1. Optimal investment strategy – log utility criterion

Suppose there is an investment opportunity that the investor will either double her investment or return nothing. The probability of the favorable outcome is p. Suppose the investor has an initial capital of X_0, and she can repeat this investment many times. How much should she invest at each time in order to maximize the growth of capital?

Statement of the problem

Let α be the proportion of capital invested during each play. The investor would like to find the optimal value of α which maximizes growth. The possible proportional changes are given by

$$\begin{cases} 1+\alpha & \text{if outcome is favorable} \\ 1-\alpha & \text{if outcome is unfavorable} \end{cases}, \quad 0 \leq \alpha \leq 1.$$

General formulation:-

Let X_k represent the capital after the k^{th} trial, then

$$X_k = R_k X_{k-1}$$

where R_k is the random return variable.

We assume that all R_k's have *identical probability distribution* and they are *mutually independent*. The capital at the end of n trials is

$$X_n = R_n R_{n-1} \cdots R_2 R_1 X_0.$$

Taking logarithm on both sides

$$\ln X_n = \ln X_0 + \sum_{k=1}^{n} \ln R_k$$

or

$$\ln \left(\frac{X_n}{X_0} \right)^{1/n} = \frac{1}{n} \sum_{k=1}^{n} \ln R_k.$$

Since the random variables $\ln R_k$ are independent and have identical probability distribution, by the law of large numbers, we have

$$\frac{1}{n} \sum_{k=1}^{n} \ln R_k \longrightarrow E[\ln R_1].$$

Remark

Since the expected value of $\ln R_k$ is independent of k, so we simply consider $E[\ln R_1]$. Suppose we write $m = E(\ln R_1)$, we have

$$\left(\frac{X_n}{X_0}\right)^{1/n} \longrightarrow e^m \quad \text{or} \quad X_n \longrightarrow X_0 e^{mn}.$$

For large n, the capital grows (roughly) exponentially with n at a rate m. Here, e^m is the growth factor for each investment period.

Log utility form

$$m + \ln X_0 = E[\ln R_1] + \ln X_0 = E[\ln R_1 X_0] = E[\ln X_1].$$

If we define the log utility form: $U(x) = \ln x$, then the problem of maximizing the growth rate m is equivalent to maximizing the expected utility $E[U(X_1)]$.

Remark

Essentially, we may treat the investment growth problem as a single-period model. The single-period maximization guarantees the maximum growth rate in the long run.

Back to the investment strategy problem, how to find the optimal value of α such that the growth factor is maximized:

$$m = E[\ln R_1] = p \ln(1 + \alpha) + (1 - p) \ln(1 - \alpha).$$

Setting $\frac{dm}{d\alpha} = 0$, we obtain

$$p(1 - \alpha) - (1 - p)(1 + \alpha) = 0$$

giving $\alpha = 2p - 1$.

Suppose we require $\alpha \geq 0$, then the existence of the above solution implicitly requires $p \geq 0.5$.

What happen when $p < 0.5$, the value for α for optimal growth is given by $\alpha = 0$?

Lesson learnt If the game is unfavorable to the player, then he should stay away from the game.

Example (volatility pumping)

Stock: In each period, its value either doubles or reduces by half.

riskless asset: just retain its value.

How to use these two instruments in combination to achieve growth?

$$\text{Return vector } \boldsymbol{R} = \begin{cases} \left(\frac{1}{2} \quad 1\right) & \text{if stock price goes down} \\ (2 \quad 1) & \text{if stock price goes up} \end{cases}.$$

Strategy:

