million, remaining life of 9 months.

- 6-month LIBOR is exchanged for a fixed rate of 10% per annum.

- $L_{\frac{1}{2}}\left(-\frac{1}{4}\right)$: 6-month LIBOR that has been set at 3 months earlier
  $L_{\frac{1}{2}}\left(\frac{1}{4}\right)$: 6-month LIBOR that will be set at 3 months later.



Cash flow of the floating rate receiver

- Market prices of unit par zero coupon bonds with maturity dates 3 months and 9 months from now are

$$B_0\left(\frac{1}{4}\right) = 0.972 \quad \text{and} \quad B_0\left(\frac{3}{4}\right) = 0.918.$$

- The 6-month LIBOR to be paid 3 months from now has been fixed 3 months earlier. This LIBOR $L_{\frac{1}{2}}\left(-\frac{1}{4}\right)$ should be reflected in the price of the *floating rate bond* maturing 3 months from now. This floating rate bond is now priced at \$0.992, and will receive $1 + \frac{1}{2}L_{\frac{1}{2}}\left(-\frac{1}{4}\right)$ at a later time 3 months from now.

- Considering the present value of amount received:

$$PV\left[1+\frac{1}{2}L_{\frac{1}{2}}\left(-\frac{1}{4}\right)\right]=0.992= \text{ price of floating rate bond.}$$

Present value of \$1 received 3 months from now $= B_0\left(\frac{1}{4}\right)$.

Hence, $PV\left[\frac{1}{2}L_{\frac{1}{2}}\left(-\frac{1}{4}\right)\right]=0.992-0.972=0.02$.

Present value to the floating rate receiver of the in-progress interest rate swap

$= PV\left[\frac{1}{2}L_{\frac{1}{2}}\left(-\frac{1}{4}\right)\right]+PV\left[\frac{1}{2}L_{\frac{1}{2}}\left(\frac{1}{4}\right)\right]-PV$ (fixed rate payments).

Note that \$1 received at 3 months later $= \$\left[1 + \frac{1}{2}L_{\frac{1}{2}}\left(\frac{1}{4}\right)\right]$ at 9 months later so that

$$
\begin{aligned}
PV\left(\frac{1}{2}L_{\frac{1}{2}}\left(\frac{1}{4}\right)\right) &= PV(\$1 \text{ at 3 months later}) - PV(\$1 \text{ at 9 months later}) \\
&= B_0\left(\frac{1}{4}\right) - B_0\left(\frac{3}{4}\right) = 0.972 - 0.918 = 0.054.
\end{aligned}
$$

$$
\begin{aligned}
PV \text{ (fixed rate payments)} &= 0.05\left[B_0\left(\frac{1}{4}\right) + B_0\left(\frac{3}{4}\right)\right] \\
&= 0.05(0.972 + 0.918) = 0.0945.
\end{aligned}
$$

The present value of the swap to the floating rate receiver

$= 0.02 + 0.054 - 0.0945 = -0.0205.$

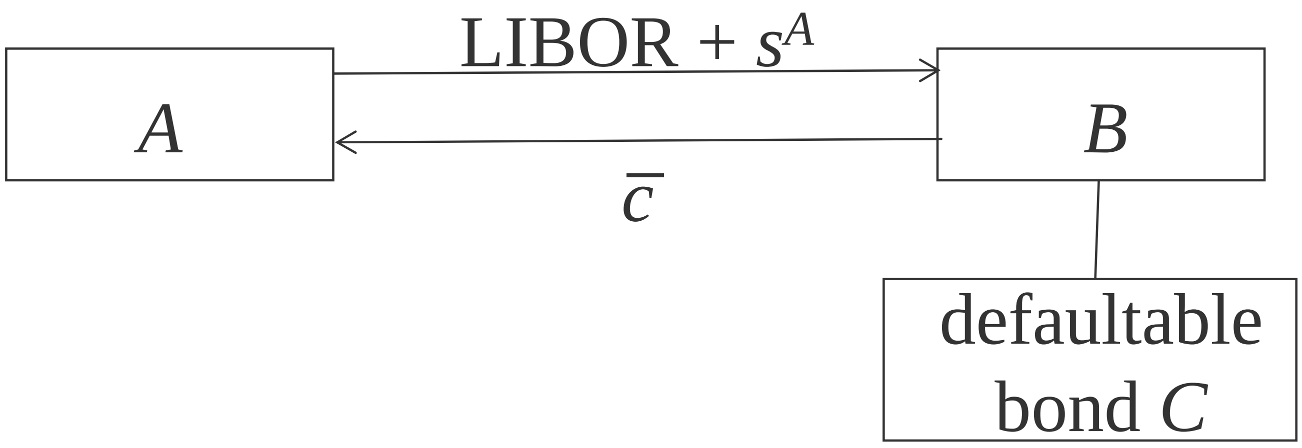
2.2 Asset swap

- Combination of a defaultable bond with an interest rate swap.

  B pays the notional amount upfront to acquire the asset swap package.

1. A fixed coupon bond issued by C with coupon $\bar{c}$ payable on coupon dates.

2. A fixed-for-floating swap.



The interest rate swap continues even after the underlying bond defaults.

The asset swap spread s^A is adjusted to ensure that the asset swap package has an initial value equal to the notional (at par value).

Asset swaps are more liquid than the underlying defaultable bonds.

- Asset swaps are done most often to achieve a more favorable payment stream.

  For example, an investor is interested to acquire the defaultable bond issuer by a company but he prefers floating rate coupons instead of fixed rate. The whole package of bond and interest rate swap is sold.

Asset swap packages

- An asset swap package consists of a defaultable coupon bond $\overline{C}$ with coupon $\overline{c}$ and an interest rate swap.

- The bond's coupon is swapped into LIBOR plus the asset swap rate s^A.

- Asset swap package is sold at par.

- Asset swap transactions are driven by the desire to strip out unwanted coupon streams from the underlying risky bond. Investors gain access to highly customized securities which target their particular cash flow requirements.

1. Default free bond

$$C(t) \;=\; \text{time-}t \text{ price of default-free bond with fixed-coupon } \overline{c}$$

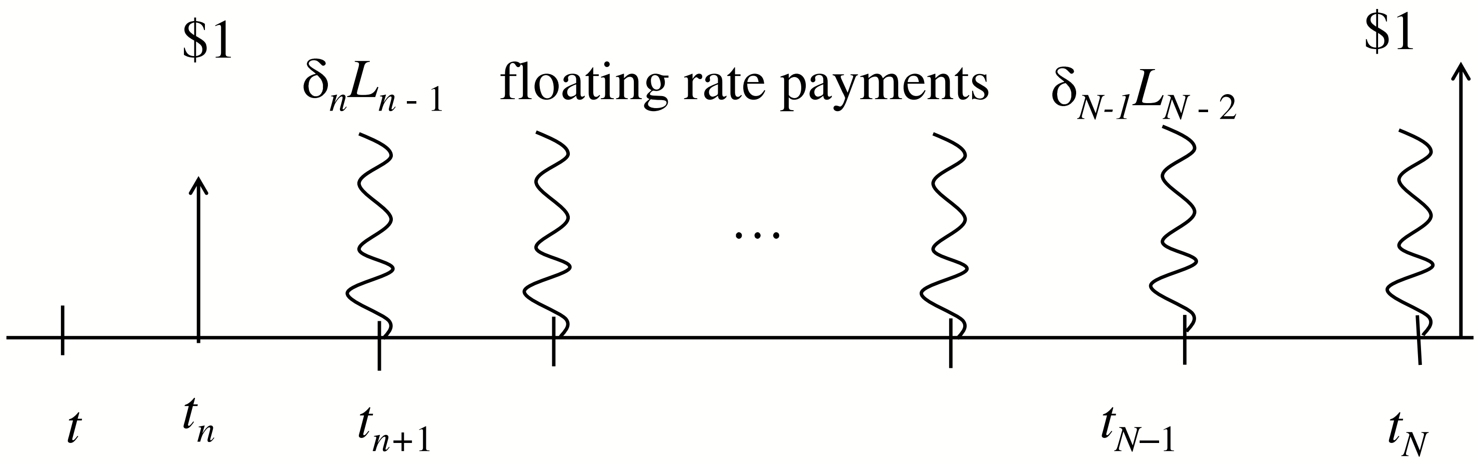
2. Defaultable bond

$$\overline{C}(t) \;=\; \text{time-}t \text{ price of defaultable bond with fixed-coupon } \overline{c}$$

The difference $C(t) - \overline{C}(t)$ reflects the premium on the potential default risk of the defaultable bond.

Let $B(t, t_i)$ be the time-t price of a unit par zero coupon bond maturing on t_i. The market-traded bond price gives the market value of the discount factor over (t, t_i). Write δ_i as the accrual period over (t_{i-1}, t_i) using a certain day count convention. Note that δ_i differs slightly from the actual length of the time period $t_i - t_{i-1}$.

Time-t value of sum of floating coupons paid at fixing dates $t_{n+1}, \cdots,$ t_N is given by $B(t,t_n) - B(t,t_N)$. This is because \$1 at t_n can generate all floating coupons over $t_{n+1}, \cdots, t_N$, plus \$1 par at t_N. This is done by putting \$1 at t_n in a money market account that earns the floating LIBOR.



3. Interest rate swap (tenor is $[t_n, t_N]$; reset dates are $t_n, \cdots, t_{N-1}$ while payment dates are $t_{n+1}, \cdots, t_N$)

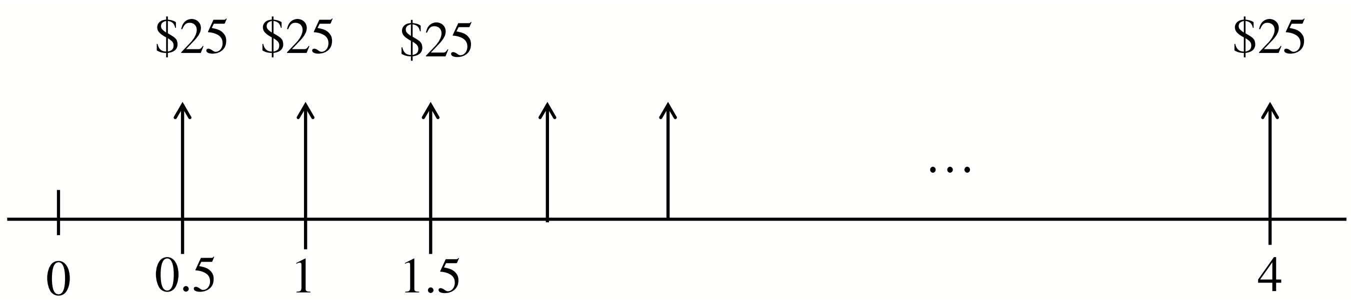
$$\begin{aligned} s(t) &= \text{forward swap rate at time } t \text{ of a standard fixed-for-floating} \\ &= \frac{B(t,t_n) - B(t,t_N)}{A(t;t_n,t_N)}, \quad t \leq t_n \end{aligned}$$

where $A(t;t_n,t_N) = \sum_{i=n+1}^{N} \delta_i B(t,t_i) =$ value of the payment stream paying δ_i on each date t_i. The first swap payment starts on t_{n+1} and the last payment date is t_N.

Theoretically, $s(t)$ is precisely determined by the market observable bond prices according to no-arbitrage argument. However, the swap market and bond market may not trade in a completely consistent manner due to liquidity and the force of supply and demand.

Fixed leg payments and annuity stream

Given the tenor of the dates of coupon payments of the underlying risky bond, the floating rate and fixed rate coupons are exchanged under the interest rate swap arrangement. The stream of fixed leg payments resemble an annuity stream. Suppose $\delta = \frac{1}{2}$ (coupons are paid semi-annually), $N = \$1,000$, and fixed rate $= 5\%$, the stream of the fixed leg payments is like an annuity that pays \$25 semi-annually (\$50 per annum).



Payoff streams to the buyer of the asset swap package ($\delta_i = 1$)

time	defaultable bond	interest rate swap	net
$t = 0^\dagger$	$-\overline{C}(0)$	$-1 + \overline{C}(0)$	-1
$t = t_i$	$\overline{c}^*$	$-\overline{c} + L_{i-1} + s^A$	$L_{i-1} + s^A + (\overline{c}^* - \overline{c})$
$t = t_N$	$(1 + \overline{c})^*$	$-\overline{c} + L_{N-1} + s^A$	$1^* + L_{N-1} + s^A + (\overline{c}^* - \overline{c})$
default	recovery	unaffected	recovery

$\star$ denotes payment contingent on survival.

$\dagger$ The value of the interest rate swap at $t = 0$ is not zero. The sum of the values of the interest rate swap and defaultable bond is equal to par at $t = 0$.

The asset swap buyer pays \$1 (notional). In return, he receives

1. risky bond whose value is $\overline{C}(0)$;

2. floating leg payments at LIBOR;

3. fixed leg payments at $s^A(0)$;

while he forfeits

4. fixed leg payments at $\overline{c}$.

The two streams of fixed leg payments can be related to annuity. The floating leg payments can be related to swap rate times annuity.

The additional asset spread s^A serves as the compensation for bearing the potential loss upon default.

$s(0)$ = fixed-for-floating swap rate (market quote)

$A(0)$ = value of an annuity paying at \$1 per annum (calculated based on the observable default free bond prices)

The value of asset swap package is set at par at $t = 0$, so that

$$\overline{C}(0) + \underbrace{A(0)s(0) + A(0)s^A(0) - A(0)\overline{c}}_{\text{swap arrangement}} = 1.$$

The present value of the floating coupons is given by $A(0)s(0)$. Since the swap continues even after default, $A(0)$ appears in all terms associated with the swap arrangement.

Solving for $s^A(0)$

$$s^A(0) = \frac{1}{A(0)}[1 - \overline{C}(0)] + \overline{c} - s(0). \tag{A}$$

The asset spread s^A consists of two parts [see Eq. (A)]:

(i) one is from the difference between the bond coupon and the par swap rate, namely, $\overline{c} - s(0)$;

(ii) the difference between the bond price and its par value, which is spread as an annuity.

- Bond price $\overline{C}(0)$ and fixed coupon rate $\overline{c}$ are known from the bond.
- $s(0)$ is observable from the market swap rate.
- $A(0)$ can be calculated from market discount rates (inferred from the market prices of discount bonds).

Rearranging the terms,

$$\overline{C}(0) + A(0)s^A(0) = \underbrace{[1 - A(0)s(0)] + A(0)\overline{c}}_{\text{default-free bond price}} \equiv C(0)$$

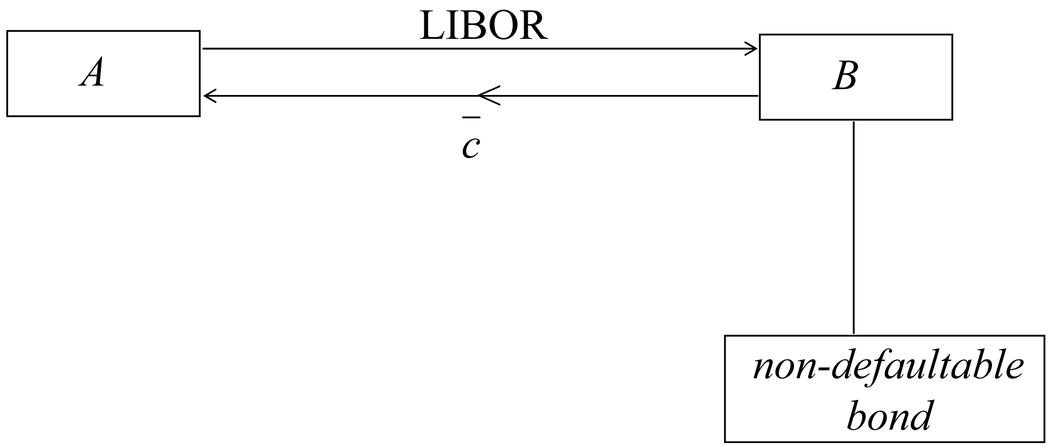
where the right-hand side gives the value of a default-free bond with coupon $\overline{c}$. Note that $1 - A(0)s(0)$ is the present value of receiving \$1 at maturity t_N. We obtain

$$s^A(0) = \frac{1}{A(0)}[C(0) - \overline{C}(0)]. \tag{B}$$

- The difference in the bond prices is equal to the present value of the annuity stream at the rate $s^A(0)$.

Alternative proof

A combination of the non-defaultable counterpart (bond with coupon rate $\overline{c}$) plus an interest rate swap (whose floating leg is LIBOR while the fixed leg is $\overline{c}$) becomes a par floater. Hence, the new asset package should also be sold at par.



The buyer receives LIBOR floating rate interests plus par.
Value of interest rate swap $= A(0)[s(0) - \overline{c}]$;
value of interest rate swap $+ C(0) = 1$ so

$$C(0) = 1 - A(0)s(0) + A(0)\overline{c}.$$

On the other hand,

$$\overline{C}(0) = 1 - A(0)s(0) - A(0)s^A(0) + A(0)\overline{c}.$$

- The two interest swaps with floating leg at LIBOR $+ s^A(0)$ and LIBOR, respectively, differ in values by $s^A(0)A(0)$.

- Let $V_{swap\_L+s^A}$ denote the value of the swap at $t = 0$ whose floating rate is set at LIBOR $+ s^A(0)$. Both asset swap packages are sold at par. We then have

$$1 = \overline{C}(0) + V_{swap\_L+s^A} = C(0) + V_{swap\_L}.$$

  Hence, the difference in $C(0)$ and $\overline{C}(0)$ is the present value of the annuity stream at the rate $s^A(0)$, that is,

$$C(0) - \overline{C}(0) = V_{swap\_L+s^A} - V_{swap\_L} = s^A(0)A(0).$$

Replication-based argument from seller's perspectives

- Under the interest rate swap, at each t_i, the seller receives $\overline{c}_i$ for sure, but must pay $L_{i-1} + s^A$.

- To replicate this payoff stream of the interest rate swap, the seller buys a default-free coupon bond with coupon size $\overline{c}_i - s^A$, and borrows \$1 at LIBOR and rolls this debt forward, paying: L_{i-1} at each t_i. At the final date t_N, the seller pays back his debt using the principal repayment of the default-free bond.

Let $C'(0)$ denote the time-0 price of the default-free coupon bond with coupon rate $\overline{c}_i - s^A$.

