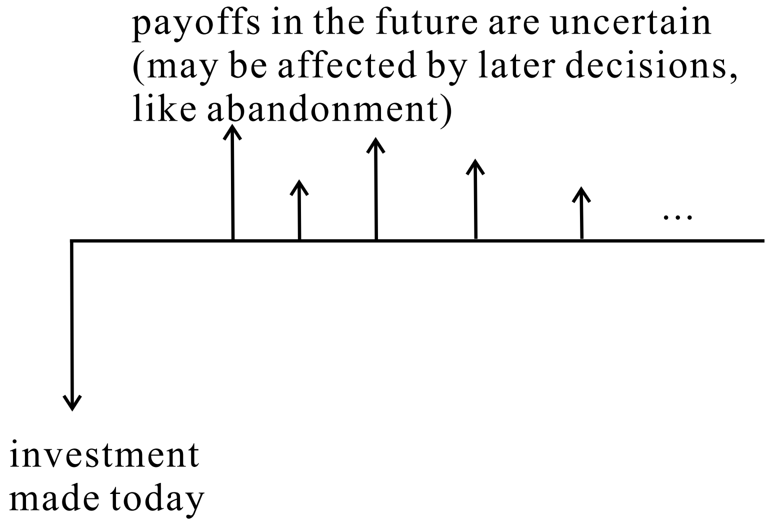
7. Real option of investment decisions



e.g. Development of a housing project, subject to uncertainty in the property prices

- ⋆ option of postponement of investment
- ⋆ sunk cost cannot be recovered (at least partly irreversible)
- ⋆ multi-stage investment decisions and abandonment rights
- ⋆ resembles the early exercise right in an American option model

Net present value rule

Investment in a project when the present value of its expected cash flow is at least as large as its cost.

- ⋆ Opportunity cost of delaying investment must be included as part of the total cost of investing.
- ⋆ The net present value rule corresponds to the assumption of zero volatility of the underlying stochastic state variable.

Sleeping patents – multi-stage decisions

- Research and Development phase – arrivals of new innovations are modeled as Poisson processes.
- After the success of R & D, investor still waits for the optimal time to launch the project – sleeping patents.
- Investment thresholds (optimal entry points) will be affected by the extent of loss of revenue flows due to the earlier entry by a competitor.

Two approaches

1. Dynamic programming; 2. contingent claims analysis

They make different assumptions about the financial markets and the discount rates that firms use to value future cash flows.

In the *dynamic programming approach*, it breaks a whole sequence of decisions into two components: the immediate decision, and a valuation function that encapsulates the consequences of all subsequent decisions. The exogenously specified discount rate ρ for future cash flows appears in the governing equation for the value of the project (or investment.) Here, ρ is the rate of return demanded by the investor on the investment project.

In contingent claims approach, there is a **spanning** requirement on the availability of traded assets that replicate the pattern of returns from the investment. If this is satisfied, then risk neutral valuation is feasible.

Uncertainty is modelled using discrete-time Markov processes. A random process is Markovian if the future of the process given the present is independent of the past.

x_t — stochastic state variable that describes the firm's current status (say, output price)

1. At any time t, x_t is known but $x_{t+1}, x_{t+2}, \cdots$ are random variables.

2. Also, some choices are available and they are represented by a control variable u. For example, u may be a binary variable where "0" represents waiting and "1" represents investing at once.

3. The state and the control at time t affect the firm's immediate profit flows, denoted by $\pi_t(x_t, u_t)$.

Basic dynamic programming procedure

$F_t(x_t) =$ expected net present value of all firm's cash flows (firm makes all decisions optimally from this point onwards)

Suppose the firm choose u_t, the immediate profit is $\pi_t(x_t, u_t)$. At the next period $t+1$, the state will be x_{t+1} and the continuation value $= E_t[F_{t+1}(x_{t+1})] = \int F_{t+1}(x_{t+1})\, d\phi_t(x_{t+1}|x_t, u_t)$.

The firm chooses u_t to maximize $\pi_t(x_t, u_t) + \dfrac{1}{1+\rho} E_t[F_{t+1}(x_{t+1})]$, and the result will be just $F_t(x_t)$. Hence,

$$F_t(x_t) = \max_{u_t} \left\{ \pi_t(x_t, u_t) + \frac{1}{1+\rho} E_t[F_{t+1}(x_{t+1})] \right\}.$$

The optimality of the remaining choices u_{t+1}, u_{t+2}, etc is subsumed in the continuation value, so only the intermediate control u_t remains to be chosen optimally. The decomposition is based on the *Bellman fundamental equation of optimality*.