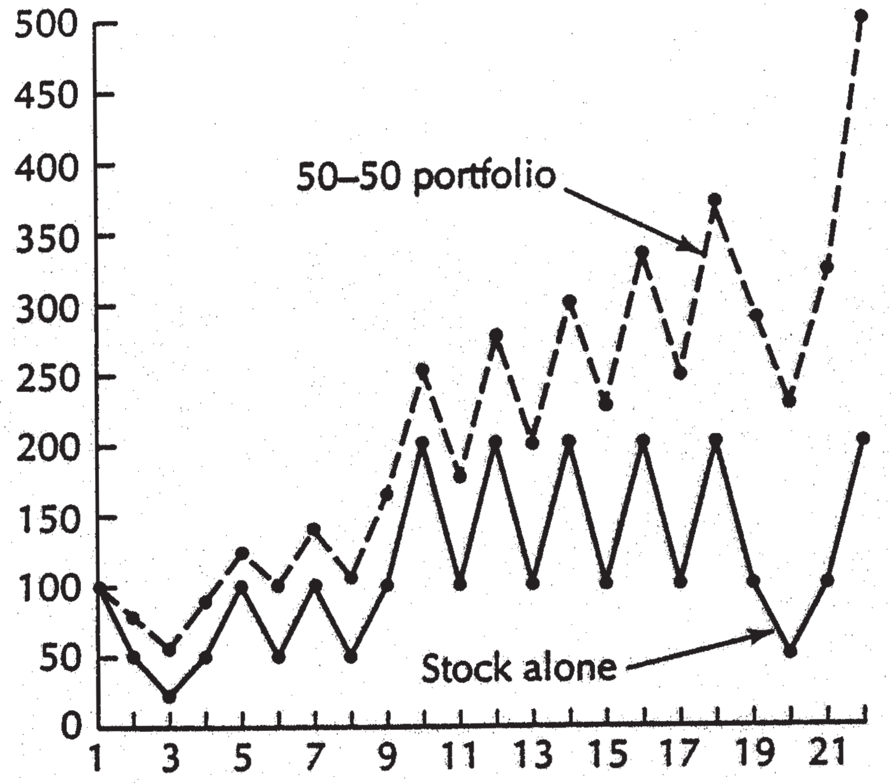
Invest *one half* of the capital in each asset for every period. Do the rebalancing at the beginning of each period so that one half of the capital is invested in each asset.

The expected growth rate

$$m = \underset{\substack{\uparrow \\ \text{prob of doubling}}}{\frac{1}{2}} \ln\left(\frac{1}{2} + 1\right) + \underset{\substack{\uparrow \\ \text{prob of halving}}}{\frac{1}{2}} \ln\left(\frac{1}{2} + \frac{1}{4}\right) \approx 0.059.$$

We obtain $e^m \approx 1.0607$, so the gain on the portfolio is about 6% per period.

Remark This strategy follows the dictum of "buy low and sell high" by the process of rebalancing.



Combination of 50-50 **portfolio of risky stock and riskless asset gives an enhanced growth.**

Example (pumping two stocks)

Both assets either double or halve in value over each period with probability 1/2; and the price moves are independent. Suppose we invest one half of the capital in each asset, and rebalance at the end of each period. The expected growth rate of the portfolio is found to be

$$m = \frac{1}{4}\ln 2 + \frac{1}{2}\ln\frac{5}{4} + \frac{1}{4}\ln\frac{1}{2} = \frac{1}{2}\ln\frac{5}{4} = 0.1116,$$

so that $e^m = \sqrt{\frac{5}{4}} = 1.118$. This gives an 11.8% growth rate for each period.

Remarks

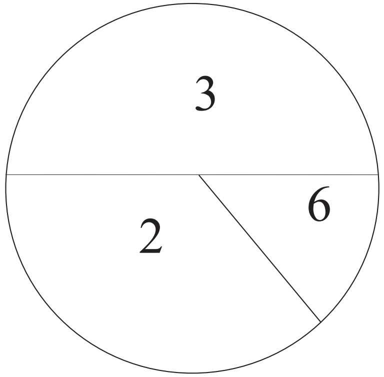
1. What would result if the two stocks happen to be HSBC and PCCW? One stock continues to perform well while the other stock continues to deteriorate.

2. Advantage of the index tracking fund, say, Dow Jones Industrial Average. The index automatically

   (i) exercises some form of volatility pumping due to stock splitting,
   (ii) get rids of the weaker performers periodically.

Investment wheel

The numbers shown are the payoffs for one-dollar investment on that sector.



