2. Utility functions

How to rank the following 4 investment choices?

<table>
<thead>
<tr>
<th>Investment A</th>
<th>Investment B</th>
<th>Investment C</th>
<th>Investment D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$p(x)$</td>
<td>$x$</td>
<td>$p(x)$</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1/2</td>
<td>10</td>
<td>1/5</td>
</tr>
<tr>
<td>40</td>
<td>1/4</td>
<td>20</td>
<td>2/5</td>
</tr>
<tr>
<td>30</td>
<td>1/5</td>
<td></td>
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</tbody>
</table>

When there is no risk, we choose the investment with the highest rate of return. — *Maximum Return Criterion*.

e.g. Investment B dominates Investment A, but this criterion fails to compare Investment B with Investment C.

*Maximum expected return criterion* (for risky investments)

Identifies the investment with the highest expected return.

$$E_C(x) = \frac{1}{4}(-5) + \frac{1}{2}(0) + \frac{1}{4}(40) = 8.75$$

$$E_D(x) = \frac{1}{5}(-10) + \frac{1}{5}(10) + \frac{2}{5}(20) + \frac{1}{5}(30) = 14.$$ 

Is such procedure well justified?
**St Petersburg paradox** (failure of Maximum Expected Return Criterion)

Tossing of a fair coin until the first head shows up. The prize is $2^{x-1}$ where $x$ is the number of tosses until the first head shows up (the game is then ended).

$$\text{Expected prize of the game} = \sum_{x=1}^{\infty} \frac{1}{2^x} 2^{x-1} = \infty.$$  

**Question**

What is the certain amount that one would be willing to accept so that it is indifferent between playing the game for free or receiving this certain sum?

This certain amount is called the *certainty equivalent* of the game.
• When people are faced with such a lottery in experimental trials, they refuse to pay more than a finite price (usually low).

• There is a very small chance to receive large sum of money. This occurs when $x$ is large.

e.g. $x = 10$, there is a chance of $\frac{1}{2^{10}}$ to receive $2^9$.

• *Alternative view* In Mark Six, people are willing to pay a few dollars to bet on winning a reasonably large sum with infinitesimally small chance.

• Certainty equivalent of the game under log utility

\[
\ln c = E[\ln R] = \sum_{x=1}^{\infty} \frac{1}{2^x} \ln 2^{x-1} = \ln 2 \sum_{x=1}^{\infty} \frac{x - 1}{2^x} = \ln 2
\]

so that $c = 2$ is the certainty equivalent.
How much the player is ready to pay to play the game?

We need to solve for

\[ U(w) = E[U(w + y - p)] \]

where

- \( w \) = initial wealth
- \( y \) = price received from the game
- \( p \) = price that the player is willing to pay to participate in the game.
Pairwise comparison

Consider the set of alternatives $B$, how to determine which element in the choice set $B$ that is preferred?

The individual first considers two arbitrary elements: $x_1, x_2 \in B$. He then picks the preferred element $x_1$ and discards the other. From the remaining elements, he picks the third one and compares with the winner. The process continues and the winner among all alternatives is identified.
Let the choice set $B$ be a convex subset of the $n$-dimensional Euclidean space. The component $x_i$ of the $n$-dimensional vector $x$ may represent $x_i$ units of commodity $i$. By convex, we mean that if $x_1, x_2 \in B$, then $\alpha x_1 + (1 - \alpha)x_2 \in B$ for any $\alpha \in [0, 1]$.

- Each individual is endowed with a preference relation, $\succeq$.

- Given any elements $x_1$ and $x_2 \in B$, $x_1 \succeq x_2$ means either that $x_1$ is preferred to $x_2$ or that $x_1$ is indifferent to $x_2$. 
Three axioms for \( \succeq \)

**Reflexivity**

For any \( x_1 \in B, x_1 \succeq x_1 \).

**Comparability**

For any \( x_1, x_2 \in B \), either \( x_1 \succeq x_2 \) or \( x_2 \succeq x_1 \).

**Transitivity**

For \( x_1, x_2, x_3 \in B \), given \( x_1 \succeq x_2 \) and \( x_2 \succeq x_3 \), then \( x_1 \succeq x_3 \).

**Remark**

1. Without the comparability axiom, an individual could not determine an optimal choice. This is because there would exist at least two elements of \( B \) between which the individual could not discriminate.
2. The transitivity axiom ensures that the choices are consistent.
Example 1
Let \( B = \{(x, y) : x \in [0, \infty) \text{ and } y \in [0, \infty)\} \) represent the set of alternatives. Let \( x \) represent ounces of orange soda and \( y \) represent ounces of grape soda. It is easily seen that \( B \) is a convex subset of \( \mathbb{R}^2 \).