Payoff streams to the seller from a default-free coupon bond investment replicating his payment obligations from the interest-rate swap of an asset swap package.

Time	Default-free bond	Funding	Net
$t=0$	$-C'(0)$	$+1$	$1-C'(0)$
$t=t_i$	$\overline{c}_i - s^A$	$-L_{i-1}$	$\overline{c}_i - L_{i-1} - s^A$
$t=t_N$	$1+\overline{c}_N - s^A$	$-L_{N-1}-1$	$\overline{c}_N - L_{N-1} - s^A$
Default	Unaffected	Unaffected	Unaffected

Day count fractions are set to one, $\delta_i = 1$ and no counterparty defaults on his payments from the interest rate swap.

1. The replication generates a cash flow of $1-C'(0)$ initially, where $1 =$ proceeds from borrowing and $C'(0) :=$ price of the default-free coupon bond with coupons $\overline{c}_i - s^A$.

2. Since the asset swap is sold at par, we have

$$\underbrace{\text{value of interest rate swap}}_{1-C'(0)} + \overline{C}(0) = 1$$

so that $C'(0) = \overline{C}(0)$. One is a defaultable bond paying coupon $\overline{c}$ while the other is default free but paying $\overline{c} - s^A$. If we promise to continue to pay the coupons even upon default, the asset swap spread s^A can be viewed as the amount by which we can reduce the coupon while still keep the price at the original price $\overline{C}(0)$.

Summary

$\overline{C}(0) =$ price of the defaultable bond with fixed coupon rate $\overline{c}$

$C(0) =$ price of the default free bond with fixed coupon rate $\overline{c}$

$C'(0) =$ price of the default free bond with coupon rate $\overline{c} - s^A$

We have shown

$$s^A(0) = \frac{1}{A(0)}[C(0) - \overline{C}(0)],$$

where $s^A(0)$ is the additional asset spread paid by the seller to compensate for potential default loss faced by the buyer. We may consider $s^A(0)$ as the credit protection premium required to safeguard against default risk. The defaultable bond with fixed coupon $\overline{c}$ may be protected against default loss by paying $s^A(0)$ periodically. Therefore, the defaultable bond with fixed coupon $\overline{c}$ has the same value as that of the default bond with fixed coupon $\overline{c} - s^A(0)$. This also explains why $\overline{C}(0) = C'(0)$.

In-progress asset swap

- At a later time $t > 0$, the prevailing asset spread is

$$s^A(t) = \frac{C(t) - \overline{C}(t)}{A(t)},$$

where $A(t)$ denotes the value of the annuity over the remaining payment dates as seen from time t.

As time proceeds, $C(t) - \overline{C}(t)$ will tend to decrease to zero, unless a default happens*. This is balanced by $A(t)$ which will also decrease.

- The original asset swap with $s^A(0) > s^A(t)$ would have a positive value. Indeed, the value of the asset swap package at time t equals $A(t)[s^A(0) - s^A(t)]$. This value can be extracted by entering into an offsetting trade.

*A default would cause a sudden drop in $\overline{C}(t)$, thus widens the difference $C(t) - \overline{C}(t)$.

2.3 Short position in defaultable bonds and total return swaps

Under a repo (repurchase agreement), an investment dealer who owns securities agrees to sell them to another company now and buy them back later at a slightly higher price.

- The counterparty is essentially providing a loan to the investment dealer.

- The difference between the price at which the securities are sold and the price at which they are repurchased is the interest it earns. This interest rate is called the *repo rate*.

This loan involves very little credit risk.

- If the borrower does not honor the agreement, the lending company keeps the securities.

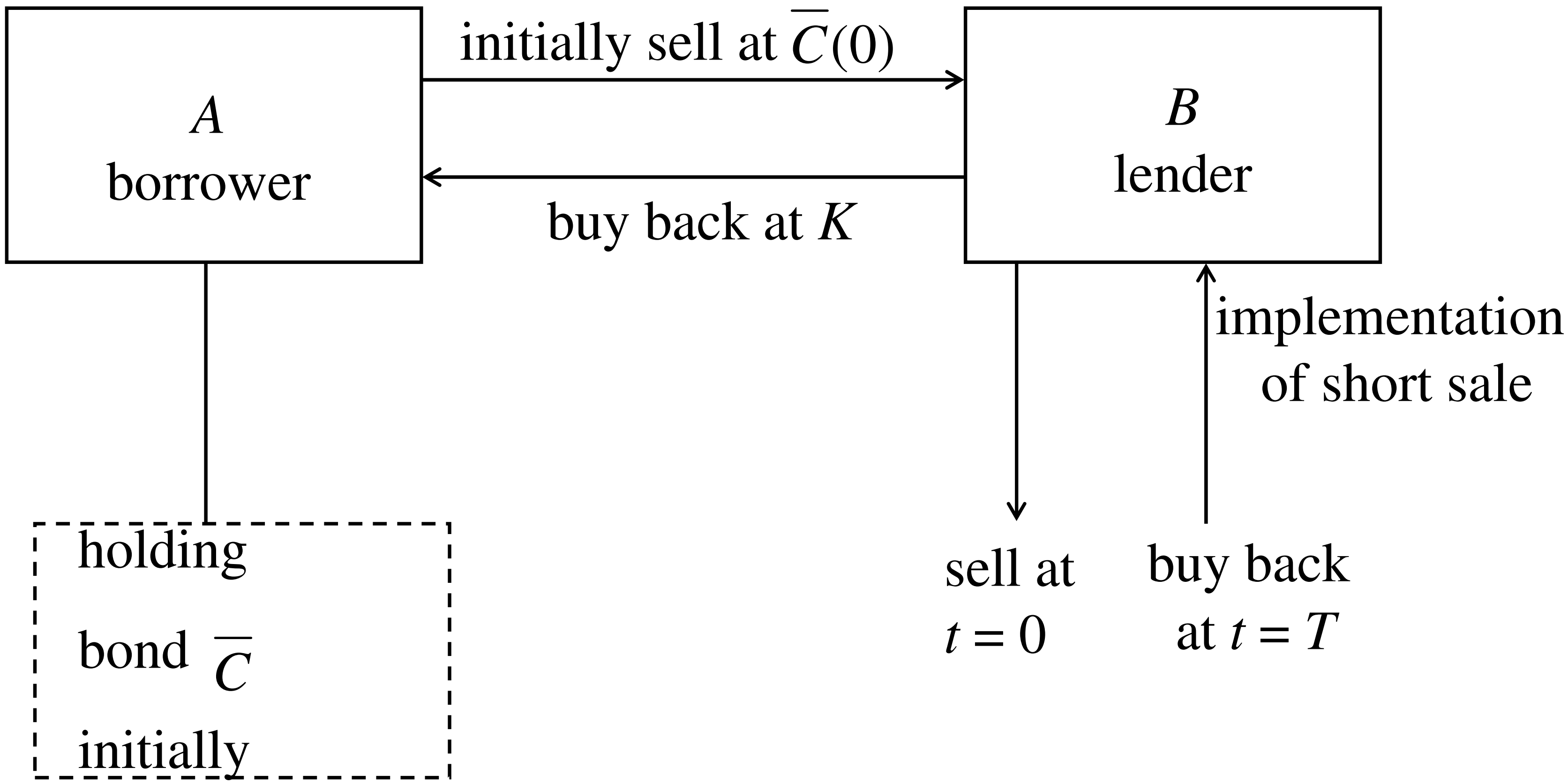
- If the lending company does not keep to its side of the agreement, the original owner of the securities keeps the cash.

- Repurchase (repo) transactions were first used in government bond markets where they are still an important instrument for funding and short sales of treasury bonds.

- A repo market for corporate bonds has developed which can be used to implement short positions in corporate bonds.

A repurchase (repo) transaction consists of a sale part and a repurchase part:

- Before the transaction, A owns the defaultable bond $\overline{C}$;
- B buys the bond from A for the price $\overline{C}(0)$;
- At the same time, A and B enter a *repurchase* agreement: B agrees to sell the bond back to A at time $t = T$ for the forward price K. A agrees to buy the bond.


initially sell at $\overline{C}(0)$
A
borrower
buy back at K
B
lender
implementation
of short sale
holding
bond $\overline{C}$
initially
sell at
$t = 0$
buy back
at $t = T$

The forward price K is the spot price $\overline{C}(0)$ of the bond, possibly adjusted for intermediate coupon payments, and increased by the *repo rate* r^{repo}:

$$K = (1 + Tr^{repo})\overline{C}(0).$$

For example, $\overline{C}(0) = \$100, T = 0.5$ (half year) and $r^{repo} = 10\%$, then

$$K = (1 + 0.5 \times 10\%)100 = \$105.$$

Short sale

To implement a short position, B does two more things:

- At time $t = 0$, B sells the bond in the market for $\overline{C}(0)$;
- At time $t = T$ (in order to deliver the bond to A), B has to buy the bond back in the market for the then current market price $\overline{C}(T)$.

- B is now exposed to the risk of price changes in $\overline{C}$ between time $t = 0$ and time $t = T$. The price difference $K - \overline{C}(T)$ is his profit or loss. For example, suppose $\overline{C}(T) = \$102$, then the profit is $\$105 - \$102 = \$3$.

- If the price falls $\overline{C}(T) < \overline{C}(0)(1 + r^{repo}T)$, then B makes a gain, because he can buy the bond back at a cheaper price. Thus, such a repo transaction is an efficient way for B to speculate on falling prices.

- To B, the repo transaction has achieved the aim of implementing a short position in the bond.

- This position is *funding-neutral* (or called unfunded transaction): he has to pay $\overline{C}(0)$ to A, but this amount he immediately gets from selling the bond in the market.

Collateralised lending transaction

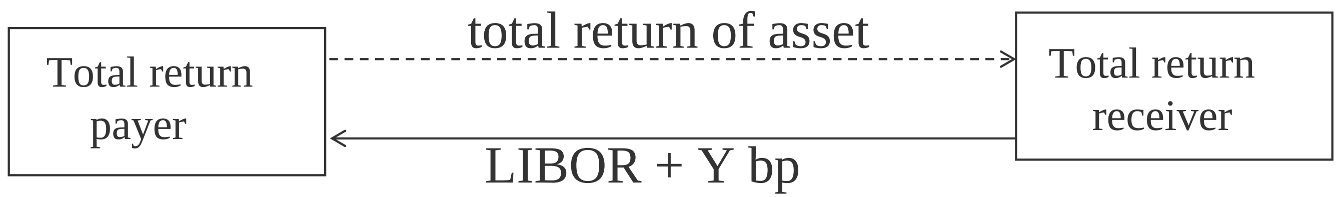
- A has borrowed from B the amount of $\overline{C}(0)$ at the rate r^{repo}, and as collateral he has delivered the bond to B. At maturity of the agreement, he will receive his bond $\overline{C}$ back after payment of K, the borrowed amount plus interest.

- To owners of securities like A, a repo transaction offers the opportunity to *refinance their position* at the repo rate. Usually, the repo rate is lower than alternative funding rates for A which makes this transaction attractive to him.

- A has given up the opportunity to get out of his position in the bond at an earlier time than T (except through another short sale in a second repo transaction). Repo borrowers are therefore usually long-term investors who did not intend to sell the bond anyway.

Total return swap

- Exchange the total economic performance of a specific asset for another cash flow.



Total return comprises the sum of interests, fees and any change-in-value payments with respect to the reference asset.

A commercial bank can hedge all credit risk on a bond/loan it has originated. The counterparty can gain access to the bond/loan on an off-balance sheet basis, without bearing the cost of originating, buying and administering the loan. The TRS terminates upon the default of the underlying asset.

Used as a financing tool

- The receiver wants financing to invest \$100 million in the reference bond. It approaches the payer (a financial institution) and agrees to the swap.
- The payer invests \$100 million in the bond. The payer retains ownership of the bond for the life of the swap and has much less exposure to the risk of the receiver defaulting (as compared to the actual loan of \$100 million).
- The receiver is in the same position as it would have been if it had borrowed money at LIBOR $+ s^{TRS}$ to buy the bond. He bears the market risk and default risk of the underlying bond.

Some essential features

1. The receiver is synthetically long the reference asset without having to fund the investment up front. He has almost the same payoff stream as if he had invested in risky bond directly and funded this investment at LIBOR $+ s^{TRS}$.
2. The TRS is marked to market at regular intervals, similar to a futures contract on the risky bond. The reference asset should be liquidly traded to ensure objective market prices for marking to market (determined using a dealer poll mechanism).
3. The TRS allows the receiver to leverage his position much higher than he would otherwise be able to (may require collateral). The TRS spread should not only be driven by the default risk of the underlying asset but also by the credit quality of the receiver.

The payments received by the total return receiver are:

1. The coupon $\overline{c}$ of the bond (if there were one since the last payment date T_{i-1}).
2. The price appreciation $(\overline{C}(T_i)-\overline{C}(T_{i-1}))^+$ of the underlying bond C since the last payment (if there were any).
3. The recovery value of the bond (if there were default).

The payments made by the total return receiver are:

1. A regular fee of LIBOR $+s^{TRS}$.
2. The price depreciation $(\overline{C}(T_{i-1}) - \overline{C}(T_i))^+$ of bond C since the last payment (if there were any).
3. The par value of the bond C (if there were a default in the meantime).

The coupon payments are netted and swap's termination date is earlier than bond's maturity.

Motivation of the receiver

1. Investors can create new assets with a specific maturity not currently available in the market.
2. Investors gain efficient off-balance sheet exposure to a desired asset class to which they otherwise would not have access.
3. Investors may achieve a higher leverage on capital – ideal for hedge funds. Otherwise, direct asset ownership is on on-balance sheet funded investment.
4. Investors can reduce administrative costs via an off-balance sheet purchase.
5. Investors can access entire asset classes by receiving the total return on an index.

Motivation of the payer

- A long-term investor, who feels that a reference asset in the portfolio may *widen in spread* in the short term but will recover later, may enter into a total return swap that is shorter than the maturity of the asset. She can gain from the price depreciation. This structure is *flexible and does not require a sale of the asset* (thus accommodates a temporary *short-term negative view* on an asset).

Differences between entering a total return swap and an outright purchase

(a) An outright purchase of the C-bond at $t = 0$ with a sale at $t = T_N$. B finances this position with debt that is rolled over at LIBOR, maturing at T_N.

(b) A total return receiver B in a TRS with the asset holder A.

1. B receives the coupon payments of the underlying security at the same time in both positions.

2. The debt service payments in strategy (a) and the LIBOR part of the funding payment in the TRS (strategy (b)) coincide, too.

Payoff streams of a total return swap to the total return receiver B (the payoffs to the total return payer A are the converse of these).

Time	Defaultable bond	TRS payments		
		Funding	Returns	Marking to market
$t = 0$	$-\overline{C}(0)$	0	0	0
$t = T_i$	$\overline{c}$	$-\overline{C}(0)(L_{i-1} + s^{TRS})$	$+\overline{c}$	$+\overline{C}(T_i) - \overline{C}(T_{i-1})$
$t = T_N$	$\overline{C}(T_N) + \overline{c}$	$-\overline{C}(0)(L_{N-1} + s^{TRS})$	$+\overline{c}$	$+\overline{C}(T_N) - \overline{C}(T_{N-1})$
Default	Recovery	$-\overline{C}(0)(L_{i-1} + s^{TRS})$	0	$-(\overline{C}(T_{i-1})$ – Recovery$)$

The TRS is unwound upon default of the underlying bond. Day count fractions are set to one, $\delta_i = 1$

The source of value difference lies in the marking-to-market of the TRS at the intermediate intervals.

Final payoff of strategy

B sells the bond in the market for $\overline{C}(T_N)$, and has to pay back his debt which costs him $\overline{C}(0)$. (The LIBOR coupon payment is already cancelled with the TRS.) This yields:

$$\overline{C}(T_N) - \overline{C}(0),$$

which is the amount that B receives at time T_N from following strategy (a), net of intermediate interest and coupon payments.

We decompose this total price difference between $t = 0$ and $t = T_N$ into the small, incremental differences that occur between the individual times T_i:

$$\begin{aligned}\overline{C}(T_N) - \overline{C}(0) &= [\overline{C}(T_i) - \overline{C}(T_{i-1})] + [\overline{C}(T_{i-1}) - \overline{C}(T_{i-2})] \\ &\quad + \cdots + [\overline{C}(T_1) - \overline{C}(0)].\end{aligned}$$

This representation allows us to distribute the final payoff of the strategy over the intermediate time intervals and to compare them to the payout of the TRS position (b).

- Each time interval $[T_{i-1}, T_i]$ contributes an amount of

$$\overline{C}(T_i) - \overline{C}(T_{i-1})$$

to the final payoff, and this amount is directly observable at time T_i.

- This payoff contribution can be converted into a payoff that occurs at time T_i by discounting it back from T_N to T_i, reaching

$$[\overline{C}(T_i) - \overline{C}(T_{i-1})]B(T_i, T_N).$$

Conversely, if we paid B the amount given in above equation at each T_i, and if B reinvested this money at the default-free interest rate until T_N, then B would have exactly the same final payoff as in strategy (a).

From the TRS position in strategy (b), B has a slightly different payoff:

$$\overline{C}(T_i) - \overline{C}(T_{i-1})$$

at all times $T_i > T_0$ net of his funding expenses.

Time value of intermediate payments

- The difference (b) − (a) is:

$$(\overline{C}(T_i) - \overline{C}(T_{i-1}))[1 - B(T_i, T_N)] = \Delta\overline{C}(T_i)[1 - B(T_i, T_N)].$$

  The above gives the excess payoff at time T_i of the TRS position over the outright purchase of the bond.

- This term will be positive if the change in value of the underlying bond $\Delta\overline{C}(T_i)$ is positive. It will be negative if the change in value of the underlying bond is negative, and zero if $\Delta\overline{C}(T_i)$ is zero.

- If the underlying asset is a bond, the likely sign of its change in value $\Delta\overline{C}(T_i)$ can be inferred from the deviation of its initial value $\overline{C}(0)$ from par. For example, if $\overline{C}(0)$ is below par, the price changes will have to be positive on average.

- The most extreme example of this kind would be a TRS on a default-free zero-coupon bond with maturity T_N.

- If we assume constant interest rates of R, this bond will always increase in value because it was issued at such a deep discount.