1. Top sector: paying 3 to 1, though the area is 1/2 of the whole wheel (favorable odds).

2. Lower left sector: paying only 2 to 1 for an area of 1/3 of wheel (unfavorable odds).

3. Lower right sector: paying 6 to 1 for an area of 1/6 of the wheel (even odds).

Aggressive strategy

Invest all money in the top sector. This produces the highest single-period expected return. This is too risky for long-term investment! Why? The investor *goes broke* half of the time and cannot continue with other spins.

Fixed proportion strategy

Prescribe wealth proportions to each sector; apportion current wealth among the sectors as bets at each spin.

$$(\alpha_1, \alpha_2, \alpha_3) \text{ where } \alpha_i \geq 0 \quad \text{and} \quad \alpha_1 + \alpha_2 + \alpha_3 \leq 1.$$

α_1: top sector

α_2: lower left sector

α_3: lower right sector

If “top” occurs, $R(\omega_1) = 1 + 2\alpha_1 - \alpha_2 - \alpha_3$.

If “bottom left” occurs, $R(\omega_2) = 1 - \alpha_1 + \alpha_2 - \alpha_3$.

If “bottom right” occurs, $R(\omega_3) = 1 - \alpha_1 - \alpha_2 + 5\alpha_3$.

We have

$$m = \frac{1}{2}\ln(1+2\alpha_1-\alpha_2-\alpha_3)+\frac{1}{3}\ln(1-\alpha_1+\alpha_2-\alpha_3)+\frac{1}{6}\ln(1-\alpha_1-\alpha_2+5\alpha_3).$$

To maximize m, we compute $\frac{\partial m}{\partial \alpha_i}, i = 1, 2, 3$ and set them be zero:

$$\frac{2}{2(1+2\alpha_1-\alpha_2-\alpha_3)} - \frac{1}{3(1-\alpha_1+\alpha_2-\alpha_3)} - \frac{1}{6(1-\alpha_1-\alpha_2+5\alpha_3)} = 0$$

$$\frac{-1}{2(1+2\alpha_1-\alpha_2-\alpha_3)} + \frac{1}{3(1-\alpha_1+\alpha_2-\alpha_3)} - \frac{1}{6(1-\alpha_1-\alpha_2+5\alpha_3)} = 0$$

$$\frac{-1}{2(1+2\alpha_1-\alpha_2-\alpha_3)} - \frac{1}{3(1-\alpha_1+\alpha_2-\alpha_3)} + \frac{5}{6(1-\alpha_1-\alpha_2+5\alpha_3)} = 0.$$

There is a whole family of optimal solutions, and it can be shown that they all give the same value for m.

(i) $\alpha_1 = 1/2, \alpha_2 = 1/3, \alpha_3 = 1/6$

One should invest in every sector of the wheel, and the bet proportions are equal to the probabilities of occurrence.

$$m = \frac{1}{2}\ln\frac{3}{2} + \frac{1}{3}\ln\frac{2}{3} + \frac{1}{6}\ln 1 = \frac{1}{6}\ln\frac{3}{2}$$

so $e^m \approx 1.06991$ (a growth rate of about 7%).

Remark: Betting on the unfavorable sector is like buying insurance.

(ii) $\alpha_1 = 5/18, \alpha_2 = 0$ and $\alpha_3 = 1/18$.

Nothing is invested on the unfavorable sector.

Log utility and growth function

Let $\boldsymbol{w}_i = (w_{i1} \cdots w_{in})$ be the weight vector of holding n risky securities at the i^{th} period, where weight is defined in terms of wealth. Write the random return vector at the i^{th} period as $\boldsymbol{R}_i = (R_{i1} \cdots R_{in})$. Here, R_{ij} is the random return of holding the j^{th} security after the i^{th} play.

Write S_n as the total return of the portfolio after n periods:

$$S_n = \prod_{i=1}^{n} \boldsymbol{w}_i \cdot \boldsymbol{R}_i.$$

Define $B = \{\boldsymbol{w} \in R^n : \mathbf{1} \cdot \boldsymbol{w} = 1 \text{ and } \boldsymbol{w} \geq \mathbf{0}\}$, where $\mathbf{1} = (1 \cdots 1)$. This represents a trading strategy that does not allow short selling. When the successive games are identical, we may drop the dependence on i.