Suppose the individual is concerned only with the total quantity of soda available, the more the better, then the individual is endowed with the following preference relation:

For \((x_1, y_1), (x_2, y_2) \in B\),

\[ (x_1, y_1) \succeq (x_2, y_2) \text{ if and only if } x_1 + y_1 \geq x_2 + y_2. \]
**Dictionary order**

Let the choice set $B = \{(x, y) : x \in [0, \infty), y \in [0, \infty)\}$, the dictionary order $\succeq$ is defined as follows:

Suppose $(x_1, y_1) \in B$ and $(x_2, y_2) \in B$, then

$$(x_1, y_1) \succeq (x_2, y_2) \text{ if and only if } [x_1 > x_2] \text{ or } [x_1 = x_2 \text{ and } y_1 \geq y_2].$$

It is easy to check that the dictionary order satisfies the three basic axioms of a preference relation.
Definition
Given \( x, y \in B \) and a preference relation \( \succeq \) satisfying the above three axioms.

1. \( x \) is indifferent to \( y \), written as
   \[ x \sim y \quad \text{if and only if} \quad x \succeq y \text{ and } y \succeq x. \]

2. \( x \) is strictly preferred to \( y \), written as
   \[ x \succ y \quad \text{if and only if} \quad x \succeq y \text{ and not } x \sim y. \]
**Axiom 4 – Order Preserving**

For any \( x, y \in B \) where \( x \succ y \) and \( \alpha, \beta \in [0, 1] \),

\[
[\alpha x + (1 - \alpha)y] \succ [\beta x + (1 - \beta)y] \quad \text{if and only if} \quad \alpha > \beta.
\]

**Example 1 revisited**

Recall the preference relation defined in Example 1, we take \((x_1, y_1), (x_2, y_2) \in B\) such that \((x_1, y_1) \succ (x_2, y_2)\) so that \(x_1 + y_1 - x_2 - y_2 > 0\).

Take \(\alpha, \beta \in [0, 1]\) such that \(\alpha > \beta\), and observe

\[
\alpha[(x_1 + y_1) - (x_2 + y_2)] > \beta[(x_1 + y_1) - (x_2 + y_2)].
\]

Adding \(x_2 + y_2\) to both sides, we obtain

\[
\alpha(x_1 + y_1) + (1 - \alpha)(x_2 + y_2) > \beta(x_1 + y_1) + (1 - \beta)(x_2 + y_2).
\]
Axiom 5 – Intermediate Value

For any $x, y, z \in B$, if $x \succ y \succ z$, then there exists a unique $\alpha \in (0, 1)$ such that

$$\alpha x + (1 - \alpha)z \sim y.$$ 

Remark

Given 3 alternatives with rankings of $x \succ y \succ z$, there exists a fractional combination of $x$ and $z$ that is indifferent to $y$. Trade-offs between the alternatives exist.
Example 1 revisited

Given $x_1 + y_1 > x_2 + y_2 > x_3 + y_3$, choose

$$\alpha = \frac{(x_2 + y_2) - (x_3 + y_3)}{(x_1 + y_1) - (x_3 + y_3)}.$$ 

Rearranging gives

$$\alpha(x_1 + y_1) + (1 - \alpha)(x_3 + y_3) = x_2 + y_2$$

so that

$$[\alpha(x_1, y_1) + (1 - \alpha)(x_3, y_3)] \sim (x_2, y_2).$$
Dictionary order does not satisfy the intermediate value axiom

Suppose \((x_1, y_1), (x_2, y_2), (x_3, y_3) \in B\) such that \((x_1, y_1) \succ (x_2, y_2) \succ (x_3, y_3)\) and \(x_1 > x_2 = x_3\) and \(y_2 > y_3\). For any \(\alpha \in (0, 1)\), we have

\[
\alpha(x_1, y_1) + (1 - \alpha)(x_3, y_3) \\
= \alpha(x_1, y_1) + (1 - \alpha)(x_2, y_3) \\
= (\alpha x_1 + (1 - \alpha)x_2, \alpha y_1 + (1 - \alpha)y_3).
\]

But for \(\alpha > 0\), we have \(\alpha x_1 + (1 - \alpha)x_2 > x_2\) so

\[
\alpha(x_1, y_1) + (1 - \alpha)(x_3, y_3) \succ (x_2, y_2) \quad \text{for all} \quad \alpha \in (0, 1).
\]

In other words, there is no \(\alpha \in (0, 1)\) such that

\[
\alpha x + (1 - \alpha)z \sim y.
\]
Axiom 6 – Boundedness

There exist \( x^*, y^* \in B \) such that \( x^* \succeq z \succeq y^* \) for all \( z \in B \).

- This Axiom ensures the existence of a most preferred element \( x^* \in B \) and a least preferred element \( y^* \in B \).

Example 1 revisited

Recall \( B = \{(x,y) : x \in [0, \infty) \text{ and } y \in [0, \infty) \} \). Given any \( (z_1, z_2) \in B \), we have

\( (z_1 + 1, z_2) \succeq (z_1, z_2) \) since \( z_1 + z_2 + 1 > z_1 + z_2 \).