- A direct investor in the bond will only realise this increase in value at maturity of the bond, while the TRS receiver effectively receives prepayments. He can reinvest these prepayments and earn an additional return.

Bonds that initially trade at a discount to par should command a positive TRS spread s^{TRS}, while bonds that trade above par should have a negative TRS spread s^{TRS}.

2.4 Swaptions

- The buyer of a swaption has the right to enter into an interest rate swap by some specified date. The swaption also specifies the maturity date of the swap.

- The buyer can be the fixed-rate receiver (put swaption) or the fixed-rate payer (call swaption).

- The writer becomes the counterparty to the swap if the buyer exercises.

- The strike rate indicates the fixed rate that will be swapped versus the floating rate.

- The buyer of the swaption either pays the premium upfront.

Uses of swaptions

Used to hedge a portfolio strategy that uses an interest rate swap but where the cash flow of the underlying asset or liability is uncertain.

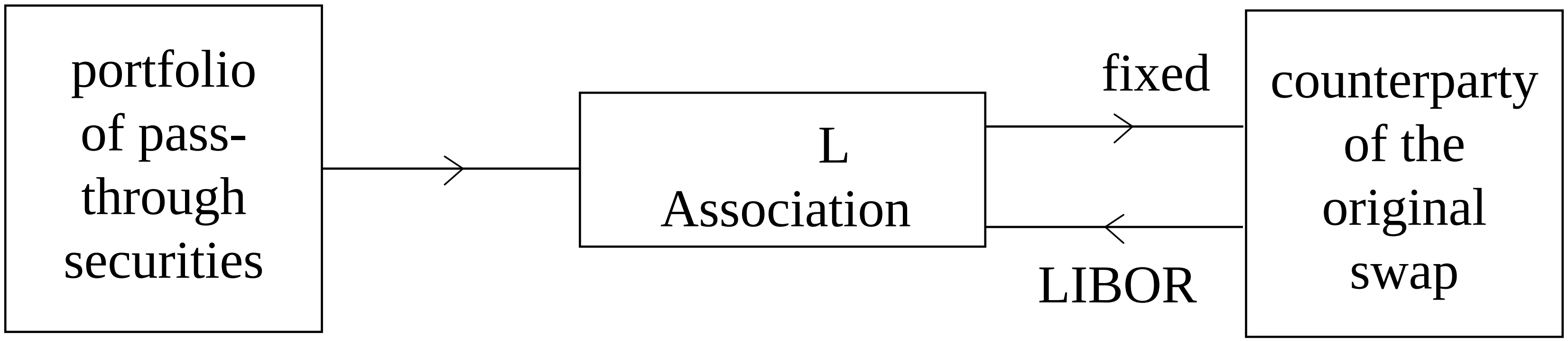
Uncertainties come from (i) callability, eg, a callable bond or mortgage loan, (ii) exposure to default risk.

Example

Consider a S & L Association entering into a 4-year swap in which it agrees to pay 9% fixed and receive LIBOR.

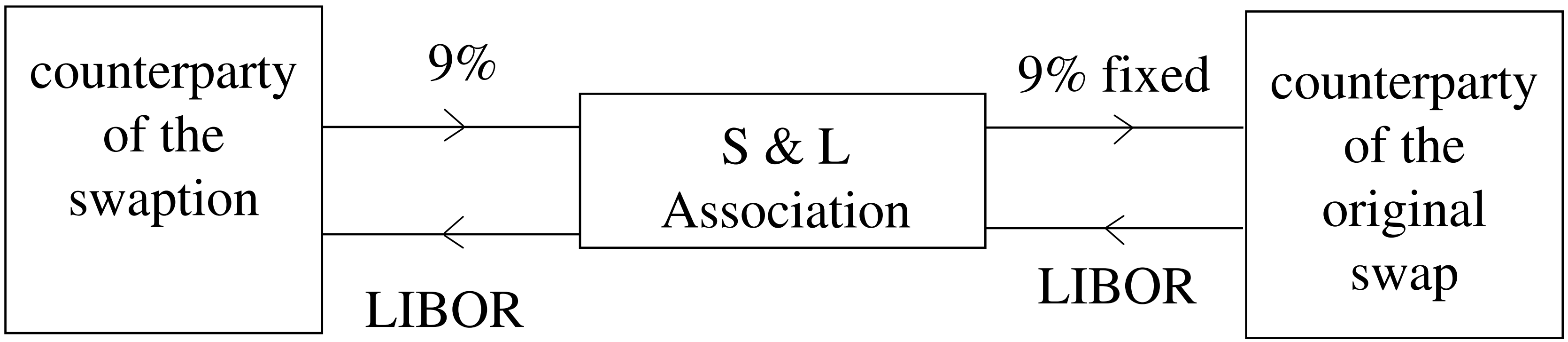
- The fixed rate payments come from a portfolio of mortgage pass-through securities with a coupon rate of 9%. One year later, mortgage rates decline, resulting in large prepayments.

- The purchase of a put swaption with a strike rate of 9% would be useful to offset the *original swap position*.



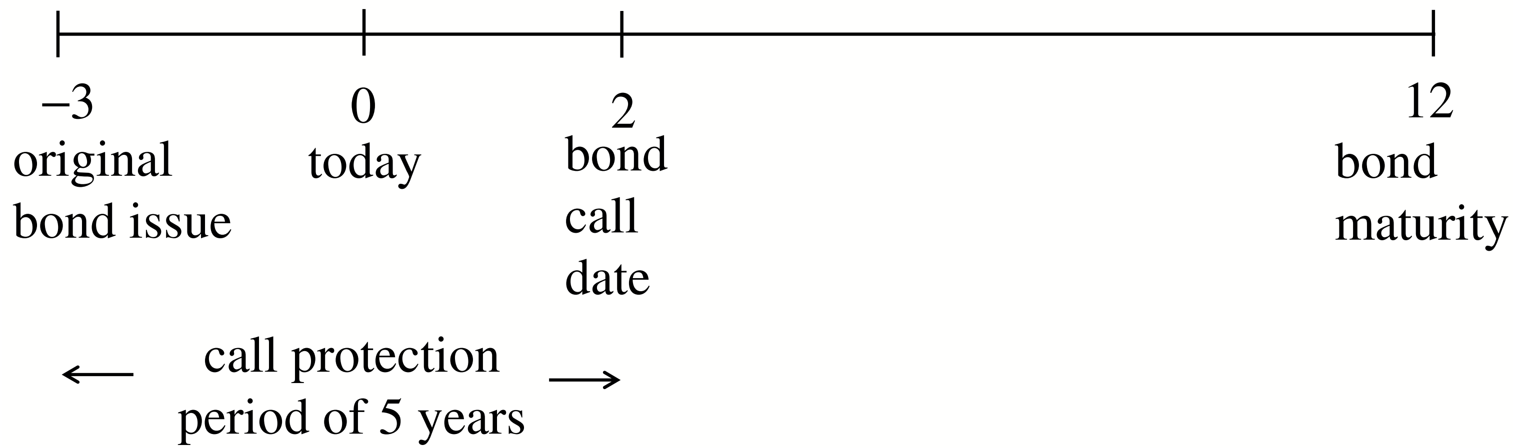
Due to decline in the interest rate, large prepayments are resulted in the mortgage pass-through securities. The source of 9% fixed payment dissipates. The swaption is in-the-money since the interest rate declines, so does the swap rate.

By exercising the put swaption, the S & L Association receives a fixed rate of 9%



Management of callable debts

Three years ago, XYZ issued 15-year fixed rate callable debt with a coupon rate of 12%.



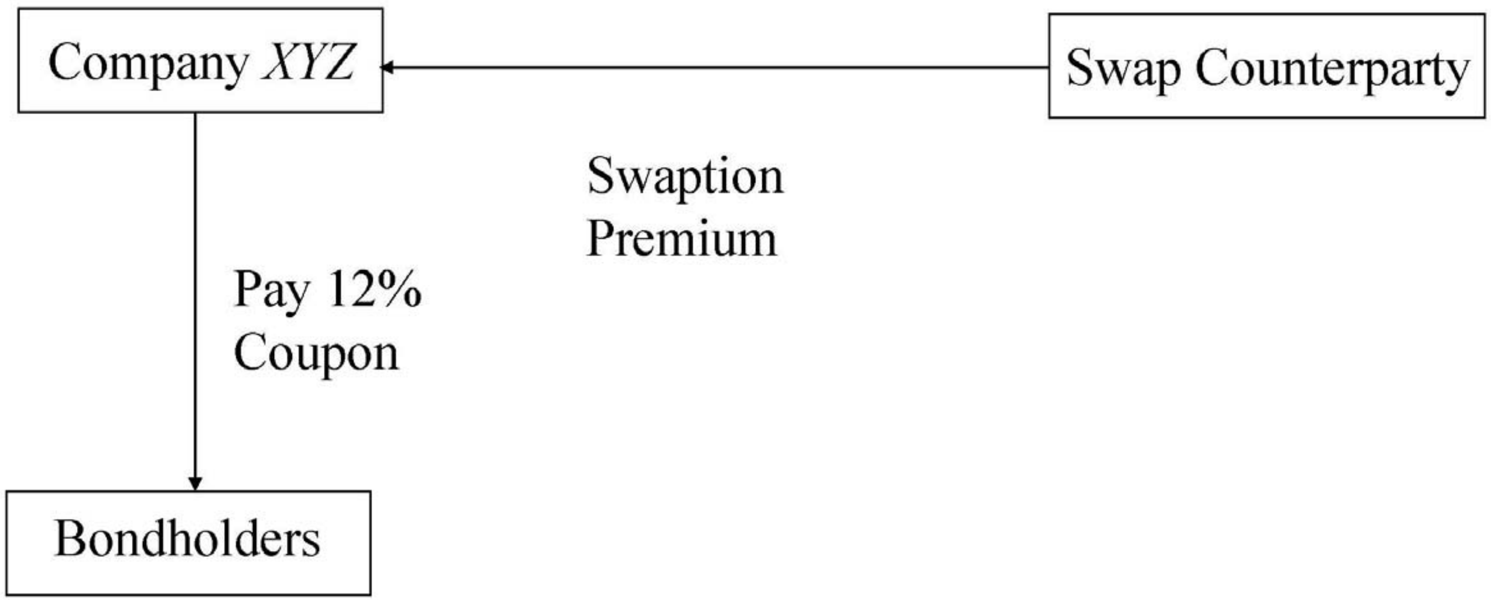
Strategy

The issuer sells a two-year fixed-rate receiver option on a 10-year swap, that gives the holder the right, but not the obligation, to receive the fixed rate of 12%.

Call monetization

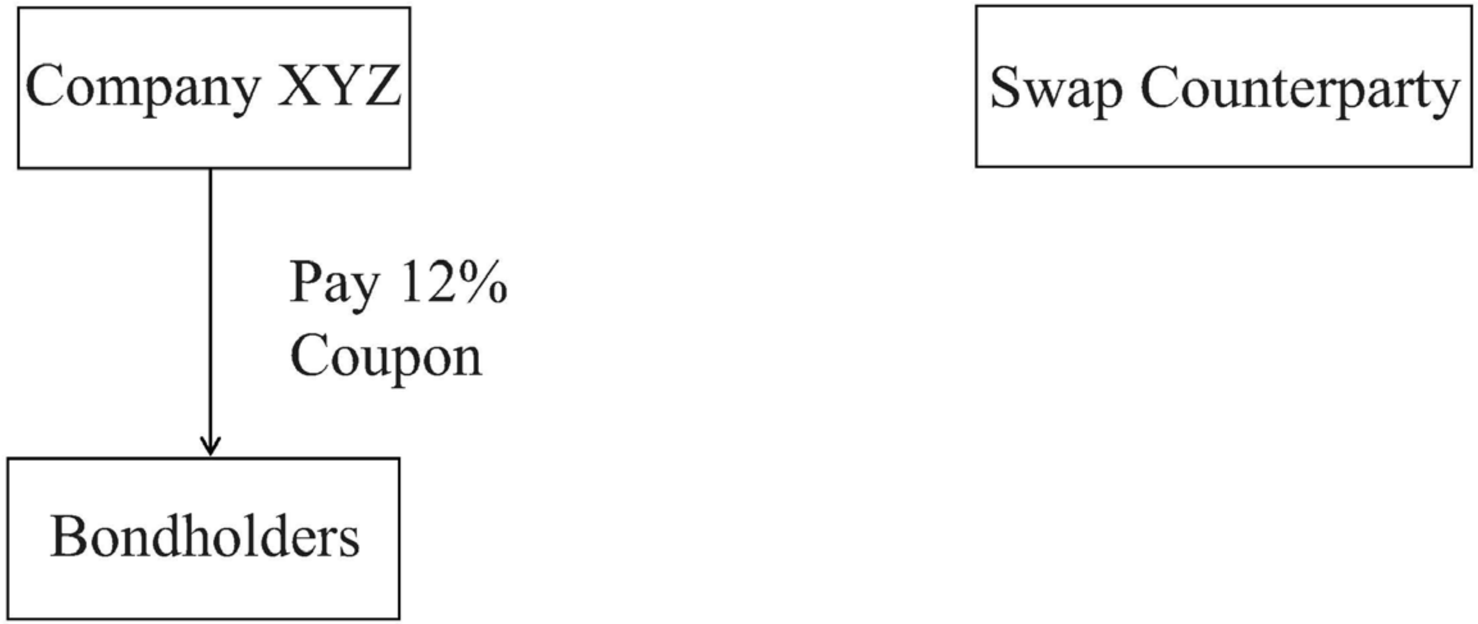
The value of the embedded callable right that can only be realized two years later is extracted today through a swaption sold today (receiving the swaption premium). The uncertainty in the cash flows due to the callable feature can be replicated by the swaption.

By selling the swaption today, the company has committed itself to paying a 12% coupon for the remaining life of the original bond.



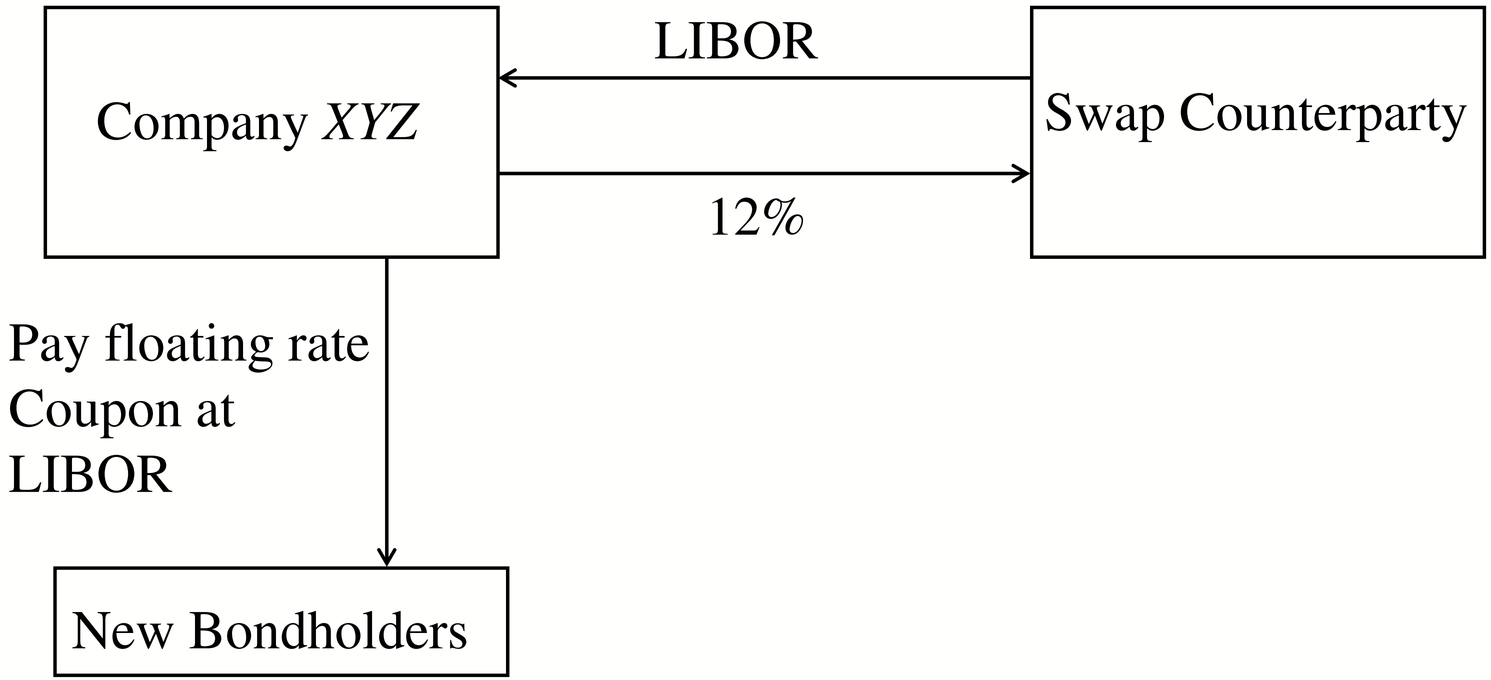
Call-Monetization cash flow: Swaption expiration date

Interest rate $\geq 12\%$



- Counterparty does not exercise the swaption
- XYZ earns the full proceed of the swaption premium

Interest rate $<$ 12%



- Counterparty exercises the swaption

- XYZ calls the bond. Once the old bond is retired, XYZ issues a new floating rate bond that pays floating rate LIBOR (funding rate depends on the creditworthiness of XYZ at that time).

Example on the use of swaption

- In August 2006 (two years ago), a corporation issued 7-year bonds with a fixed coupon rate of 10% payable semiannually on Feb 15 and Aug 15 of each year.

- The debt was structured to be callable (at par) offer a 4-year deferment period and was issued at par value of $100 million.

- In August 2008, the bonds are trading in the market at a price of 106, reflecting the general decline in market interest rates and the corporation's recent upgrade in its credit quality.

Question

The corporate treasurer believes that the current interest rate cycle has bottomed. If the bonds were callable today, the firm would realize a considerable savings in annual interest expense.

Considerations

- The bonds are still in their call protection period.
- The treasurer's fear is that the market rate might rise considerably prior to the call date in August 2010.

Notation

$T =$ 3-year Treasury yield that prevails in August, 2010

$T + BS =$ refunding rate of corporation, where BS is the company specific bond credit spread; $T + SS =$ prevailing 3-year swap fixed rate, where SS stands for the swap spread.