Based on the *log-utility* criterion, we define the *growth function* by

$$W(\boldsymbol{w};F) = E[\ln(\boldsymbol{w}\cdot\boldsymbol{R})] = \int \ln(\boldsymbol{w}\cdot\boldsymbol{R})\,dF(\boldsymbol{R}),$$

where $F(\boldsymbol{R})$ is the distribution function of the stochastic return vector $\boldsymbol{R}$. The growth function is seen to be a function of the *trading strategy* $\boldsymbol{\omega}$ together with dependence on F. The optimal growth function is defined by

$$W^*(F) = \max_{\boldsymbol{w}\in B} W(\boldsymbol{w};F).$$

Lemmas

1. For a given $\boldsymbol{w}$, $W(\boldsymbol{w}; F)$ is a linear function of the distribution function F. This follows directly from the linearity property of the expectation integral.
2. For a given function $F, W(\boldsymbol{w}; F)$ is a concave function on $\boldsymbol{w}$; and $W^*(F)$ is a convex function on F.

Proof

From the concave property of the logarithmic function, we have

$$\ln(\lambda \boldsymbol{w}_1 + (1-\lambda)\boldsymbol{w}_2) \cdot \boldsymbol{R} \geq \lambda \ln \boldsymbol{w}_1 \cdot \boldsymbol{R} + (1-\lambda) \ln \boldsymbol{w}_2 \cdot \boldsymbol{R}.$$

We then take the expectation on both side and obtain the concave property on $\boldsymbol{w}$.

To show the convexity property of $\boldsymbol{w}^*$, we consider two distribution functions F_1 and F_2. Let the corresponding optimal weights be denoted by $\boldsymbol{w}^*(F_1)$ and $\boldsymbol{w}^*(F_2)$.

Write $\boldsymbol{w}^*(\lambda F_1+(1-\lambda)F_2)$ as the optimal weight vector corresponding to $\lambda F_1 + (1-\lambda)F_2$. Now, we consider

$$
\begin{aligned}
& W^*(\lambda F_1 + (1-\lambda)F_2) \\
= \; & W(\boldsymbol{w}^*(\lambda F_1 + (1-\lambda)F_2); \lambda F_1 + (1-\lambda)F_2) \\
= \; & \lambda W(\boldsymbol{w}^*(\lambda F_1 + (1-\lambda)F_2); F_1) + (1-\lambda)W(\boldsymbol{w}^*(\lambda F_1 + (1-\lambda)F_2; F_2) \\
\leq \; & \lambda W(\boldsymbol{w}^*(F_1); F_1) + (1-\lambda)W(\boldsymbol{w}^*(F_2); F_2) \\
= \; & \lambda W^*(F_1) + (1-\lambda)W^*(F_2).
\end{aligned}
$$

The inequality holds since $\boldsymbol{w}^*(F_1)$ and $\boldsymbol{w}^*(F_2)$ are the weights that lead to the maximization of $W(\boldsymbol{w}; F_1)$ and $W(\boldsymbol{w}; F_2)$, respectively.

Lemma

The log-utility optimal portfolio $\boldsymbol{w}^*$ that maximizes the growth function $W(\boldsymbol{w}; F)$ satisfies

$$E\left(\frac{R_j}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right) \leq 1.$$

Proof

Note that $W(\boldsymbol{w}; F)$ is a concave function on $\boldsymbol{w}$, and the domain of definition of $\boldsymbol{w}$ is a simplex. The necessary and sufficient condition for $\boldsymbol{w}^*$ to be an optimal solution is that the directional derivative of $W(\boldsymbol{w})$ at $\boldsymbol{w}^*$ along any path must be non-positive.

Let $\boldsymbol{w}_\lambda = (1-\lambda)\boldsymbol{w}^* + \lambda \boldsymbol{w}, 0 \leq \lambda \leq 1$, where $\boldsymbol{w}_\lambda$ represents an element in B that moves from $\boldsymbol{w}^*$ to an arbitrary vector $\boldsymbol{w}$ in B.