Therefore, a maximum does not exist.
Motivation for defining utility

Knowledge of the preference relation $\succeq$ effectively requires a complete listing of preferences over all pairs of elements from the choice set $B$. We define a utility function that assigns a numeric value to each element of the choice set such that larger numeric value implies higher preference.
Theorem – Existence of Utility Function

Let $B$ denote the set of payoffs from a finite number of securities, also being a convex subset of $\mathbb{R}^n$. Let $\succeq$ denote a preference relation on $B$. Suppose $\succeq$ satisfies the following axioms

(i) $\forall x \in B, x \succeq x$.
(ii) $\forall x, y \in B, x \succeq y$ or $y \succeq x$.
(iii) For any $x, y, z \in B$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.
(iv) For any $x, y \in B, x \succeq y$ and $\alpha, \beta \in [0, 1],$

\[ \alpha x + (1 - \alpha)y \succeq \beta x + (1 - \beta)y \quad \text{if and only if} \quad \alpha > \beta. \]

(v) For any $x, y, z \in B$, suppose $x \succ y \succ z$, then there exists a unique $\alpha \in (0, 1)$ such that $\alpha x + (1 - \alpha)z \sim y$.
(vi) There exist $x^*, y^* \in B$ such that $\forall z \in B, x^* \succeq z \succeq y^*$.

Then there exists a utility function $U : B \rightarrow \mathbb{R}$ such that

(a) $x \succ y$ iff $U(x) > U(y)$.
(b) $x \sim y$ iff $U(x) = U(y)$. 
To show the existence of $U : B \to \mathbb{R}$, we write down one such function and show that it satisfies the stated conditions.

Based on Axiom 6, we choose $x^*, y^* \in B$ such that

$$x^* \succeq z \succeq y^* \quad \text{for all} \quad z \in B.$$  

Without loss of generality, let $x^* \succ y^*$. [Otherwise, $x^* \sim z \sim y^*$ for all $z \in B$. In this case, $U(z) = 0$ for all $z \in B$, which is a trivial utility function that satisfies conditions (a) and (b).]

Consider an arbitrary $z \in B$. There are 3 possibilities:

1. $z \sim x^*$;
2. $x^* \succ z \succ y^*$;
3. $z \sim y^*$. 
We define $U$ by giving its value under all 3 cases:

1. $U(z) = 1$

2. By Axiom 5, there exists a unique $\alpha \in (0, 1)$ such that 
   
   $[\alpha x^* + (1 - \alpha)y^*] \sim z$.

   Define $U(z) = \alpha$.

3. $U(z) = 0$.

Such $U$ satisfies properties (a) and (b).
Proof of property (a)

Necessity

Suppose $z_1, z_2 \in B$ are such that $z_1 \succ z_2$, we need to show

$$U(z_1) > U(z_2).$$

Consider the four possible cases.

1. $z_1 \sim x^* \succ z_2 \succ y^*$
2. $z_1 \sim x^* \succ z_2 \sim y^*$
3. $x^* \succ z_1 \succ z_2 \succ y^*$
4. $x^* \succ z_1 \succ z_2 \sim y^*$.

**Case 1**  By definition, $U(z_1) = 1$ and $U(z_2) = \alpha$, where $\alpha \in (0, 1)$ uniquely satisfies

$$\alpha x^* + (1 - \alpha) y^* \sim z_2.$$

Now, $U(z_1) = 1 > \alpha = U(z_2)$. 
Case 2  By definition, $U(z_1) = 1 > 0 = U(z_2)$.

Case 3  By definition, $U(z_i) = \alpha_i$, where $\alpha_i \in (0, 1)$ uniquely satisfies
\[\alpha_i x^* + (1 - \alpha_i)y^* \sim z_i,\]
so that
\[z_1 \sim [\alpha_1 x^* + (1 - \alpha)y^*] \sim [\alpha_2 x^* + (1 - \alpha_2)y^*] \sim z_2.\]
We claim $\alpha_1 > \alpha_2$. Assume not, then $\alpha_1 \leq \alpha_2$. By Axiom 4,
\[[\alpha_2 x^* + (1 - \alpha_2)y^*] \succeq [\alpha_1 x^* + (1 - \alpha_1)y^*].\]
This is a contradiction. Hence, $\alpha_1 > \alpha_2$ is true and
\[U(z_1) = \alpha_1 > U(z_2) = \alpha_2.\]

Case 4  By definition, $U(z_1) = \alpha_1$, where $\alpha_1 \in (0, 1)$ uniquely satisfies
\[\alpha_1 x^* + (1 - \alpha_1)y^* \sim y_1 \quad \text{and} \quad U(z_2) = 0.\]
We have
\[U(z_1) = \alpha_1 > 0 = U(z_2).\]
Sufficiency

Suppose, given $z_1, z_2 \in B$, that $U(z_1) > U(z_2)$, we would like to show $z_1 \succ z_2$. Consider the following 4 cases

1. $U(z_1) = 1$ and $U(z_2) = \alpha_2$, where $\alpha_2 \in (0, 1)$ uniquely satisfies
   
   \[ [\alpha_2 x^* + (1 - \alpha_2) y^*] \sim z_2. \]

2. $U(z_1) = 1$, where $z_1 \sim x^*$ and $U(z_2) = 0$, where $z_2 \sim y^*$.