Strategy I. Enter on off-market forward swap as the fixed rate payer

Agreeing to pay 9.5% *(rather than the at-market rate of* 8.55%*) for a three-year swap, two years forward.*

Initial cash flow: Receive \$2.25 million since the the fixed rate is above the at-market rate.

Assume that the corporation's refunding spread remains at its current 100 bps level and the 3-year swap spread over Treasuries remains at 50 bps.

Gains and losses

August 2010 decisions:

- Gain on refunding (per settlement period): embedded callable right

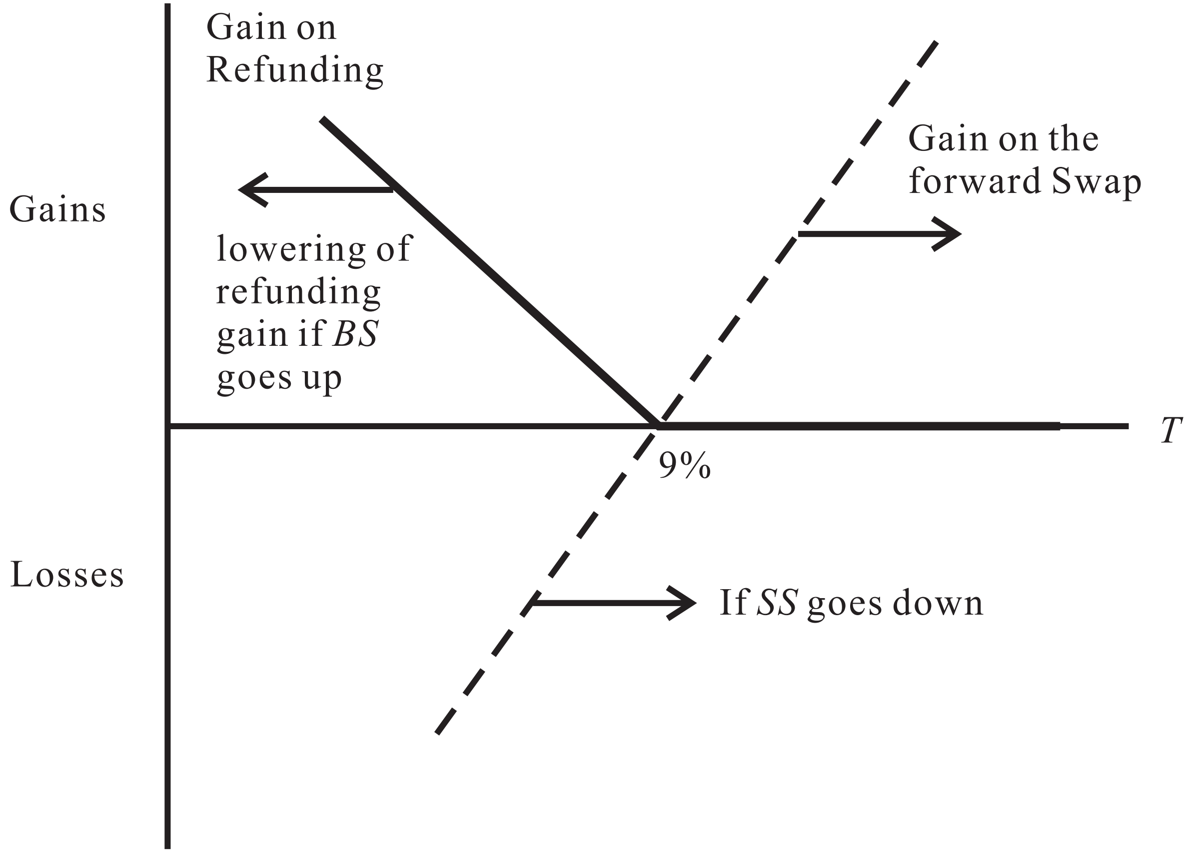
$$\begin{cases} [10 \text{ percent } -(T+BS)] & \text{if } T+BS < 10 \text{ percent}, \\ 0 & \text{if } T+BS \geq 10 \text{ percent}. \end{cases}$$

- Gain (or loss) on the swap forward (per settlement period):

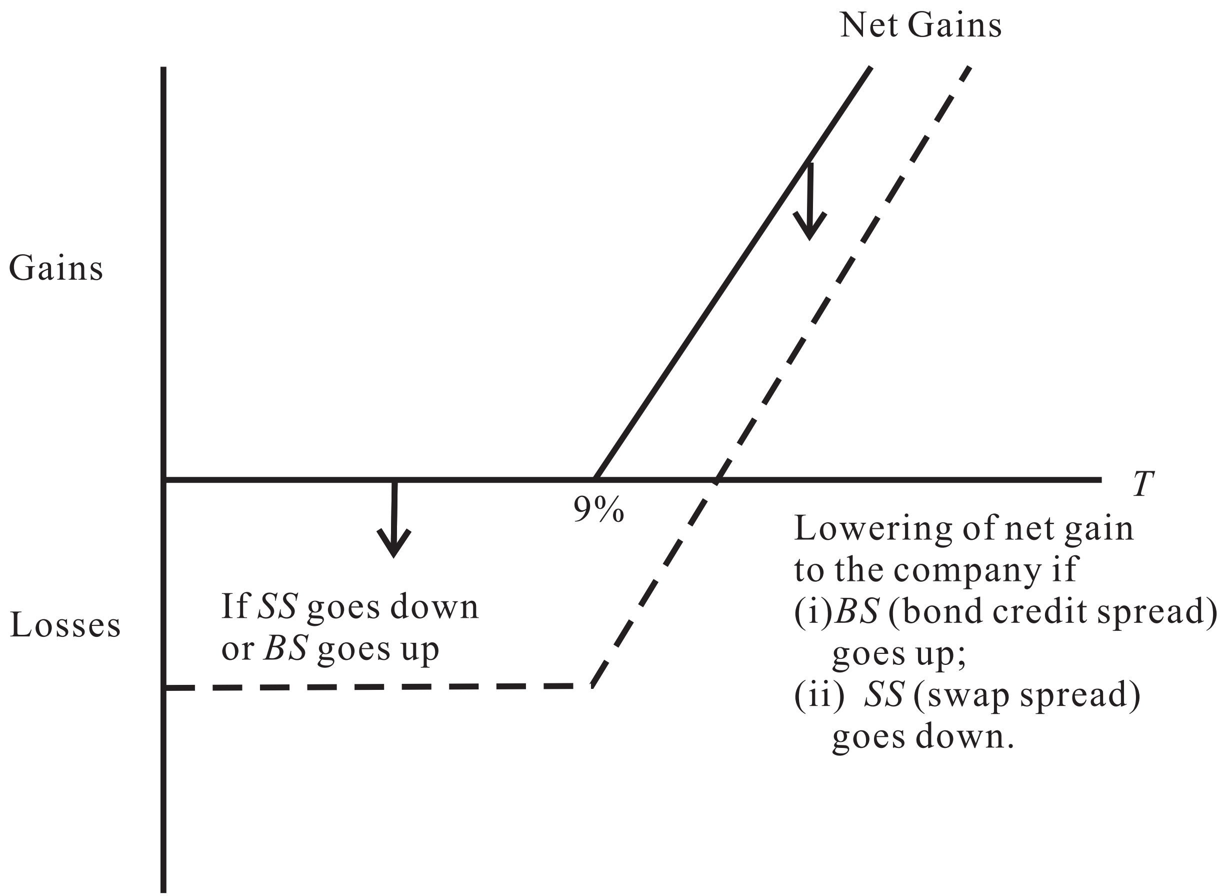
$$\begin{cases} -[9.50\text{percent} - (T+SS)] & \text{if } T+SS < 9.50\text{percent}, \\ [(T+SS) - 9.50 \text{ percent}] & \text{if } T+SS \geq 9.50\text{percent}. \end{cases}$$

  Assuming that $BS = 1.00$ percent and $SS = 0.50$ percent, these gains and losses in 2010 are:

Callable debt management with forward swap



- Refunding payoff resembles a put payoff on T
- Forward swap payoff resembles a forward payoff on T



Since the company stands to gain in August 2010 if rates rise, it has not fully monetized the embedded callable right. This is because a symmetric payoff instrument (a forward swap) is used to hedge an asymmetric payoff (option).

Strategy II. Buy payer swaption expiring in two years with a strike rate of 9.5%.

Initial cash flow: Pay \$1.10 million as the cost of the swaption (the swaption is out-of-the-money)

August 2010 decisions:

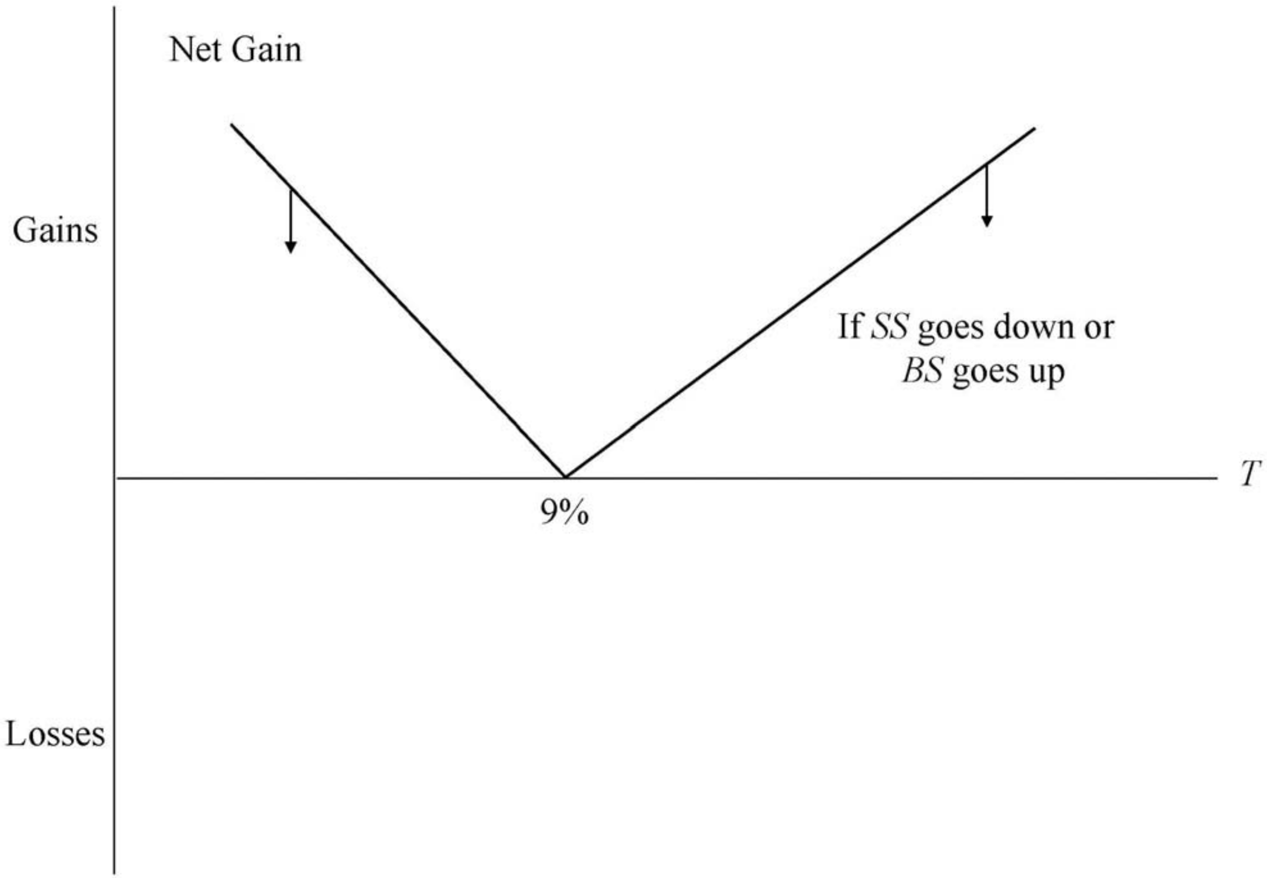
- Gain on refunding (per settlement period):

$$\begin{cases} 10 \text{ percent } - (T + BS) & \text{if } T + BS < 10 \text{ percent}, \\ 0 & \text{if } T + BS \geq 10 \text{ percent}. \end{cases}$$

- Gain (or loss) on unwinding the swap (per settlement period):

$$\begin{cases} (T + SS) - 9.50 \text{ percent} & \text{if } T + SS > 9.50 \text{ percent}, \\ 0 & \text{if } T + SS \leq 9.50 \text{ percent}.. \end{cases}$$

With $BS = 1.00$ percent and $SS = 0.50$ percent, these gains and losses in 2010 are:



Comment on the strategy (too conservative)

The company will benefit from Treasury rates being either higher or lower than 9% in August 2010. However, the treasurer had to spend \$1.1 million to lock in this straddle.

Strategy III. Sell a receiver swaption at a strike rate of 9.5% expiring in two years

Initial cash flow: Receive \$2.50 million (in-the-money swaption)

August 2010 decisions:

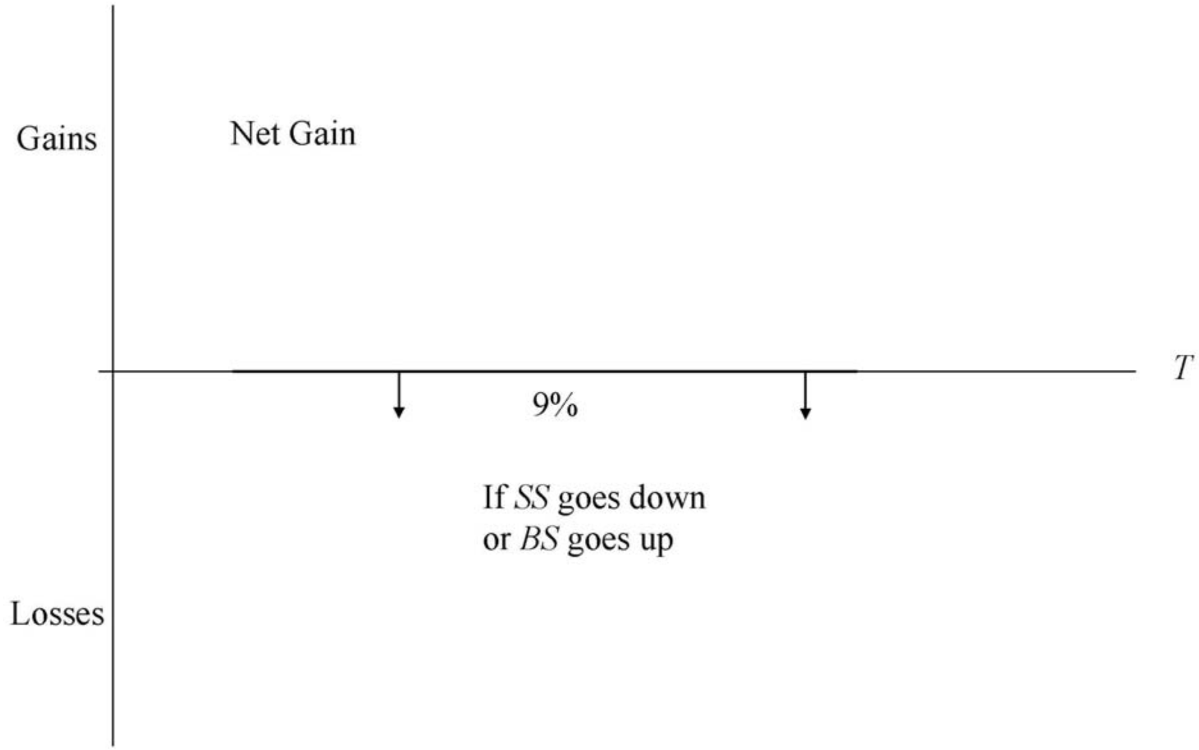
- Gain on refunding (per settlement period):

$$\begin{cases} [10 \text{ percent } - (T + BS)] & \text{if } T + BS < 10 \text{ percent}, \\ 0 & \text{if } T + BS \geq 10 \text{ percent}. \end{cases}$$

- Loss on unwinding the swap (per settlement period):

$$\begin{cases} [9.50 \text{ percent } -(T + SS)] & \text{if } T + SS < 9.50 \text{ percent}, \\ 0 & \text{if } T + SS \geq 9.50 \text{ percent}. \end{cases}$$

With $BS = 1.00$ percent and $SS = 0.50$ percent, these gains and losses in 2010 are:



Comment on the strategy

By selling the receiver swaption, the company has been able to simulate the sale of the embedded call feature of the bond, thus fully monetizing that option. The only remaining uncertainty is the basis risk associated with unanticipated changes in swap and bond spreads.

Cancelable swap

- A cancelable swap is a plain vanilla interest rate swap where one side has the option to terminate on one or more payment dates.
- Terminating a swap is the same as entering into the offsetting (opposite) swaps.
- If there is only one termination date, a cancelable swap is the same as a regular swap plus a position in a European swaption.

Example

- A ten-year swap where Microsoft will receive 6% and pay LIBOR. Suppose that Microsoft has the option to terminate at the end of six years.

- The swap is a regular ten-year swap to receive 6% and pay LIBOR plus long position in a six-year European option to enter a four-year swap where 6% is paid and LIBOR is received (the latter is referred to as a 6×4 European swaption).

- When the swap can be terminated on a number of different payment dates, it is a regular swap plus a Bermudan-style swaption.

Relation of swaptions to bond options

- An interest rate swap can be regarded as an agreement to exchange a fixed-rate bond for a floating-rate bond. At the start of a swap, the value of the floating-rate bond paying LIBOR always equals the notional principal of the swap.