The above necessary and sufficent condition can be represented by

$$\frac{d}{d\lambda} W(\boldsymbol{w}_\lambda)\Big|_{\lambda=0^+} \leq 0 \quad \text{for all } \boldsymbol{w} \in B.$$

Consider

$$\begin{aligned} & \frac{d}{d\lambda} E[\ln(\boldsymbol{w} \cdot \boldsymbol{R})]\Big|_{\lambda=0^+} \\ = & \lim_{\Delta\lambda \to 0^+} \frac{1}{\Delta\lambda} E\left[\ln\left(\frac{(1-\Delta\lambda)\boldsymbol{w}^* \cdot \boldsymbol{R} + \Delta\lambda \boldsymbol{w} \cdot \boldsymbol{R}}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right)\right] \\ = & E\left[\lim_{\Delta\lambda \to 0^+} \frac{1}{\Delta\lambda} \ln\left(1 + \Delta\lambda\left(\frac{\boldsymbol{w} \cdot \boldsymbol{R}}{\boldsymbol{w}^* \cdot \boldsymbol{R}} - 1\right)\right)\right] \\ = & E\left[\frac{\boldsymbol{w} \cdot \boldsymbol{R}}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right] - 1 \leq 0. \end{aligned}$$

In particular, when $\boldsymbol{w}^*$ is an interior point of B, then

$$E\left[\frac{\boldsymbol{w} \cdot \boldsymbol{R}}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right] = 1 \quad \text{for all } \boldsymbol{w} \in B.$$

Suppose we take $\boldsymbol{w} = \boldsymbol{e}_j$, we then deduce that

$$E\left[\frac{R_j}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right] = 1, \quad j = 1, 2, \cdots, n.$$

Let P_j^0 be the price of security j at time 0 and P_j be the random payout of security j. The return of security j is

$$R_j = P_j / P_j^0$$

so that

$$P_j^0 = E\left[\frac{P_j}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right].$$

Note that $\boldsymbol{w}^* \cdot \boldsymbol{R}$ is the return on the log-optimal portfolio. Here, P_j^0 can be interpreted as the fair price of security j based on the knowledge of F.

Remark

Let w_j^* be the optimal weight invested on asset j, and R_j is its return per unit dollar betted. The random weight of asset j after one investment period is

$$\frac{w_j^* R_j}{w_1^* R_1 + \cdots + w_n^* R_n}.$$

Taking the expectation

$$E\left[\frac{w_j^* R_j}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right] = w_j^* E\left[\frac{R_j}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right] = w_j^*.$$

when $\boldsymbol{w}^*$ is an interior point of B. The expected weight of asset j after the game under the optimal trading strategy is simply the original optimal weight.

Betting wheel revisited

Let the payoff upon the occurrence of the i^{th} event (pointer landing on the i^{th} sector) be $(0 \cdots a_i \cdot 0)$ with probability p_i. That is, $\boldsymbol{R}(\boldsymbol{\omega}_i) = (0 \cdots a_i \cdot 0)$. Take the earlier example, when the pointer lands on the bottom left sector, the return vector is given by

$$\begin{aligned} \boldsymbol{R}(\boldsymbol{\omega}_1) &= (3 \quad 0 \quad 0) \\ \boldsymbol{R}(\boldsymbol{\omega}_2) &= (0 \quad 2 \quad 0) \\ \boldsymbol{R}(\boldsymbol{\omega}_3) &= (0 \quad 0 \quad 6). \end{aligned}$$

In general, for this betting wheel game, the gambler betting on the i^{th} sector (equivalent to investment on security i) is paid a_i if the pointer lands on the i^{th} sector and loses the whole bet if otherwise.

Suppose the gambler chooses $\boldsymbol{w} = (w_1 \cdots w_n)$ as the betting strategy with $\sum_{i=1}^{n} w_i = 1$, then

$$\begin{aligned} W(\boldsymbol{w}; F) &= \sum_{i=1}^{n} p_i \ln(\boldsymbol{w} \cdot \boldsymbol{R}(\omega_i)) = \sum_{i=1}^{n} p_i \ln w_i a_i \\ &= \sum_{i=1}^{n} p_i \ln \frac{w_i}{p_i} + \sum_{i=1}^{n} p_i \ln p_i + \sum_{i=1}^{n} p_i \ln a_i, \end{aligned}$$

where the last two terms are known quantities.