3. $U(z_i) = \alpha_i$, where $\alpha_i \in (0, 1)$ uniquely satisfies
   
   \[ [\alpha_i x^* + (1 - \alpha_i) y^*] \sim z_i. \]

4. $U(z_1) = \alpha_1$ and $U(z_2) = 0$, where $z_2 \sim y^*$. 

Case 1 \[ z_1 \sim x^* \sim [1 \cdot x^* + 0 \cdot y^*] \text{ and } z_2 \sim [\alpha_2 x^* + (1 - \alpha_2) y^*]. \]

By Axiom 4, \( 1 > \alpha_2 \) so that \( z_1 \succ z_2 \).

Case 2 \[ z_1 \sim x^* \succ y^* \sim z_2. \]

Case 3 \[ z_1 \sim [\alpha_1 x^* + (1 - \alpha_1) y^*] \]

\[ z_2 \sim [\alpha_2 x^* + (1 - \alpha_2) y^*] \]

Since \( \alpha_1 > \alpha_2 \), by Axiom 4, \( z_1 \succ z_2 \).

Case 4 \[ z_1 \sim [\alpha_1 x^* + (1 - \alpha_1) y^*] \text{ and } \]

\[ z_2 \sim y^* \sim [0 x^* + (1 - 0) y^*]. \]

By Axiom 4 and Axiom 3, since \( \alpha_1 > 0 \), \( z_1 \succ z_2 \).
Proof of Property (b)

Necessity

Suppose \( z_1 \sim z_2 \) but \( U(z_1) \neq U(z_2) \), then

\[
U(z_1) > U(z_2) \quad \text{or} \quad U(z_2) > U(z_1).
\]

By property (a), this implies \( z_1 \succ z_2 \) or \( z_2 \succ z_1 \), a contradiction. Hence,

\[
U(z_1) = U(z_2).
\]

Sufficiency

Suppose \( U(z_1) = U(z_2) \), but \( z_1 \succ z_2 \) or \( z_1 \prec z_2 \). By property (a), this implies \( U(z_1) > U(z_2) \) or \( U(z_2) > U(z_1) \), a contradiction. Hence, \( z_1 \sim z_2 \).
Choices among lotteries

How to make a choice between the following two lotteries:

\[ L_1 = \{p_1, A_1; p_2, A_2; \cdots; p_n, A_n\} \]

\[ L_2 = \{q_1, A_1; q_2, A_2; \cdots; q_n, A_n\} \]

The outcomes are \( A_1, \cdots, A_n \); \( p_i \) and \( q_i \) are the probabilities of occurrence of \( A_i \) in \( L_1 \) and \( L_2 \), respectively. These outcomes are mutually exclusive and only one outcome can be realized under each investment.

**Comparability**

When faced by two monetary outcomes \( A_i \) and \( A_j \), the investor must say \( A_i \succ A_j \), \( A_j \succ A_i \) or \( A_i \sim A_j \).
Continuity

If $A_3 \succeq A_2$ and $A_2 \succeq A_1$, then there exists $U(A_2)$ \([0 \leq U(A_2) \leq 1]\) such that

$$L = \{[1 - U(A_2)], A_1; U(A_2), A_3\} \sim A_2.$$

For a given set of outcomes $A_1$, $A_2$ and $A_3$, these probabilities are a function of $A_2$, hence the notation $U(A_2)$.

Why is it called continuity axiom? When $U(A_2) = 1$, we obtain $L = A_3 \succ A_2$; when $U(A_2) = 0$, we obtain $L = A_1 \prec A_2$. If we increase $U(A_2)$ continuously from 0 to 1, we hit a value $U(A_2)$ such that $L \sim A_2$.

Remark
Though $U(A_2)$ is a probability value, we will see that it is also the investor’s utility function.
Interchangeability

Given $L_1 = \{p_1, A_1; p_2, A_2; p_3, A_3\}$ and $A_2 \sim A = \{q, A_1; (1 - q), A_2\}$, the investor is indifferent between $L_1$ and $L_2 = \{p_1, A_1; p_2, A; p_3, A_3\}$.

Transitivity

Given $L_1 \succ L_2$ and $L_2 \succ L_3$, then $L_1 \succ L_3$.

Also, if $L_1 \sim L_2$ and $L_2 \sim L_3$, then $L_1 \sim L_3$.

Decomposability

A complex lottery has lotteries as prizes. A simple lottery has monetary values $A_1, A_2$ etc as prizes.