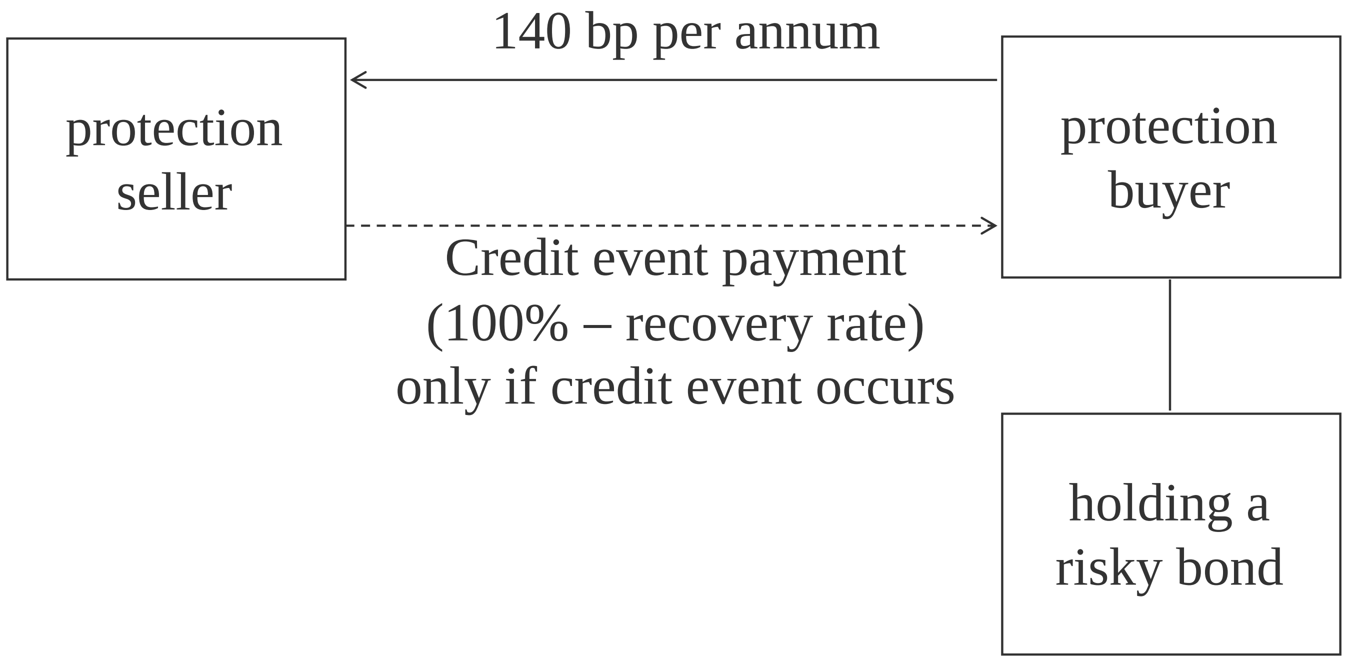
- A swaption can be regarded as an option to exchange a fixed-rate bond for the notional principal of the swap.

- If a swaption gives the holder the right to pay fixed and receive floating, it is a put option on the fixed-rate bond with strike price equal to the notional principal.

- If a swaption gives the holder the right to pay floating and receive fixed, it is a call option on the fixed-rate bond with a strike price equal to the notional principal.

2.5 Credit default swaps

The protection seller receives fixed periodic payments from the protection buyer in return for making a single contingent payment covering losses on a reference asset following a default.



Protection seller

- earns premium income with no funding cost
- gains customized, synthetic access to the risky bond

Protection buyer

- hedges the default risk on the reference asset

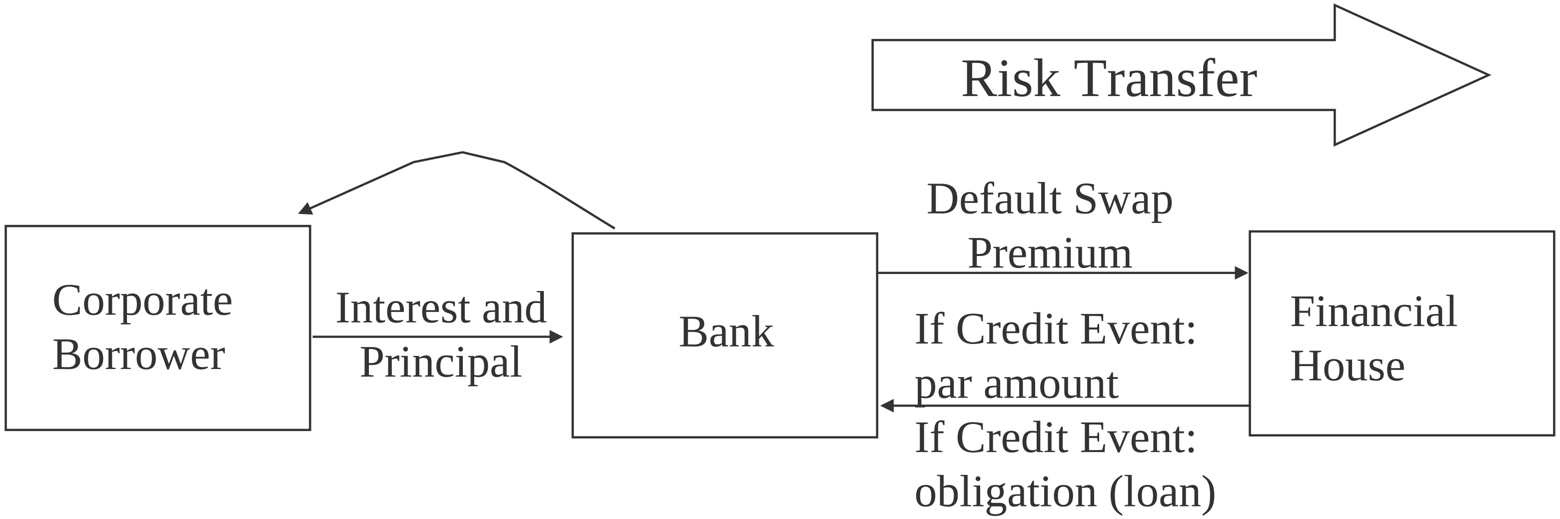
1. Very often, the bond tenor is longer than the swap tenor. In this way, the protection seller does not have exposure to the full period of the bond.

2. Basket default swap – gain additional yield by selling default protection on several assets.

A bank lends 10mm to a corporate client at L + 65bps. The bank also buys 10mm default protection on the corporate loan for 50bps.

Objective achieved

- maintain relationship
- reduce credit risk on a new loan



Settlement of compensation payment

1. Physical settlement:

   The defaultable bond is put to the Protection Seller in return for the par value of the bond.

2. Cash compensation:

   An independent third party determines the loss upon default at the end of the settlement period (say, 3 months after the occurrence of the credit event).

   $$\text{Compensation amount} = (1 - \text{recovery rate}) \times \text{bond par.}$$

Selling protection

To receive credit exposure for a fee or in exchange for credit exposure to better diversify the credit portfolio.

Buying protection

To reduce either individual credit exposure or credit concentrations in portfolios. Synthetically to take a short position in an asset which are not desired to sell outright, perhaps for relationship or tax reasons.

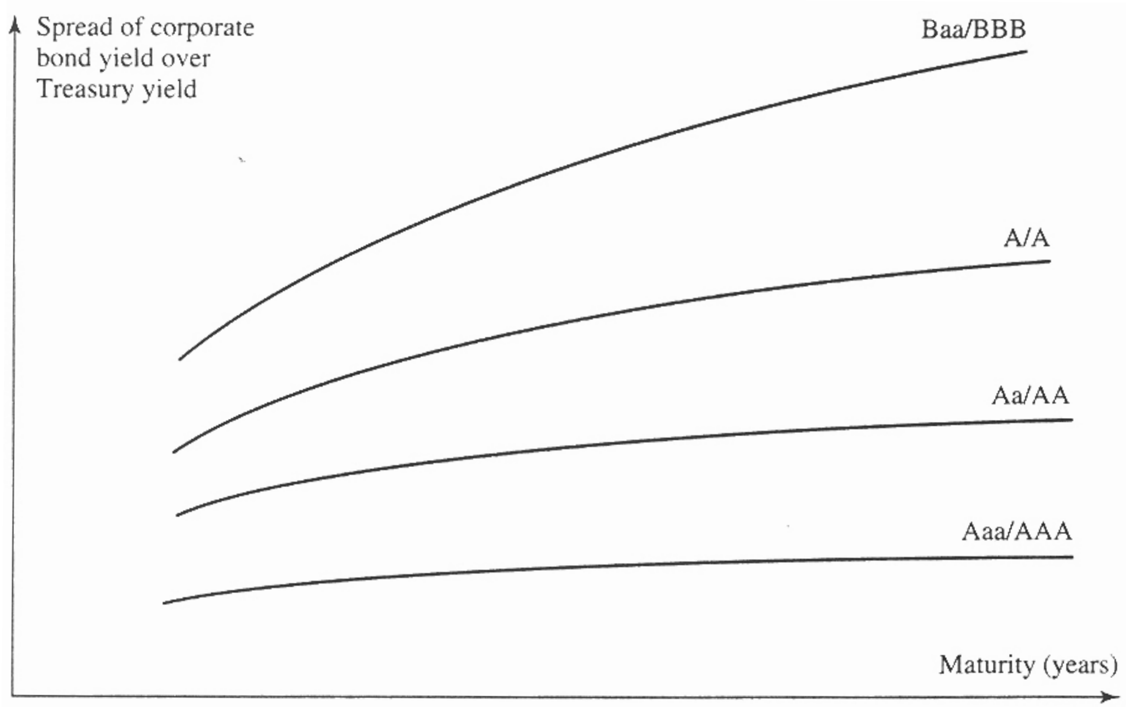
The price of a corporate bond must reflect not only the spot rates for default-free bonds but also a risk premium to reflect default risk and any options embedded in the issue.

Credit spreads: compensate investor for the risk of default on the underlying securities

Construction of a credit risk adjusted yield curve is hindered by

1. The general absence in money markets of liquid traded instruments on credit spread. For liquidly traded corporate bonds, we may have good liquidity on trading of credit default swaps whose underlying is the credit spread.

2. The absence of a complete term structure of credit spreads as implied from traded corporate bonds. At best we only have infrequent data points.



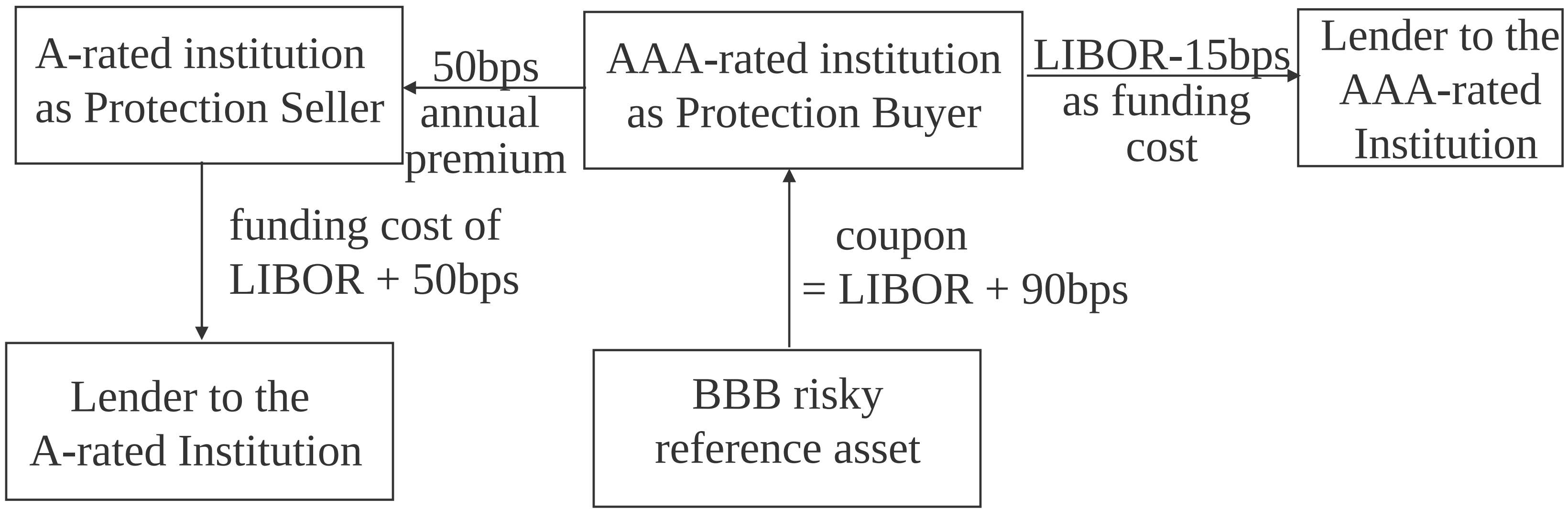
Spread over Treasury zero curve for corporate bond yields

- The spread increases as the rating declines. It also increases with maturity.

- The spread tends to increase faster with maturity for low credit ratings than for high credit ratings.

Funding cost arbitrage

Should the Protection Buyer look for a Protection Seller who has a higher/lower credit rating than himself?



The combined risk faced by the Protection Buyer:

- default of the BBB-rated bond
- default of the Protection Seller on the contingent payment

Consider the S&P's Ratings for jointly supported obligations (the two credit assets are uncorrelated)

	A_+	A	A_-	BBB_+	BBB
A_+	AA_+	AA_+	AA_+	AA	AA
A	AA_+	AA	AA	AA_-	AA_-

The AAA-rated Protection Buyer creates a synthetic AA_-asset with a coupon rate of LIBOR + 90bps − 50bps = LIBOR + 40bps. This is better than LIBOR + 30bps, which is the coupon rate of a AA_-asset (net gains of 10bps).

For the A-rated Protection Seller, it gains synthetic access to a BBB-rated asset with earning of net spread of

$$50\text{bps} - [\underbrace{(\text{LIBOR} + 90\text{bps}) - (\text{LIBOR} + 50\text{bps})}_{\substack{\text{the A-rated Protection Seller earns} \\ \text{40bps if it owns the BBB asset directly}}}] = 10\text{bps}$$

- Funding cost of the A-rated Protection Seller = LIBOR + 50bps
- Coupon from the underlying BBB bond = LIBOR + 90bps
- Credit swap premium earned = 50bps

In order that the credit arbitrage works, the funding cost of the default protection seller must be higher than that of the default protection buyer.

Example

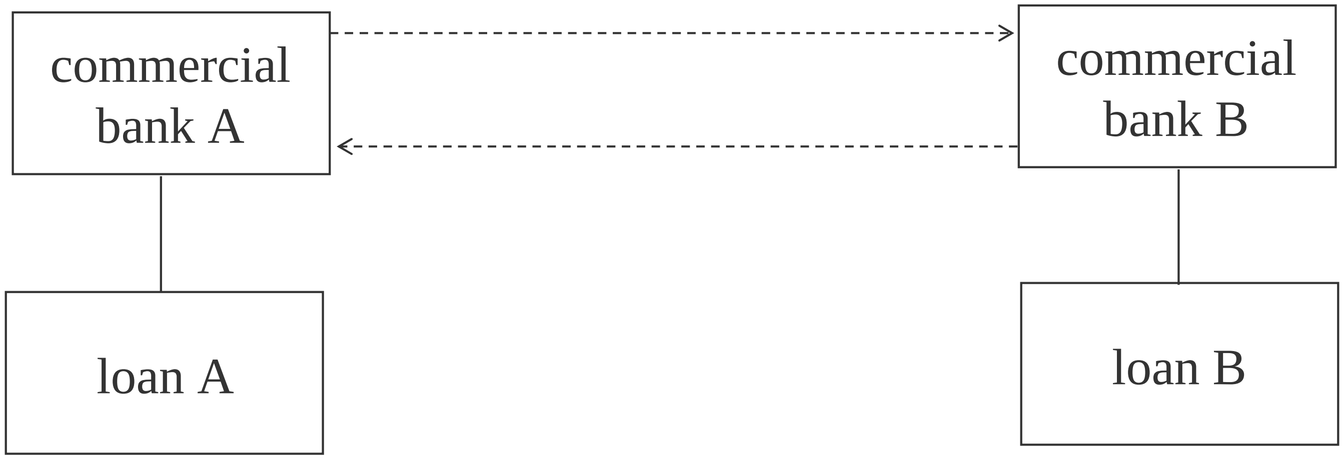
Suppose the A-rated institution is the Protection Buyer, and assume that it has to pay 60bps for the credit default swap premium (higher premium since the AAA-rated institution has lower counterparty risk).

$$
\begin{aligned}
& \text{spread earned from holding the risky bond} \\
= \; & \text{coupon from bond} - \text{funding cost} \\
= \; & (\text{LIBOR} + 90\text{bps}) - (\text{LIBOR} + 50\text{bps}) = 40\text{bps}
\end{aligned}
$$

which is lower than the credit swap premium of 60bps paid for hedging the credit exposure. No deal is done!

Credit default exchange swaps

Two institutions that lend to different regions or industries can diversify their loan portfolios in a single non-funded transaction – hedging the concentration risk on the loan portfolios.

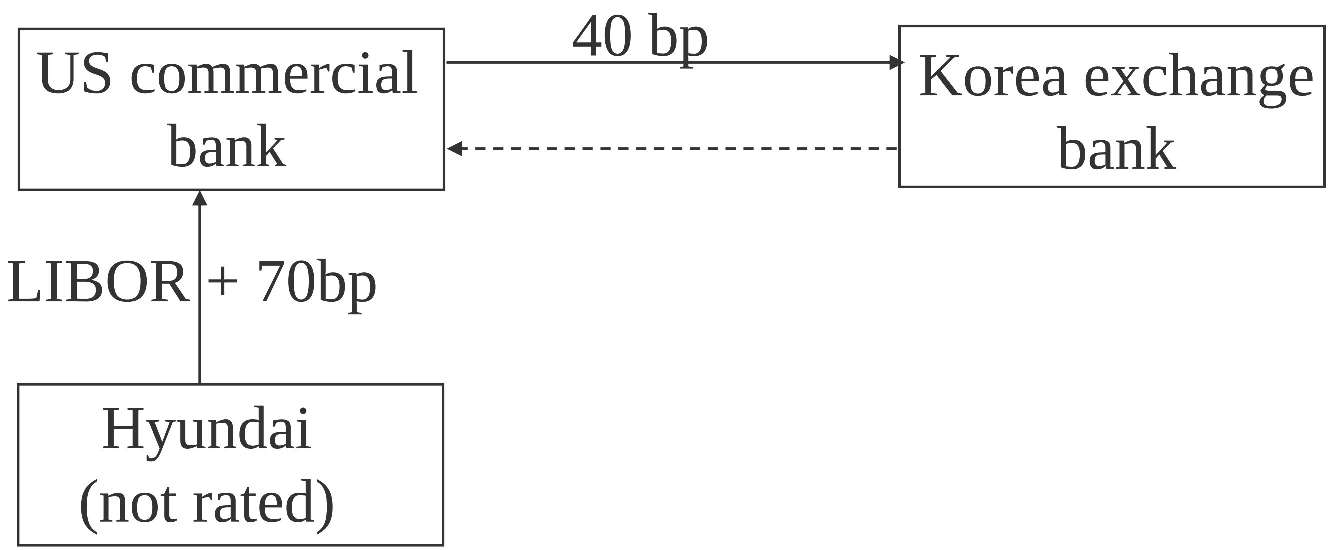


- contingent payments are made only if credit event occurs on a reference asset

- periodic payments may be made that reflect the different risks between the two reference loans

Counterparty risk in CDS

Before the Fall 1997 crisis, several Korean banks were willing to offer credit default protection on other Korean firms.