Using the inequality: $\ln x \leq x - 1$ for $x \geq 0$, with equality holds when $x = 1$, we have

$$\sum_{i=1}^{n} p_i \ln \frac{w_i}{p_i} \leq \sum_{i=1}^{n} p_i \left(\frac{w_i}{p_i} - 1 \right) = \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} p_i = 0$$

with equality holds if and only if $w_i = p_i$. Hence, an optimal portfolio is $w_i^* = p_i$, for all i.

Continuous-time version

Let S_i denote the price of the i^{th} asset, $i = 1, 2, \cdots, n$, which is governed by the stochastic differential equation

$$\frac{dS_i}{S_i} = \mu_i\, dt + \sigma_i\, dZ_i.$$

Let the correlation structure be defined by

$$\text{cov}(dZ_i, dZ_j) = E[dZ_i\, dZ_j] = \rho_{ij}\, dt, \quad \text{where} \quad E[dZ_i] = 0.$$

The mean and variance of the log return of asset i are

$$E\left[\ln \frac{S_i(t)}{S_i(0)}\right] = \left(\mu_i - \frac{\sigma_i^2}{2}\right) t \quad \text{and} \quad \text{var}\left[\ln \frac{S_i(t)}{S_i(0)}\right] = \sigma_i^2 t.$$

Portfolio dynamics

Let w_i denote the weight of asset i with $\sum_{i=1}^{n} w_i = 1$. Let V be the value of the portfolio, where $V = \sum_{i=1}^{n} n_i S_i$. Note that the number of units n_i of asset i is changing at all the times due to rebalancing.

Consider the differential dV, where

$$dV = \sum_{i=1}^{n} n_i\, dS_i + \sum_{i=1}^{n} S_i\, dn_i.$$

We assume self-financing strategy where there is no additional fund added or withdrawn so that the purchase of new units of one asset is financed by the sale of other assets. Under self-financing strategy, we have

$$\sum_{i=1}^{n} S_i\, dn_i = 0.$$

Now

$$\frac{dV}{V} = \sum_{i=1}^{n} \frac{n_i\, dS_i}{V} = \sum_{i=1}^{n} w_i \frac{dS_i}{S_i} = \sum_{i=1}^{n} [w_i \mu_i\, dt + w_i \sigma_i\, dZ_i],$$

where the weight w_i is given by $\dfrac{n_i S_i}{V}$.

Volatility pumping – the weights are fixed at all times

For fixed value of w_i, the variance of the stochastic term is

$$E\left[\left(\sum_{i=1}^{n} w_i \sigma_i \, dZ_i\right)^2\right] = \sum_{i,j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j \, dt.$$

Hence, $V(t)$ is lognormal with

$$E\left[\ln \frac{V(t)}{V(0)}\right] = \sum_{i=1}^{n} w_i \mu_i t - \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j t.$$

Note that $\frac{1}{t} E\left[\ln \frac{V(t)}{V(0)}\right]$ gives the growth rate of the portfolio.

Consider the simpler case where all n assets have the same mean μ and variance σ and they are all uncorrelated so that $\rho_{ij} = 0, i \neq j$. The expected growth rate of each stock is $\mu - \frac{\sigma^2}{2}$.

Suppose these n stocks are each included in a portfolio with weight $1/n$. In this case, $-\frac{1}{2}\sum_{ij=1}^{n} w_i w_j \rho_{ij}\sigma_i\sigma_j = -\frac{\sigma^2}{2n}$ since $w_i = \frac{1}{n}, \rho_{ij} = 1$ and $\rho_{ij} = 0$ for $i \neq j$. Hence, the expected growth rate of the portfolio is $\mu - \frac{\sigma^2}{2n}$. By pumping, the growth rate has increased over that of a single stock by

$$\left(\mu - \frac{\sigma^2}{2n}\right) - \left(\mu - \frac{\sigma^2}{2}\right) = \frac{1}{2}\left(1 - \frac{1}{n}\right)\sigma^2 = \frac{1}{2}\frac{n-1}{n}\sigma^2.$$

Remark

The pumping effect is obviously most dramatic when the original variance is high. Under this scenario, volatility is not the same as risk, rather it represents opportunity.