Consider a complex lottery $L^* = (q, L_1; (1 - q), L_2)$, where

$L_1 = \{p_1, A_1; (1 - p_1), A_2\}$ and $L_2 = \{p_2, A_1; (1 - p_2), A_2\}$,

$L^*$ can be decomposed into a simply lottery $L = \{p^*, A_1; (1 - p^*), A_2\}$, with $A_1$ and $A_2$ as prizes where $p^* = qp_1 + (1 - q)p_2$. 
**Monotonicity**

(a) For monetary outcomes, \( A_2 > A_1 \implies A_2 > A_1 \).

(b) For lotteries

(i) Let \( L_1 = \{p, A_1; (1-p), A_2\} \) and \( L_2 \{p, A_1; (1-p), A_3\} \). If \( A_3 > A_2 \), then \( A_3 > A_2 \); and \( L_2 > L_1 \).

(ii) Let \( L_1 = \{p, A_1; (1-p), A_2\} \) and \( L_2 = \{q, A_1; (1-q), A_2\} \), also \( A_2 > A_1 \) (hence \( A_2 > A_1 \)). If \( p < q \), then \( L_1 > L_2 \).
Theorem

The optimal criterion for ranking alternative investments is the expected utility of the various investments.

Proof

How to make a choice between $L_1$ and $L_2$

\[
L_1 = \{p_1, A_1; p_2, A_2; \cdots; p_n, A_n\}
\]
\[
L_2 = \{q_1, A_1; q_2, A_2; \cdots; q_n, A_n\}
\]

$A_1 < A_2 < \cdots < A_n$, where $A_i$ are various monetary outcomes?

* By comparability axiom, we can compare $A_i$. Further, by monotonicity axiom, we determine that

$A_1 < A_2 < \cdots < A_n$ implies $A_1 \prec A_2 \prec \cdots \prec A_n$.

* Define $A_i^* = \{[1 - U(A_i)], A_1; U(A_i), A_n\}$ where $0 \leq U(A_i) \leq 1$. 


By continuity axiom, for every $A_i$, there exists $U(A_i)$ such that $A_i \sim A_i^*$.

For $A_1$, $U(A_1) = 0$, hence $A_1^* \sim A_1$; for $A_n$, $U(A_n) = 1$. For other $A_i$, $0 < U(A_i) < 1$. By the monotonicity and transitivity axioms, $U(A_i)$ increases from zero to one as $A_i$ increases from $A_1$ to $A_n$.

* Substitute $A_i$ by $A_i^*$ in $L_1$ successively and by the interchangeability axiom,

$$L_1 \sim \tilde{L}_1 = \{p_1, A_1^*; p_2, A_2^*; \cdots; p_n, A_n^*\}.$$
By the decomposability axiom,

\[ L_1 \sim \tilde{L}_1 \sim L_1^* = \{ \Sigma p_i[1 - U(A_i)], A_1; \Sigma p_i U(A_i), A_n \} \].

Similarly

\[ L_2 \sim L_2^* = \{ \Sigma q_i[1 - U(A_i)], A_1; \Sigma q_i U(A_i), A_n \} \].

By the monotonicity axiom, \( L_1^* \succ L_2^* \) if

\[ \Sigma p_i U(A_i) > \Sigma q_i U(A_i) \].

This is precisely the expected utility criterion. The same conclusion applies to \( L_1 \succ L_2 \), due to transitivity.
Remark

Why does $U(A_i)$ reflect the investor’s utility?

Recall $A_i \sim A_i^* = \{[1 - U(A_i)], A_1; U(A_i), A_n\}$, such a function $U(A_i)$ always exists, though not all investors would agree on the specific value of $U(A_i)$.

- By the monotonicity axiom, utility is non-decreasing.
- A utility function is determined up to a positive linear transformation, so its value is not limited to the range $[0, 1]$. “Determined” means that the ranking of the projects by the MEUC does not change.
- The absolute difference or ratio of the utilities of two investment choices gives no indication of the degree of preference of one over the other since utility values can be expanded or suppressed by a linear transformation.
Characterization of utility functions

1. More is being preferred to less: \( u'(w) > 0 \)

2. Investors’ taste for risk
   - averse to risk (reject a fair gamble)
   - neutral toward risk (indifferent to a fair gamble)
   - seek risk (select a fair gamble)

3. Investors’ preference changes with a change in wealth. Percentage of wealth invested in risky asset changes as wealth changes.

*Jensen’s inequality*

Suppose \( u''(w) \leq 0 \) and \( X \) is a random variable, then

\[
u(E[X]) \geq E[u(X)].\]
Write $E[X] = \mu$; since $u(w)$ is concave, we have

$$u(w) \leq u(\mu) + u'(\mu)(w - \mu) \quad \text{for all values of } w.$$
Risk aversion

Replace $w$ by $X$ and take the expectation on each side

$$E[u(X)] \leq u(\mu).$$

Interpretation

$E[u(X)]$ represents the expected utility of the gamble associated with $X$. The investor prefers a sure wealth of $\mu = E[X]$ rather than playing the game, if $u''(w) \leq 0$. This indicates risk aversion.
Insurance premium

Individual's total initial wealth is \( w \), and the wealth is subject to random loss \( Y \) during the period, \( 0 \leq Y < w \).