⋆ Higher geographic risks lead to higher default correlations.

Advice: Go for a European bank to buy the protection.

How does the inter-dependent default risk structure between the Protection Seller and the Reference Obligor affect the swap rate?

1. *Replacement cost* (Seller defaults earlier)

   - If the Protection Seller defaults prior to the Reference Entity, then the Protection Buyer renews the CDS with a new counterparty.

   - Supposing that the default risks of the Protection Seller and Reference Entity are positively correlated, then there will be an increase in the swap rate of the new CDS.

2. *Settlement risk* (Reference Entity defaults earlier)

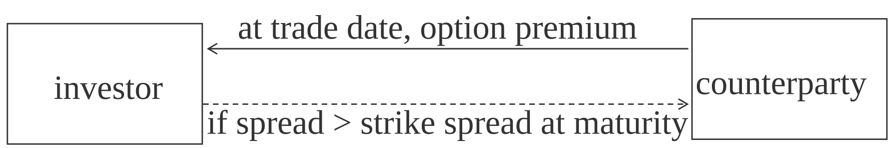
   - The Protection Seller defaults during the settlement period after the default of the Reference Entity.

Credit spread option

- hedge against rising credit spreads;
- target the future purchase of assets at favorable prices.

Example

An investor wishing to buy a bond at a price below market can sell a credit spread option to target the purchase of that bond if the credit spread increases (earn the premium if spread narrows).



Payout = notional $\times$ (final spread $-$ strike spread)$^+$

- It may be structured as a put option that protects against the drop in bond price – right to sell the bond when the spread moves above a target strike spread.

Example

The holder of the put spread option has the right to sell the bond at the strike spread (say, spread = 330 bps) when the spread moves above the strike spread (corresponding to a drop of the bond price).

May be used to target the future purchase of an asset at a favorable price.

The investor intends to purchase the bond below current market price (300 bps above US Treasury) in the next year and has targeted a forward purchase price corresponding to a spread of 350 bps. She sells for 20 bps a one-year credit spread put struck at 330 bps to a counterparty who is currently holding the bond and would like to protect the market price against spread above 330 bps.

- spread < 330; investor earns the option premium
- spread > 330; investor acquires the bond at 350 bps

Hedge strategy using fixed-coupon bonds

Portfolio 1

- One defaultable coupon bond $\overline{C}$; coupon $\overline{c}$, maturity t_N.
- One CDS on this bond, with CDS spread $\overline{s}$

The portfolio is unwound after a default.

Portfolio 2

- One default-free coupon bond C: with the same payment dates as the defaultable coupon bond and coupon size $\overline{c} - \overline{s}$.

The default free bond is sold after default of the defaultable counterpart.

Comparison of cash flows of the two portfolios

1. In survival, the cash flows of both portfolio are identical.

	Portfolio 1	*Portfolio 2*
$t = 0$	$-\overline{C}(0)$	$-C(0)$
$t = t_i$	$\overline{c} - \overline{s}$	$\overline{c} - \overline{s}$
$t = t_N$	$1 + \overline{c} - \overline{s}$	$1 + \overline{c} - \overline{s}$

2. At default, portfolio 1's value = par = 1 (full compensation by the CDS); that of portfolio 2 is $C(\tau)$, τ is the time of default.

   The price difference at default $= 1 - C(\tau)$. This difference is very small when the default-free bond is a par bond.

Remark

The issuer can choose $\overline{c}$ to make the bond be a par bond such that the initial value of the bond is at par.

This is an approximate replication.

Recall that the value of the CDS at time 0 is zero. Neglecting the difference in the values of the two portfolios at default, the no-arbitrage principle dictates

$$\overline{C}(0) = C(0) = B(0, t_N) + \overline{c}A(0) - \overline{s}A(0).$$

Here, $(\overline{c} - \overline{s})A(0)$ is the sum of present value of the coupon payments at the fixed coupon rate $\overline{c} - \overline{s}$. The equilibrium CDS rate $\overline{s}$ can be solved:

$$\overline{s} = \frac{B(0, t_N) + \overline{c}A(0) - \overline{C}(0)}{A(0)}.$$

$B(0, t_N) + \overline{c}A(0)$ is the time-0 price of a default free coupon bond paying coupon at the rate of $\overline{c}$.

Cash-and-carry arbitrage with par floater

A par floater $\overline{C}'$ is a defaultable bond with a floating-rate coupon of $\bar{c}_i = L_{i-1} + s^{par}$, where the par spread s^{par} is adjusted such that at issuance the par floater is valued at par.

Portfolio 1

- One defaultable par floater $\overline{C}'$ with spread s^{par} over LIBOR.
- One CDS on this bond: CDS spread is $\overline{s}$.

The portfolio is unwound after default.

Portfolio 2

- One default-free floating-coupon bond C': with the same payment dates as the defaultable par floater and coupon at LIBOR, $c_i = L_{i-1}$.

The bond is sold after default.

Time	*Portfolio 1*	*Portfolio 2*
$t = 0$	-1	-1
$t = t_i$	$L_{i-1} + s^{par} - \overline{s}$	L_{i-1}
$t = t_N$	$1 + L_{N-1} + s^{par} - \overline{s}$	$1 + L_{N-1}$
τ (default)	1	$C'(\tau) = 1 + L_i(\tau - t_i)$

The hedge error in the payoff at default is caused by accrued interest $L_i(\tau - t_i)$, accumulated from the last coupon payment date t_i to the default time τ. If we neglect the *small* hedge error at default, then

$$s^{par} = \overline{s}.$$

Remarks

- The non-defaultable bond becomes a par bond (with initial value that equals the par value) when it pays the floating rate that equals LIBOR. The extra coupon s^{par} paid by the defaultable par floater represents the credit spread demanded by the investor due to the potential credit risk. The above result shows that the credit spread s^{par} is just equal to the CDS spread $\overline{s}$.

- The above analysis neglects the counterparty risk of the Protection Seller of the CDS. Due to potential counterparty risk, the actual CDS spread will be lower.

Forward probability of default

Year	Cumulative default probability (%)	Forward default probability in year (%)	Survival probability (%)
1	0.2497	0.2497	99.7503
2	0.9950	0.7453	99.0050
3	2.0781	1.0831	97.9219
4	3.3428	1.2647	96.6582
5	4.6390	1.2962	95.3610

$$0.2497 + (1 - 0.2497) \times 0.7453 = 0.9950$$
$$0.9950 + (1 - 0.9950) \times 1.0831 = 2.0781$$

$$\begin{aligned} & \text{Survival probability up to Year 3} \\ = \; & 1 - \text{ cumulative default probability up to Year 2} \\ = \; & 1 - 0.009950 = 0.990050. \end{aligned}$$

Probability of default assuming no recovery

Define

$y(T)$:	Yield on a T-year corporate zero-coupon bond
$y^*(T)$:	Yield on a T-year risk-free zero-coupon bond
$Q(T)$:	Probability that corporation will default between time zero and time T
τ :	Random time of default

- The value of a T-year risk-free zero-coupon bond with a principal of 100 is $100e^{-y^*(T)T}$ while the value of a similar corporate bond is $100e^{-y(T)T}$.

- Present value of expected loss from default is

$$100[e^{-y^*(T)T} - e^{-y(T)T}].$$

There is a probability $Q(T)$ that the corporate bond will be worth zero at maturity and a probability $1 - Q(T)$ that it will be worth 100. The value of the bond is

$$\{Q(T) \times 0 + [1 - Q(T)] \times 100\}e^{-y^*(T)T} = 100[1 - Q(T)]e^{-y^*(T)T}.$$

The yield on the bond is $y(T)$, so that

$$100e^{-y(T)T} = 100[1 - Q(T)]e^{-y^*(T)T}$$

or

$$Q(T) = 1 - e^{-[y(T)-y^*(T)]T}.$$

Assuming zero recovery upon default, the survival probability as implied from the bond prices is

$$\begin{aligned} 1 - Q(T) &= \frac{\text{price of defaultable bond}}{\text{price of default free bond}} \\ &= e^{-\text{credit spread}\times T}, \end{aligned}$$

where credit spread $= y(T) - y^*(T)$.

Example

Suppose that the spreads over the risk-free rate for 5-year and a 10-year BBB-rated zero-coupon bonds are 130 and 170 basis points, respectively, and there is no recovery in the event of default.

$$\begin{aligned} Q(5) &= 1 - e^{-0.013 \times 5} = 0.0629 \\ Q(10) &= 1 - e^{-0.017 \times 10} = 0.1563. \end{aligned}$$

The probability of default between five years and ten years is $Q(5; 10)$ where

$$Q(10) = Q(5) + [(1 - Q(5)]Q(5; 10)$$

or

$$Q(5; 10) = \frac{0.01563 - 0.0629}{1 - 0.0629}.$$

Recovery rates

Amounts recovered on corporate bonds as a percent of par value from Moody's Investor's Service

Class	**Mean (%)**	**Standard derivation (%)**
Senior secured	52.31	25.15
Senior unsecured	48.84	25.01
Senior subordinated	39.46	24.59
Subordinated	33.17	20.78
Junior subordinated	19.69	13.85

The amount recovered is estimated as the market value of the bond one month after default.

- Bonds that are newly issued by an issuer must have seniority below that of existing bonds issued earlier by the same issuer.

Finite recovery rate

- In the event of a default the bondholder receives a proportion R of the bond's no-default value. If there is no default, the bondholder receives 100.

- The bond's no-default value is $100e^{-y^*(T)T}$ and the probability of a default is $Q(T)$. The value of the bond is

$$[1 - Q(T)]100e^{-y^*(T)T} + Q(T)100Re^{-y^*(T)T}$$

so that

$$100e^{-y(T)T} = [1 - Q(T)]100e^{-y^*(T)T} + Q(T)100Re^{-y^*(T)T}.$$

This gives

$$Q(T) = \frac{1 - e^{-[y(T)-y^*(T)]T}}{1 - R}.$$

Numerical example

Suppose the 1-year default free bond price is \$100 and the 1-year defaultable XYZ corporate bond price is \$80.

(i) Assuming $R = 0$, the probability of default of XYZ as implied by bond prices is

$$Q_0(1) = 1 - \frac{80}{100} = 20\%.$$

(ii) Assuming $R = 0.6$,

$$Q_R(1) = \frac{1 - \frac{80}{100}}{1 - 0.6} = \frac{20\%}{0.4} = 50\%.$$

The ratio of $Q_0(1) : Q_R(1) = 1 : \frac{1}{1-R}$.

Implied default probabilities (equity-based versus credit-based)

- Recovery rate has a significant impact on the defaultable bond prices. The forward probability of default as implied from the defaultable and default free bond prices requires estimation of the expected recovery rate (an almost impossible job).

- The industrial code $mKMV$ estimates default probability using stock price dynamics – equity-based implied default probability.

For example, the JAL stock price dropped to ¥1 in early 2010. Obviously, the equity-based default probability over one year horizon is close to 100% (stock holders receive almost nothing upon JAL's default). However, the credit-based default probability as implied by the JAL bond prices is less than 30% since the bond par payments are somewhat partially guaranteed even in the event of default.

Valuation of Credit Default Swap

- Suppose that the probability of a reference entity defaulting during a year *conditional on no earlier default* is 2%.

- Table 1 shows survival probabilities and forward default probabilities (i.e., default probabilities as seen at time zero) for each of the 5 years. The probability of a default during the first year is 0.02 and the probability that the reference entity will survive until the end of the first year is 0.98.

- The forward probability of a default during the second year is $0.02 \times 0.98 = 0.0196$ and the probability of survival until the end of the second year is $0.98 \times 0.98 = 0.9604$.

Table 1 Forward default probabilities and survival probabilities

Time (years)	*Forward default probability*	*Survival probability*
1	0.0200	0.9800
2	0.0196	$0.9604 = 0.98^2$
3	0.0192	$0.9412 = 0.98^3$
4	0.0188	$0.9224 = 0.98^4$
5	0.0184	$0.9039 = 0.98^5$

Forward default probability of default during the fourth year (as seen at Year Zero)
= survival probability until end of Year 3 X conditional probability of default in Year 4
$= 0.98^3 \times 0.02 = 0.9412 \times 0.02 = 0.0188.$

Assumptions on default and recovery rate

We will assume the defaults always happen halfway through a year and that payments on the credit default swap are made once a year, at the end of each year. We also assume that the risk-free (LIBOR) interest rate is 5% per annum with continuous compounding and the recovery rate is 40%.

Expected present value of CDS premium payments

Table 2 shows the calculation of the expected present value of the payments made on the CDS assuming that payments are made at the rate of s per year and the notional principal is \$1.

For example, there is a 0.9412 probability that the third payment of s is made (recall survival probability until the end of Year 3 $= 0.9412$). The expected payment is therefore $0.9412s$ and its present value is $0.9412se^{-0.05\times 3} = 0.8101s$. The total present value of the expected payments is $4.0704s$.

Table 2 Calculation of the present value of expected payments. Payment $= s$ per annum.

Time (years)	*Probability of survival*	*Expected payment*	*Discount factor*	*PV of expected payment*
1	0.9800	$0.9800s$	0.9512	$0.9322s$
2	0.9604	$0.9604s$	0.9048	$0.8690s$
3	0.9412	$0.9412s$	0.8607	$0.8101s$
4	0.9224	$0.9224s$	0.8187	$0.7552s$
5	0.9039	$0.9039s$	0.7788	$0.7040s$
Total				$4.0704s$

Table 3 Calculation of the present value of expected payoff. Notional principal = $1.

Time (years)	Probability of default	*Recovery rate*	*Expected payoff ($)*	*Discount factor*	*PV of expected payoff* ($)
0.5	0.0200	0.4	0.0120	0.9753	0.0117
1.5	0.0196	0.4	0.0118	0.9277	0.0109
2.5	0.0192	0.4	0.0115	0.8825	0.0102
3.5	0.0188	0.4	0.0113	0.8395	0.0095
4.5	0.0184	0.4	0.0111	0.7985	0.0088
Total					0.0511

For example, there is a 0.0192 probability of a payoff halfway through the third year. Given that the recovery rate is 40%, the expected payoff at this time is $0.0192 \times 0.6 \times 1 = 0.0115$. The present value of the expected payoff is $0.0115e^{-0.05\times2.5} = 0.0102$.

The total present value of the expected payoffs is $0.0511.

Table 4 Calculation of the present value of accrual payment.

Time (years)	*Probability of default*	*Expected accrual payment*	*Discount factor*	*PV of expected accrual payment*
0.5	0.0200	$0.0100s$	0.9753	$0.0097s$
1.5	0.0196	$0.0098s$	0.9277	$0.0091s$
2.5	0.0192	$0.0096s$	0.8825	$0.0085s$
3.5	0.0188	$0.0094s$	0.8395	$0.0079s$
4.5	0.0184	$0.0092s$	0.7985	$0.0074s$
Total				$0.0426s$

As a final step we evaluate in Table 4 the accrual payment made in the event of a default.

- There is a 0.0192 probability that there will be a final accrual payment halfway through the third year.
- The accrual payment is $0.5s$.
- The expected accrual payment at this time is therefore $0.0192 \times 0.5s = 0.0096s$.
- Its present value is $0.0096se^{-0.05\times 2.5} = 0.0085s$.
- The total present value of the expected accrual payments is $0.0426s$.

From Tables 2 and 4, the present value of the expected payment is

$$4.0704s + 0.0426s = 4.1130s.$$

Equating expected CDS premium payments and expected compensation payment

From Table 3, the present value of the expected payoff is 0.0511. Equating the two, we obtain the CDS spread for a new CDS as

$$4.1130s = 0.0511$$

or $s = 0.0124$. The mid-market spread should be 0.0124 times the principal or 124 basis points per year.

In practice, we are likely to find that calculations are more extensive than those in Tables 2 to 4 because

(a) payments are often made more frequently than once a year

(b) we might want to assume that defaults can happen more frequently than once a year.

Impact of expected recovery rate R on credit swap premium s

Recall that the expected compensation payment paid by the Protection Seller is $(1-R)\times$ notional. Therefore, the Protection Seller charges a higher s if her estimation of the recovery rate R is lower. Let s_R denote the credit swap premium when the recovery rate is R. We deduce that

$$\frac{s_{10}}{s_{50}} = \frac{(100-10)\%}{(100-50)\%} = \frac{90\%}{50\%} = 1.8.$$

Remark

A binary credit default swap pays the full notional upon default of the reference asset. The credit swap premium of a binary swap depends only on the estimated default probability but not on the recovery rate.

Marking-to-market a CDS

- At the time it is negotiated, a CDS, like most swaps, is worth close to zero. Later it may have a positive or negative value.

- Suppose, for example the credit default swap in our example had been negotiated some time ago for a spread of 150 basis points, the present value of the payments by the buyer would be $4.1130 \times 0.0150 = 0.0617$ and the present value of the payoff would be 0.0511.

- The value of swap to the seller would therefore be $0.0617 - 0.0511$, or 0.0166 times the principal.

- Similarly the mark-to-market value of the swap to the buyer of protection would be -0.0106 times the principal.

2.6 Currency swaps

Currency swaps originally were developed by banks in the UK to help large clients circumvent UK exchange controls in the 1970s.

- UK companies were required to pay an exchange equalization premium when obtaining dollar loans from their banks.

How to avoid paying this premium?