Let \( \pi \) be the insurable premium payable at time 0 that fully reimburses the loss (neglecting the time value of money).

1. If the individual decides not to buy insurance, then the expected utility is \( E[u(w - Y)] \). The expectation is based on investor's own subjective assessment of the loss.

2. If he buys the insurance, the utility at time 1 is \( u(w - \pi) \). Note that \( w - \pi \) is the sure wealth.
If the individual is risk averse \( u''(w) \leq 0 \), then from Jensen’s inequality (change \( X \) to \( w - Y \)), we obtain

\[
u(w - E[Y]) \geq E[u(w - Y)].
\]

The fair value of insurance premium \( \pi \) is determined by

\[
u(w - \pi) = E[u(w - Y)]
\]

so that we can deduce that \( \pi \geq E[Y] \). Suppose the higher moments of \( Y \) are negligible, it can be deduced that the maximum price that a risk-averse individual with wealth \( w \) is willing to pay to avoid a possible loss of \( Y \) is approximately

\[
\pi \approx \mu_Y + \frac{\sigma_Y^2}{2} R_A(w - \mu),
\]

where \( R_A(w) = -u''(w)/u'(w) \), \( 0 \leq Y < w \) and \( \mu = E[Y] < w \). With higher \( R_A(w) \), the individual is willing to pay a higher premium to avoid risk.
Proof

We start from the governing equation for $\pi$

$$u(w - \pi) = E[u(w - Y)].$$
Write $Y = \mu + zV$, where $V$ is a random variable with zero mean. Here, $z$ is a small perturbation parameter. We then have

$$u(w - \pi) = E[u(w - \mu - zV)]. \quad (1)$$

We are seeking the perturbation expansion of $\pi$ in powers of $z$ in the form

$$\pi = a + bz + cz^2 + \cdots$$

(i) Setting $z = 0$, $u(w - a) = E[u(w - \mu)] = u(w - \mu)$ so that

$$a = \mu.$$

(ii) Differentiating (1) with respect to $z$ and setting $z = 0$,

$$-\pi'(0)u'(w - \pi) = E[-Vu'(w - \mu)] \quad (2)$$

since $E(V) = 0$ and $\pi'(0) = b$, so $b = 0$. 
(iii) Differentiating (1) twice with respect to $z$ and setting $z = 0$

\[-\pi''(0)u'(w - \pi) = E[V^2u''(w - \mu)]\]

\[c = -\frac{\text{var}(V)u''}{2u'|w-\mu}.\]

Define the absolute risk aversion coefficient: $R_A(w) = -\frac{u''(w)}{u'(w)}$, we have

\[\pi \approx \mu + \frac{R_A(w - \mu)}{2}z^2\text{var}(V)\]

\[= \mu + \frac{\sigma_Y^2}{2}R_A(w - \mu).\]

$\pi - \mu = \frac{\sigma_Y^2}{2}R_A(w - \mu)$ is called the risk premium. For low level of risks, $\pi - \mu$ is proportional to the product of variance of the loss distribution and individual's absolute risk aversion.
Relative risk aversion coefficient

Let $X$ be a fair game with $E[X] = 0$ and $\text{var}(X) = \sigma_X^2$. The whole wealth $w$ is invested into the game.

<table>
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<th>Choice A</th>
<th>Choice B</th>
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<tbody>
<tr>
<td>$w + X$</td>
<td>$w_C$ (with certainty)</td>
</tr>
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</table>

The investor is indifferent to these two positions iff

$$E[u(w + X)] = u(w_C).$$

Note that $w_C = w - (w - w_C)$, indicating the payment of $w - w_C$ for Choice B. The difference $w - w_C$ represents the maximum amount the investor would be willing to pay in order to avoid the risk of the game.
Let $q$ be the fraction of wealth an investor is giving up in order to avoid the gamble; then $q = \frac{w - w_C}{w}$ or $w_C = w(1 - q)$. Let $Z$ be the return per dollar invested so that for a fair gamble, $E[Z] = 1$. Write $\text{var}(Z) = \sigma_Z^2$.