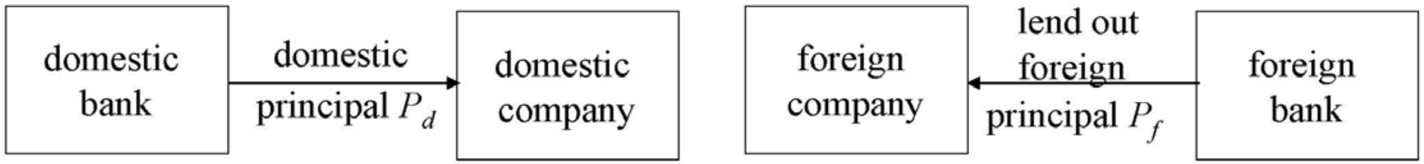
An agreement would then be negotiated whereby

- The UK organization borrowed sterling and lent it to the US company's UK subsidiary.
- The US organization borrowed dollars and lent it to the UK company's US subsidiary.

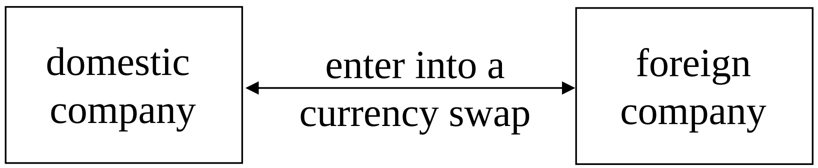
These arrangements were called back-to-back loans or parallel loans.

Exploiting comparative advantages

A domestic company has a comparative advantage in domestic loan but it wants to raise foreign capital. The situation for a foreign company happens to be reversed.



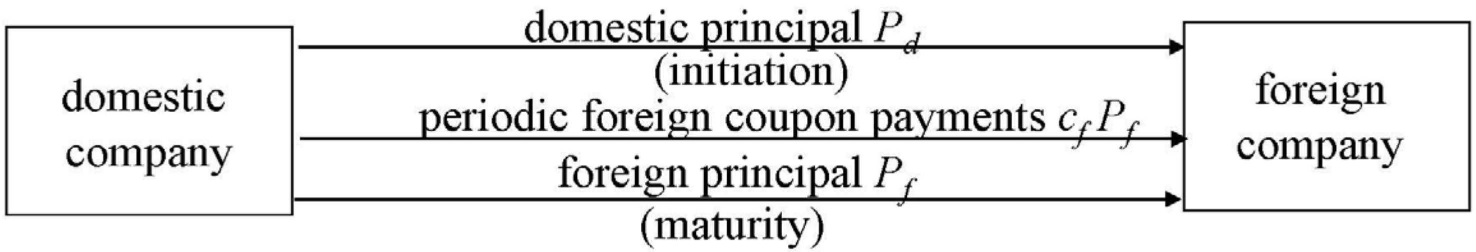
$$P_d = F_0 P_f$$

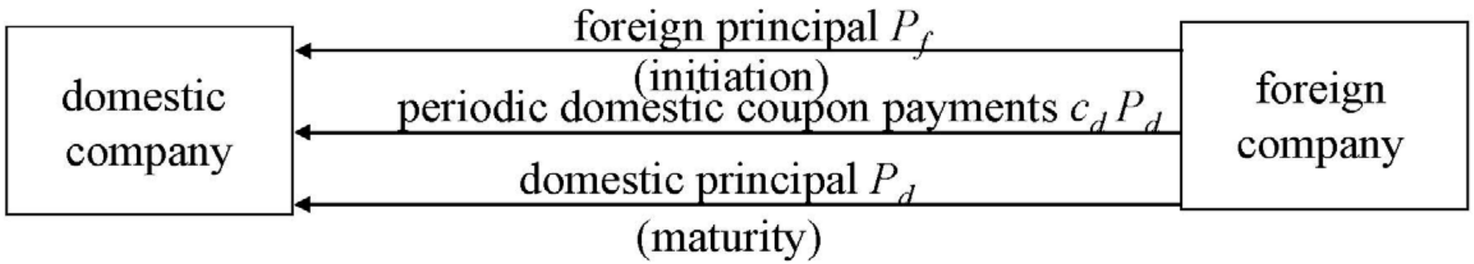


Goal:

To exploit the comparative advantages in borrowing rates for both companies in their domestic currencies.

Cashflows between the two currency swap counterparties (assuming no intertemporal default)





Settlement rules

Under the full (limited) two-way payment clause, the non defaulting counterparty is required (not required) to pay if the final net amount is favorable to the defaulting party.

Arranging finance in different currencies using currency swaps

The company issuing the bonds can use a currency swap to issue debt in one currency and then swap the proceeds into the currency it desires.

- To obtain lower cost funding:

  Suppose there is a strong demand for investments in currency A, a company seeking to borrow in currency B could issue bonds in currency A at a low rate of interest and swap them into the desired currency B.

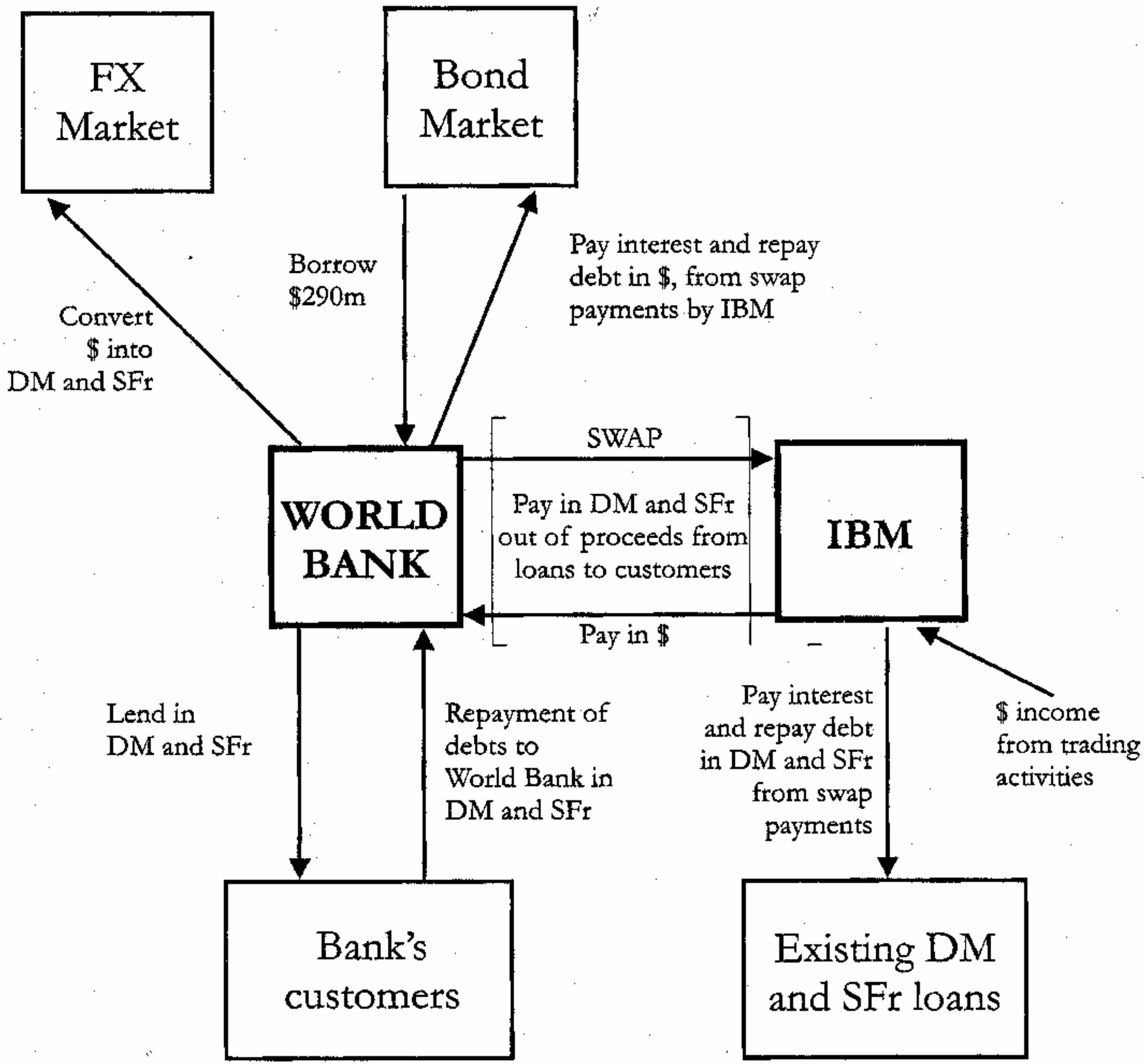
- To obtain funding in a form not otherwise available:

IBM/World Bank with Salomon Brothers as intermediary

- IBM had existing debts in DM and Swiss francs. This had created a FX exposure since IBM had to convert USD into DM and Swiss Francs regularly to make the coupon payments. Due to a depreciation of the DM and Swiss francs against the dollar, IBM could realize a large foreign exchange gain, but only if it could eliminate its DM and Swiss francs liabilities and "lock in" the gain and remove any future exposure.

- The World Bank was raising most of its funds in DM (interest rate = 12%) and Swiss francs (interest rate = 8%). It did not borrow in dollars, for which the interest rate cost was about 17%. Though it wanted to lend out in DM and Swiss francs, the bank was concerned that saturation in the bond markets could make it difficult to borrow more in these two currencies at a favorable rate. Its objective, as always, was to raise cheap funds.

World Bank/IBM Currency Swap, 1981



IBM/World Bank

- IBM was willing to take on dollar liabilities and made dollar payments (periodic coupons and principal at maturity) to the World Bank since it could generate dollar income from normal trading activities.

- The World Bank could borrow dollars, convert them into DM and SFr in FX market, and through the swap take on payment obligations in DM and SFr.

1. The foreign exchange gain on dollar appreciation is realized by IBM through the negotiation of a favorable swap rate in the swap contract.

2. The swap payments by the World Bank to IBM were scheduled so as to allow IBM to meet its debt obligations in DM and SFr.

Under this currency swap

- IBM pays regular US coupons and US principal at maturity.

- World Bank pays regular DM and SFr coupons together with DM and SFr principal at maturity.

Note that there is no exchange of principals at initiation, as in most conventional currency swaps. Now IBM converted its DM and SFr liabilities into USD, and the World Bank effectively raised hard currencies at a cheap rate. Both parties achieved their objectives!

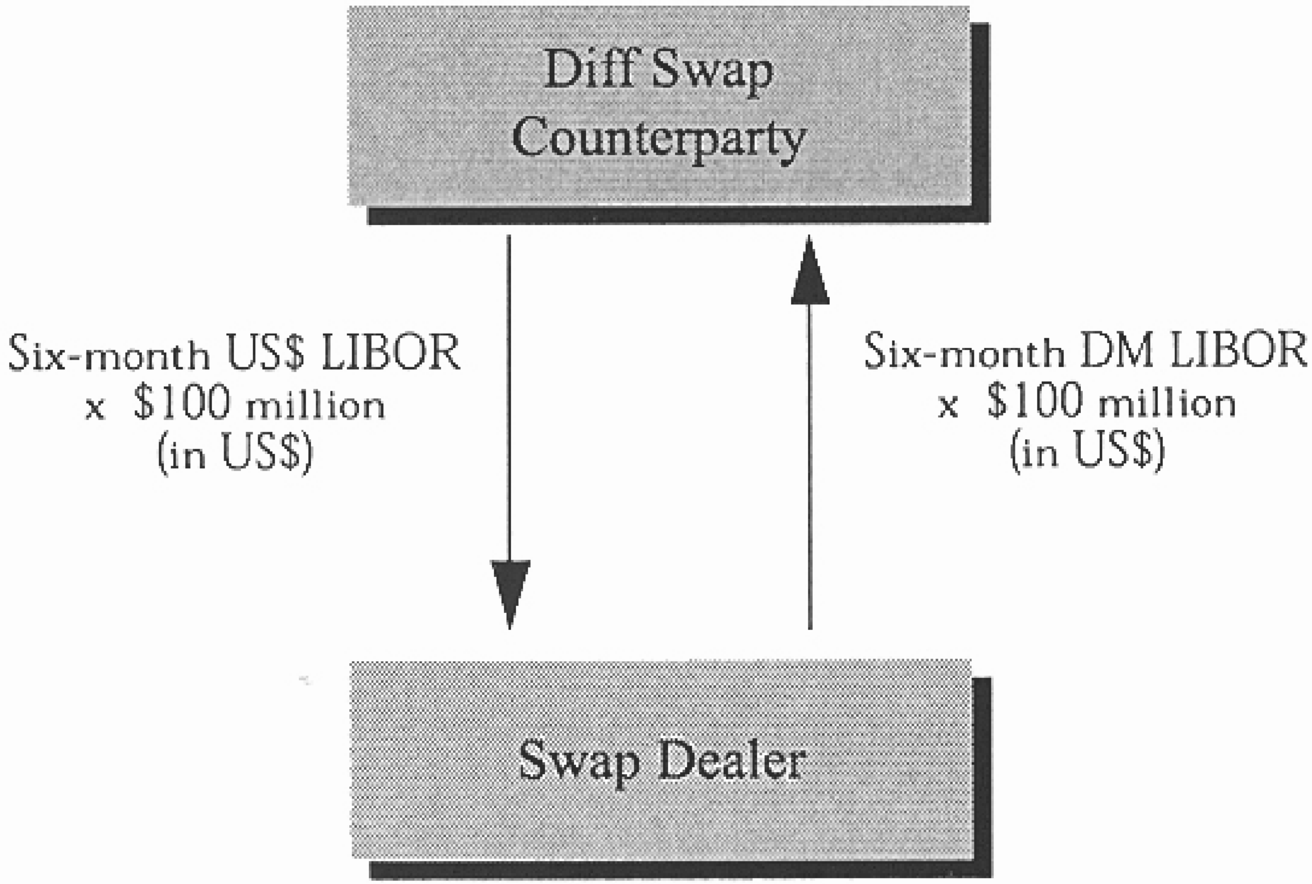
Differential Swap (Quanto Swap)

A special type of floating-against-floating currency swap that does not involve any exchange of principals, not even at maturity.

- Interest payments are exchanged by reference to a floating rate index in one currency and another floating rate index in a second currency. Both interest rates are applied to the same notional principal in one currency.
- Interest payments are made in the same currency.

Apparently, the risk factors are a floating domestic interest rate and a floating foreign interest rate. However, since foreign floating rates are applied on domestic payments, the correlation between exchange rate and foreign floating rate poses correlation risk.

Exhibit 1
DIFFERENTIAL SWAP: GENERIC CASH FLOWS



All cash flows are denominated in the same currency.

Rationale

To exploit large differential in floating interest rates across major currencies without directly holding the foreign currency.

Applications

- Money market investors use diff swaps to take advantage of the high yield if they expect yields to persist in this discount currency.
- Corporate borrowers with debt in a discount currency can use diff swaps to lower their effective borrowing costs from the expected persistence of a low nominal interest rate in the premium currency. Pay out the lower floating rate in the premium currency in exchange to receive the high floating rate in the discount currency.

- The value of a diff swap in general would not be zero at initiation. The value is settled either as an upfront premium payment or amortized over the whole life as a margin over the floating rate index.

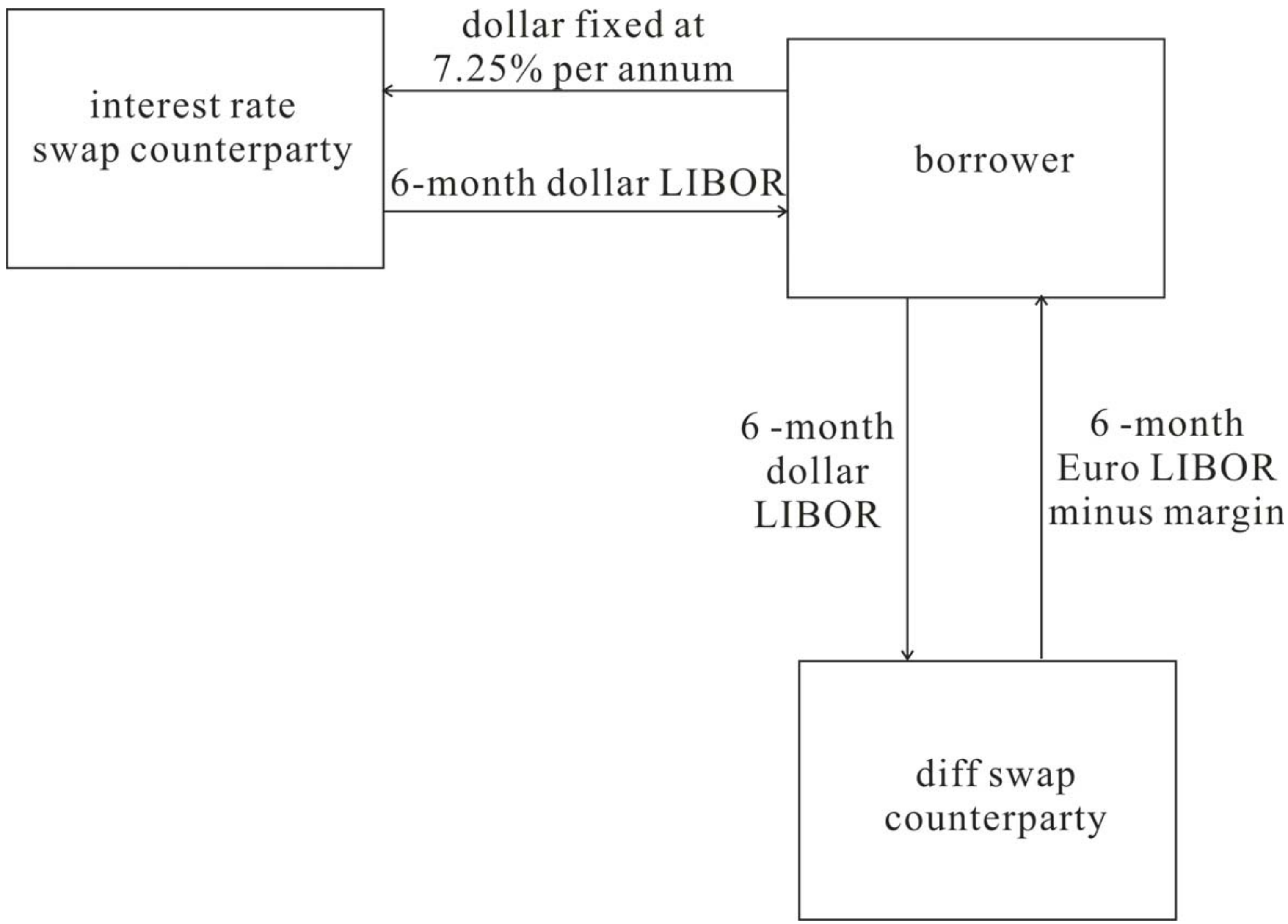
Uses of a differential swap

Suppose a company has hedged its liabilities with a dollar interest rate swap serving as the fixed rate payer, the shape of the yield curve in that currency will result in substantial extra costs. The cost is represented by the differential between the short-term 6-month dollar LIBOR and medium to long-term implied LIBORs payable in dollars – upward sloping yield curve.