Suppose we invest $w$ dollars, the return would be $wZ$. Expand $u(wZ)$ around $w$:

$$u(wZ) = u(w) + u'(w)(wZ - w) + \frac{u''(w)}{2}(wZ - w)^2 + \cdots$$

$$E[u(wZ)] = u(w) + 0 + \frac{u''(w)}{2}w^2\sigma_Z^2 + \cdots$$
On the other hand,

\[ u(w_C) = u(w(1 - q)) = u(w) - qwu'(w) + \cdots. \]

Equating \( u(w_C) \) with \( E[u(wZ)] \), we obtain

\[ \frac{u''(w)}{2}w^2\sigma_Z^2 = -u'(w)qw \]

so that

\[ q = -\frac{\sigma_Z^2}{w} \frac{u''(w)}{u'(w)}. \]

Define \( R_R(w) = \) coefficient of relative risk aversion \( = -w \frac{u''(w)}{u'(w)}, \)

then \( q = \frac{w - w_C}{w} = \) percentage of risk premium \( = \frac{\sigma_Z^2}{2}R_R(w). \)
Types of utility functions

1. *Exponential utility*

\[
\begin{align*}
u(x) &= 1 - e^{-ax}, \quad x > 0 \\
u'(x) &= ae^{-ax} \\
u''(x) &= -a^2e^{-ax} < 0 \quad \text{(risk aversion)}
\end{align*}
\]

so that \( R_A(x) = a \) for all wealth level \( x \).

2. *Power utility*

\[
\begin{align*}
u(x) &= \frac{x^\alpha - 1}{\alpha}, \quad \alpha \leq 1 \\
u'(x) &= x^{\alpha - 1} \\
u''(x) &= (\alpha - 1)x^{\alpha - 2}
\end{align*}
\]

\[
R_A(x) = \frac{1 - \alpha}{x} \quad \text{and} \quad R_R(x) = 1 - \alpha.
\]
3. *Logarithmic utility* (corresponds to $\alpha \to 0^+$ in power utility)

$$u(x) = a \ln x + b, \quad a > 0$$

$$u'(x) = a/x$$

$$u''(x) = -a/x^2$$

$$R_A(x) = \frac{1}{x} \text{ and } R_R(x) = 1.$$
Properties of power utility functions: $U(x) = x^{\gamma}/\gamma, \gamma \leq 1$

(i) $\gamma > 0$, aggressive utility

Consider $\gamma = 1$, corresponding to $U(x) = x$. This is the expected value criterion.

Recall that the strategy that maximizes the expected value bets all capital on the most favorable sector – prone to early bankruptcy.

For $\gamma = 1/2$; consider two opportunities:

(a) capital will double with a probability of 0.9 or it will go to zero with probability 0.10,
(b) capital will increase by 25% with certainty.

Since $0.9 \times \sqrt{2} > \sqrt{1.25}$, so opportunity (a) is preferred to (b). However, opportunity (a) is certain to go bankrupt.
(ii) $\gamma < 0$, conservative utility

For $\gamma = -1/2$, consider two opportunities

(a) capital quadruples in value with certainty
(b) with probability 0.5 capital remains constant and with probability 0.5 capital is multiplied by 10 million.

Since $-4^{-1/2} > -0.5 - 0.5(10,000,000)^{-1/2}$, opportunity (a) is preferred to (b).

Apparently, the best choice for $\gamma$ may be negative, but close to zero. This utility function is close to the logarithm function.
Example

An investor has an initial wealth of \( w \) and can allocate funds between two assets: a risky asset and a riskless asset.

\[
m = \text{expected rate of return on the risky security} = p i_u + (1 - p) i_d
\]

so that

\[
p = \frac{m - i_d}{i_u - i_d}, \quad 1 - p = \frac{i_u - m}{i_u - i_d}.
\]
• Impose the condition: \( m > i_f \), otherwise a rational investor will never invest a positive amount in the risky asset.

• No arbitrage conditions: \( i_u > i_f > i_d \). Also, \( i_u > m > i_d \) for \( 0 < p < 1 \).

• Let \( x \) be the fraction of initial wealth placed on the risky asset. Choose the power utility function: \( w^{\frac{\alpha}{\alpha}}, 0 < \alpha < 1 \).

Investor’s expected utility:

\[
p \frac{\{w[(1 + i_f) + x(i_u - i_f)]\}^\alpha}{\alpha} + (1 - p) \frac{\{w[1 + i_f) + x(i_d - i_f)]\}^\alpha}{\alpha}.
\]

Since the utility function is concave, the second order condition for a maximum is automatically satisfied.
Optimal proportion \( x^* = \frac{(1 + i_f)(\theta - 1)}{(i_u - i_f) + \theta(i_f - i_d)} \)

where \( \theta = \frac{[p(i_u - i_f)]^{1/(1-\alpha)}}{[(1 - p)(i_f - i_d)]^{1/(1-\alpha)}}. \)

The no-arbitrage condition leads to \( \theta \geq 1. \)

(i) \( x^* > 0 \iff \theta > 1. \)

(ii) As \( m \to i_f, \theta \to 1 \) and \( x^* \to 0. \)

A risk-averse investor will prefer the riskless asset if it has the same return as the expected return on the risky asset.

(iii) \( x^* < 1 \) as long as \( \theta < \frac{1 + i_u}{1 + i_d}. \)
Continuous-time limit of the discrete-time model

Assume stationarity of the return distribution and no transaction costs, the multi-period case collapses to a series of identical single-period problems. The optimal $x^*$ is the same at each time interval and it is the same as that of the one-period case.