- The borrower enters into a dollar interest rate swap whereby it pays a fixed rate and receives a floating rate (6-month dollar LIBOR).

- Simultaneously, it enters into a diff swap for the same dollar notional principal amount whereby the borrower agrees to pay 6-month dollar LIBOR and receive 6-month Euro LIBOR less a margin.

The result is to increase the floating rate receipts under the dollar interest rate swap so long as 6-monthly Euro LIBOR, adjusted for the diff swap margin, exceeds 6-month LIBOR. This has the impact of lowering the effective fixed rate cost to the borrower.



The borrower has been forced to pay a high fixed rate of 7.25% due to the upward sloping yield curve of LIBOR. On the other hand, this may help the borrower to obtain a lower margin. The borrower gains if the upward trend of LIBOR is not realized.

Exhibit 2

DIFFERENTIAL SWAP: AFTER DEALER HEDGES WITH INTEREST RATE SWAPS

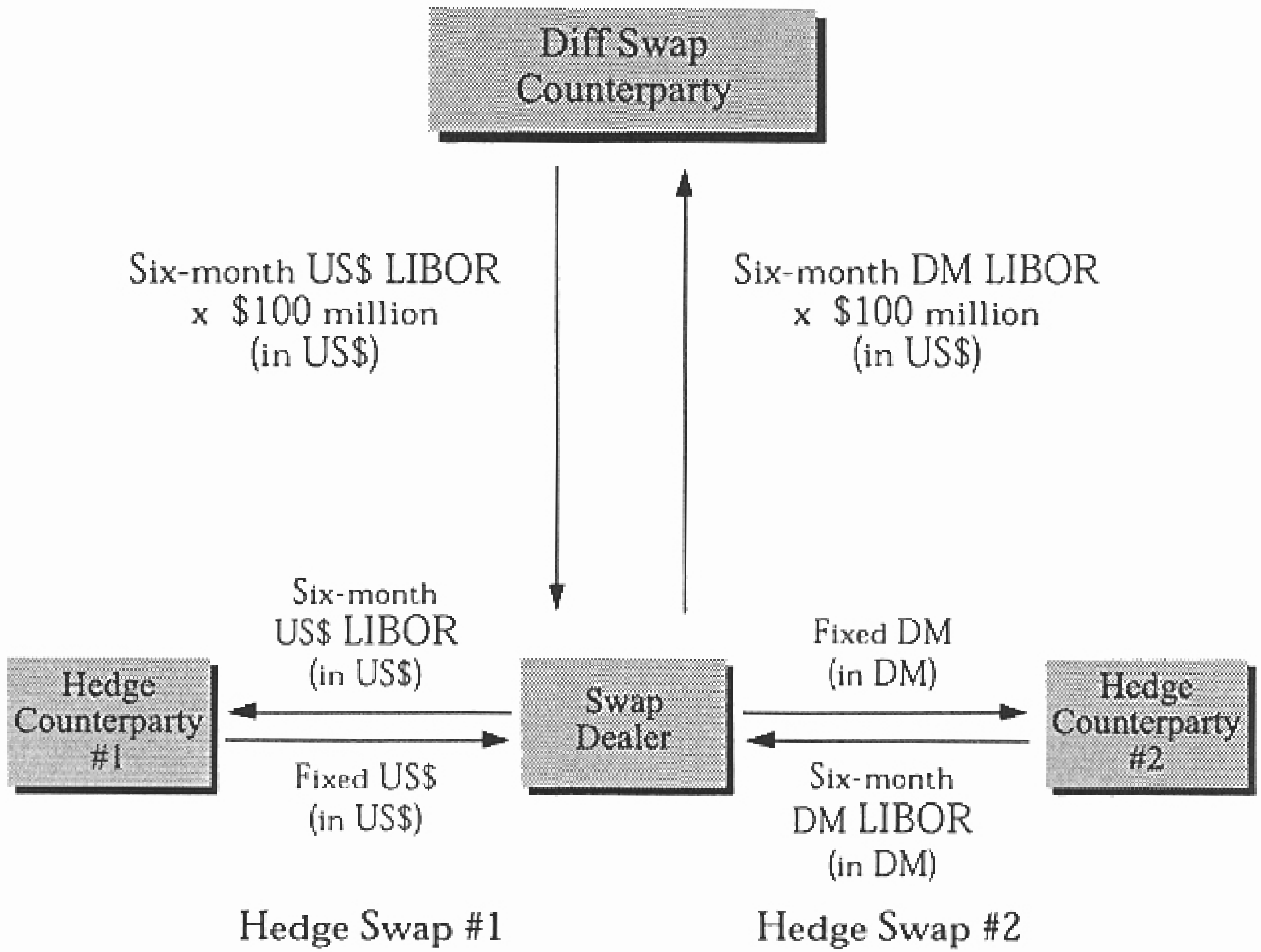


Exhibit 3

Cash Flows of a Diff Swap to and from Dealer

All cash flows take place at t=12 months based on rates at t=6 months

Diff swap:	
Inflow:	\$100 million x $\tilde{r}_{\mathit{US\$\ LIBOR}}$
Outflow:	\$100 million x $\tilde{r}_{\mathit{DM\ LIBOR}}$
Hedge swap #1:	
Inflow:	\$100 million x 6%
Outflow:	\$100 million x $\tilde{r}_{\mathit{US\$\ LIBOR}}$
Hedge swap #2:	
Inflow:	DM 160 million x $\tilde{r}_{\mathit{DM\ LIBOR}}$
Outflow:	DM 160 million x 8%
Net cash flows:	
Inflow:	\$100 million x 6% + DM 160 million x $\tilde{r}_{\mathit{DM\ LIBOR}}$
Outflow:	\$100 million x $\tilde{r}_{\mathit{DM\ LIBOR}}$ + DM 160 million x 8%

* Fixed US rate = 6%, fixed DM rate = 8%

- The combination of the diff swap and the two hedging swaps does not eliminate all price risk.
- To determine the value of the residual exposure that occurs in one year, the dealer converts the net cash flows into U.S. dollars at the exchange rate prevailing at $t = 6$ months, $\tilde{q}_{DM/\$}$:

  $$\$100m \times (6\% - \tilde{r}_{DMLIBOR}) + DM160m \times (\tilde{r}_{DMLIBOR} - 8\%)/\tilde{q}_{DM/\$}$$

  which can be simplified to:

  $$(\$100m - DM160m/\tilde{q}_{DM/\$}) \times (8\% - \tilde{r}_{DMLIBOR}) - \$100m \times 2\%.$$

- Simultaneous movements in the foreign interest rate and exchange rate will determine the sign — positive or negative — of the cash flow.

- Assume that the deutsche mark LIBOR decreases and the deutsche mark/U.S. dollar exchange rate increases (the deutsche mark depreciates relative to the U.S. dollar). Because the movements in the deutsche mark LIBOR and the deutsche mark/U.S. dollar exchange rate are negatively correlated, both terms will be positive, and the dealer will receive a positive cash flow.

- The *correlation* between the risk factors determines whether the cash flow of the diff swap will be positive or negative. The interest rate risk and the exchange rate risk are *non-separable*. This is because the two random factors: $\widetilde{q}_{\$/\mathrm{DM}}$ and $\widetilde{r}_{\mathrm{DM\ LIBOR}}$ are multiplied rather than summed or differenced.

- Non-perfect hedge using the above simple strategy arises from the payment of DM LIBOR interest settled in US dollars.

2.7 Constant Maturity Swaps

- An Interest Rate Swap where the floating rate on one leg is reset periodically but with reference to a market swap rate rather than LIBOR.

- The other leg of the swap is generally LIBOR but may be a fixed rate or potentially another Constant Maturity Rate.

- Constant Maturity Swaps can either be single currency or Cross Currency Swaps.

- The prime factor for a Constant Maturity Swap is the shape of the forward implied yield curves.

Example – Investor bets on flattening of yield curve

- The GBP yield curve is currently positively sloped with the current 6-month LIBOR at 5.00% and the 3-year swap rate at 6.50%, the 5-year swap rate at 8.00% and the 7-year swap rate at 8.50%.

- The current differential between the 3-year swap and 6-month LIBOR is therefore +150bp.

- At this moment, the investor is unsure as to when the expected flattening will occur, but believes that the differential between 3-year swap rate and LIBOR (now 150bp) will average 50bp over the next 2 years.

In order to take advantage of this view, the investor can use the Constant Maturity Swap. He can enter the following transaction for 2 years:

Investor Receives:	6-month GBP LIBOR
Investor Pays:	GBP 3-year Swap mid rate less 105bp (semi annually)

- Each six months, if the 3-year Swap rate minus LIBOR is less than 105bp, the investor will receive a net positive cashflow, and if the differential is greater than 105bp, pay a net cashflow.

- As the current spread is 150bp, the investor will be required to pay 45bp for the first 6 months. If the investor is correct and the differential does average 50bp over the two years, this will result in a net flow of 55bp to the investor.

Example – Corporate aims at maintaining stable debt duration

- In the past, the company has used the Interest Rate Swap market to convert LIBOR based funding into fixed rate and as swap transactions mature has sought to replace them with new 3, 5 and 7-year swaps.

Remark

Duration is the weighted average of the times of payment of cash flows, weighted according to the present value of the cash flow. Suppose cash amount c_i is paid at time $t_i, i = 1, 2, \cdots, n$, then

$$\text{duration} \approx \frac{\sum_{i=1}^{n} PV(c_i)t_i}{\sum_{i=1}^{n} PV(c_i)}.$$

- When the company transacts a 5-year swap, while the duration of the swap starts at around 3.3 yrs, the duration shortens as the swap gets closer to maturity, making it difficult for the company to maintain a stable debt duration.

- The debt duration of the company is therefore quite volatile as it continues to shorten until new transactions are booked when it jumps higher.

The Constant Maturity Swap can be used to alleviate this problem. If the company is seeking to maintain duration at the same level as say a 5 year swap, instead of entering into a 5 yr swap, they can enter the following Constant Maturity swap:

Investor Receives:	6 month Euro LIBOR
Investor Pays:	Euro 5-year Swap mid rate less 35bp (semi annually)

- The "duration" of the transaction is almost always at the same level as a 5-year swap and as time goes by, the duration remains the same unlike the traditional swap.

Replication of the CMS leg payments

Recall the put-call parity relation:

$$\underbrace{S_T - K}_{\text{forward}} = (S_T - K)^+ - (K - S_T)^+$$

where K is the strike price in the call or put while K is the delivery price in the forward contract.

Take S_T to be the constant maturity swap rate. The CMS payment can be replicated by longing a CMS caplet, shorting CMS floorlet and longing a bond.

- Interestingly, we replicate the underlying swap rate using derivative products of the swap rate.

- A CMS caplet $c_i(t; K)$ with reset date T_i and payment date T_{i+1} and whose underlying is the swap rate $S_{i,i+n}$ is a call option on the swap rate with terminal payoff at T_{i+1} defined by

$$\delta \max(S_{i,i+n}(T_i) - K, 0),$$

  where K is the strike and $S_{i,i+n}(T_i)$ is the swap rate with tenor $[T_i, T_{i+1}, \cdots, T_{i+n}]$ observed at T_i, δ is the accrual period.

- As the CMS caplet is not a liquid instrument, we may use a portfolio of swaptions of varying strike rates to replicate a CMS caplet. We maintain a dynamically rebalancing portfolio of swaptions so that the present value at T_i of the payoff from the caplet with varying values of the swap rate $S_{i,i+n}(T_i)$ matches with that of the portfolio of swaptions. Swaptions are derivatives whose underlying is the swap rate. They are used as the replication instruments since swaptions are the liquidly traded derivatives.

- The replicating portfolio consists of a series of payer swaptions with strike price K, $K + \Delta, K + 2\Delta, \cdots$ where Δ is a small step increment. The strike price K is chosen such that the corresponding swaptions are most liquid in the market.

- Recall that a payer swaption with strike K gives the holder the right but not the obligation to enter into a swap such that the holder pays the fixed rate K and receives floating rate LIBOR. All these payer swaptions have the same maturity T_i and the underlying swap has a tenor of $[T_i, T_{i+n}]$, where payments are made on $T_{i+1}, T_{i+2}, \cdots, T_{i+n}$. If the prevailing swap rate at T_i is higher than the fixed strike K, the payoff to the holder of the put swaption is

$$[S_{i,i+n}(T_i) - K] \sum_{k=1}^{n} \delta_i B(T_i, T_{i+k}).$$

Dynamic replication

How many units of swaptions have to be held in the portfolio such that the present value at T_i of the payoff of the CMS caplet and the portfolio of swaptions match exactly when the swap rate $S_{i,i+n}(T_i)$ falls on $K + \Delta, K + 2\Delta, \cdots$.

Let $N_j(t)$ be the number of units of payer swaption with strike $K + j\Delta$ to be held in the portfolio, $j = 0, 1, 2, \cdots$. The replication is dynamic since the notional amount $N_j(t)$ changes with time t.

- When $S_{i,i+n}(T_i) \leq K$, all payer swaptions are not in-the-money and the CMS caplet expires at zero value at T_i.

- We determine $N_0(t), N_1(t), \cdots$, successively such that the portfolio of payer swaptions and CMS caplet match in their present values of the payoff at T_i when $S_{i,i+n}(T_i)$ assumes a value equals either $K + \Delta$ or $K + 2\Delta$ or $K + 3\Delta$, etc.

- This is an approximate replication. The accuracy of the replication improves when we choose Δ to be sufficiently small in value.

(i) $S_{i,i+n}(T_i) = K + \Delta$

Only the payer swaption with strike K is in-the-money, all other payer swaptions become worthless. The payoff of the CMS caplet at T_{i+1} is $\delta\Delta$. Consider their present values at T_i:

- Holder of the K-strike payer swaption receives $\delta\Delta$ at $T_{i+1}, \cdots, T_{i+n}$ so that the present value of $N_0(T_i)$ units of K-strike payer swaption is

$$N_0(T_i)\delta\Delta \sum_{k=1}^{n} B(T_i, T_{i+k}).$$

- The holder of the CMS caplet receives $\delta\Delta$ at T_{i+1} so that its present value at time T_i is $\delta\Delta B(T_i, T_{i+1})$. Though both the K-strike payer swaption and the CMS caplet share the same underlying $S_{i,i+n}$, they have different payoff structure: swaption is related to an annuity and caplet has single payout $\delta\Delta$.

- When the swap rate $S_{i,i+n}(T_i)$ equals $K+\Delta$, this would implicitly imply

$$K + \Delta = \frac{B(T_i, T_i) - B(T_i, T_{i+n})}{\sum_{k=1}^{n} \delta B(T_i, T_{i+k})}, \text{ with } B(T_i, T_i) = 1. \tag{1}$$

Thus, the annuity $\sum_{k=1}^{n} \delta B(T_i, T_{i+k})$ can be related to $K + \Delta$, also T_i-maturity and T_{i+n}-maturity discount bond prices.

- We hold $N_0(t)$ dynamically according to

$$N_0(t) = \frac{B(t, T_{i+1})}{B(t, T_i) - B(t, T_{i+n})}(K + \Delta)\delta$$

so that at $t = T_i$,

$$N_0(T_i) = \frac{B(T_i, T_{i+1})}{1 - B(T_i, T_{i+n})}(K + \Delta)\delta. \tag{2}$$

Note that $N_0(t)$ is adjusted accordingly when the discount bond prices evolve with time t.

- It is then observed that

$$
\begin{aligned}
& N_0(T_i)\delta\Delta \sum_{k=1}^{n} B(T_i, T_{i+k}) \\
= \; & (K+\Delta)\delta\Delta \frac{B(T_i, T_{i+1})}{1 - B(T_i, T_{i+1})} \sum_{k=1}^{n} \delta B(T_i, T_{i+k}) \\
= \; & \delta\Delta B(T_i, T_{i+1}), \quad \text{by virtue of (1).}
\end{aligned}
$$

Hence, the present values of caplet and protfolio of payer swaptions match at time T_i.

(ii) $S_{i,i+n}(T_i) = K + 2\Delta$

Now, the payer swaptions with respective strike K and $K + \Delta$ are in-the-money, while all other payer swaptions become zero value. The payoff of the CMS caplet at T_{i+1} is $2\delta\Delta$. We find $N_1(t)$ such that at T_i, we have

$$[2N_0(T_i)\delta\Delta + N_1(T_i)\delta\Delta] \sum_{k=1}^{n} B(T_i, T_{i+k}) = 2\delta\Delta B(T_i, T_{i+1}).$$

Recall that when $S_{i,i+n}(T_i) = K + 2\Delta$, then

$$K + 2\Delta = \frac{B(T_i, T_i) - B(T_i, T_{i+n})}{\sum_{k=1}^{n} \delta B(T_i, T_{i+k})}.$$

Suppose we choose $N_1(t)$ dynamically such that

$$N_1(t) = 2\Delta \frac{B(t, T_{i+1})}{B(t, T_i) - B(t, T_{i+n})} \delta$$

so that

$$N_1(T_i) = 2\Delta \frac{B(T_i, T_{i+1})}{1 - B(T_i, T_{i+n})} \delta,$$

then it can be shown that the present values of the portfolio of payer swaptions and caplet match at T_i.

Deductively, it can be shown that

$$N_\ell(t) = N_1(t), \quad \ell = 2, 3, \cdots,$$

we achieve matching of the present values at T_i of the caplet and the portfolio of swaptions when $S_{i,i+n}(T_i)$ assumes value equals either $K + \Delta$, or $K + 2\Delta, \cdots$, etc.

- In the replicating portfolio consisting of swaptions with varying strikes, the K-strike swaption is dominant since its notional amount is $(K + \Delta)/\Delta$ times the notional of any of the other swaptions.