Assuming discrete random walk of proportional jumps, with $h$ = time step and $\sigma^2$ as the variance rate, we obtain

$$1 + i_u = e^{\sigma \sqrt{h}}, \quad 1 + i_d = e^{-\sigma \sqrt{h}}, \quad 1 + m = e^{\mu h}, \quad e^{rh} = 1 + r_f.$$
Express \( i_u, i_d, i_f \) and \( p \) in terms of \( h \). Taking \( h \to 0 \),

\[
x^* = \frac{\mu - r}{\sigma^2(1 - \alpha)} = \frac{\text{risk premium on risky asset}}{\text{variance} \times \text{relative risk aversion}}.
\]

- Merton assumes GBM for the asset price process with mean \( \mu \) and variance \( \sigma^2 \); and power utility function.

- It is optimal for the investor to maintain a constant proportion of wealth in the risky asset – continuous rebalancing of portfolio is required (though at no transaction cost).
Minimum expected excess return required to induce the individual to invest on the risky asset

\[
\tilde{r} = \text{random rate of return of the single risky asset}
\]

\[
x = \text{proportion of wealth invested in the risky asset}
\]

\[
w_0 = \text{initial wealth}
\]

\[
a = \text{wealth invested in the risky asset}
\]

\[
\tilde{w} = \text{random wealth at the end of the investment period}
\]

\[
r = \text{riskfree interest rate}
\]

\[
\tilde{w} = (w_0 - a)(1 + r) + a(1 + \tilde{r}) = w_0(1 + r) + a(\tilde{r} - r), \quad a = xw_0
\]

For the individual to invest at least \(xw_0\) amount of his wealth on the risky asset, it must be that

\[
\frac{d}{da}\{E[u(\tilde{w})]\} \geq 0 \quad \text{or}
\]

\[
E[u'(w_0(1 + r) + xw_0(\tilde{r} - r))(\tilde{r} - r)] \geq 0.
\]
Performing the Taylor expansion of \( u'(w_0(1 + r) + xw_0(\tilde{r} - r)) \) at \( w_0(1 + r) \), we obtain

\[
E[u'(w_0(1 + r) + xw_0(\tilde{r} - r))(\tilde{r} - r)]
= u'(w_0(1 + r))E[\tilde{r} - r] + u''(w_0(1 + r))xw_0E[(\tilde{r} - r)^2]
+ o(E[(\tilde{r} - r)^2]).
\]

Assuming low level of risk of \( \tilde{r} \) so that \( o(E[(\tilde{r} - r)^2]) \) can be neglected. The minimum risk premium required to induce \( x \) portion of wealth invested on the risky asset is

\[
E[\tilde{r} - r] \geq -xw_0 \frac{u''(w_0(1 + r))}{u'(w_0(1 + r))} E[(\tilde{r} - r)^2], \quad \text{for all} \quad 0 < x \leq 1.
\]
Log utility and maximization of the geometric mean return

Geometric mean return = $\bar{r}_G = \prod_{j=1}^{n}(1 + r_j)^{p_j} - 1$, where $r_j$ is the rate of return if outcome $j$ occurs with probability $p_j$.

Taking logarithm on both sides

$$\ln(1 + \bar{r}_G) = \sum_{j=1}^{n} p_j \ln(1 + r_j).$$

Consider the log utility function: $\max E \ln w_1$, where $w_1$ is the end-of-period wealth.
Let $r$ be the random rate of return.

Maximization of the expected log utility is given by

$$\max E[\ln w_1 - \ln w_0] = \max E \ln \frac{w_1}{w_0} = \max E[\ln(1 + r)] = \max \sum_{j=1}^{n} p_j \ln(1 + r_j).$$

Maximization of the geometric mean return is equivalent to the maximization of the expected utility, where the logarithmic function is used as the utility function.
Summary on risk aversion coefficients

**Absolute risk aversion**

\[ A(W) = -\frac{U''(W)}{U'(W)} \]

If \( A(W) \) has the same sign for all values of \( W \), then the investor has the same risk preference (risk averse, neutral or seeker) for all values of \( W \) (global).

**Relative risk aversion**

\[ R(W) = -\frac{WU''(W)}{U'(W)} \]

Note that utility functions are only unique up to a strictly positive affine transformation. The second derivative alone cannot be used to characterize the intensity of risk averse behaviors. The risk aversion coefficients are invariant to a strictly positive affine transformation of individual utility function.
## Changes in Absolute Risk Aversion with Wealth

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<td>$A'(W) &lt; 0$</td>
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Changes in Relative Risk Aversion with Wealth

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<td>$R'(W) = 0$</td>
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<td>Decreasing relative risk aversion</td>
<td>Percentage invested in risky assets increases as wealth increases</td>
<td>$R'(W) &lt; 0$</td>
<td>$-e^{2W^{-1/2}}$</td>
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