The action taken in the current period t could depend on the knowledge of the current state x_t but not on the random future state x_{t+1}.

In continuous time, we require the uncertainty to be "continuous from the right" in time while the strategies are "continuous from the left". That is, any jumps in the stochastic process of the state variable occur at an instant, while the actions cannot change until *just after the instant*.

1. Finite time horizon

$$F_{T-1}(x_{T-1}) = \max_{u_{T-1}} \left\{ \pi(x_{T-1}, u_{T-1}) + \frac{1}{1+\rho} E_{T-1}[\Omega_T(x_T)] \right\},$$

where $\Omega_T(x_T)$ is the terminal payoff.

Thus, we know the value function at $T-1$. That in turn allows us to solve the maximization problem for u_{T-2}, leading to the value function $F_{T-2}(x_{T-2})$, and so on. This is called the *backward induction*.

2. Infinite time horizon (no time dependency)

$$F(x) = \max_u \left\{ \pi(x,u) + \frac{1}{1+\rho} E_t[F(x')|x,u] \right\}.$$

This is a functional equation, with F as its unknown.

Iterative procedure

- Start with any guess $F^{(1)}(x)$, solve for the corresponding optimal choice rule u^1 (expressed as a function of x). Substitute into RHS and obtain $F^{(2)}(x)$, and repeat the procedure.

- The limiting function $F(x)$ can be visualized as a fixed point of the iteration step. The factor $\frac{1}{1+\rho}$, which is less than one, leads to the desirable property of geometric reduction in errors (contraction mapping property).

Continuous time framework

Now ρ is the discount rate per unit time and $\pi(x,u,t)$ is the rate of profit flow. Bellman equation becomes

$$F(x,t) = \max_u \left\{ \pi(x,u,t)\Delta t + \frac{1}{1+\rho\Delta t} E_t[F(x',t+\Delta t)|x,u] \right\}.$$

Multiply by $1+\rho\Delta t$ and rearrange to give

$$\begin{aligned} \rho\Delta t F(x,t) &= \max_u \{\pi(x,u,t)\Delta t(1+\rho\Delta t) + E_t[F(x',t+\Delta t) - F(x,t)]\} \\ &= \max_u \{\pi(x,u,t)\Delta t(1+\rho\Delta t) + E_t[\Delta F]\}. \end{aligned}$$

Taking the limit $\Delta t \to 0$, we obtain

$$\rho F(x,t) = \max_u \left\{ \pi(x,u,t) + \frac{1}{dt} E_t[dF] \right\}.$$

The entitlement to the flow of profits can be considered as an asset, and that $F(x,t)$ is its value.

$\rho F(x,t) =$ normal return per unit time that a decision maker (with ρ as discount rate) would require for holding this asset

$\pi(x,u,t) =$ immediate payout or dividend from the asset per unit time

$\frac{1}{dt}E_t[dF] =$ expected rate of capital gain.

- ⋆ The maximization with respect to the control u means that the current operation of the asset is being managed optimally.

- ⋆ The equilibrium condition is expressed by the equality, revealing the investor's view of balancing the normal return with the sum of profit flow and expected rate of capital gain.

Assume the following Ito process for x:

$$dx = a(x,u,t)\ dt + b(x,u,t)\ dZ$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} E_t[dF] = F_t(x,t) + a(x,u,t)F_x(x,t) + \frac{b^2(x,u,t)}{2} F_{xx}(x,t).$$

(i) finite time horizon

$$\begin{aligned} \rho F(x,t) \ &= \ \max_u \Big\{ \pi(x,u,t) + F_t(x,t) + a(x,u,t)F_x(x,t) \\ &\quad + \ \frac{b^2(x,u,t)}{2} F_{xx}(x,t) \Big\} \end{aligned}$$

with $F(x,T) = \Omega(x,T)$ for all x.

(ii) infinite time horizon

$$\rho F(x) = \max_u \left\{ \pi(x,u) + a(x,u)F'(x) + \frac{b^2(x,u)}{2} F''(x) \right\}.$$

Derivation of governing equation for the value of asset

Let $\pi(x, t)$ be the profit (revenue) flow, where x is the underlying stochastic state variable. Let $\Omega(x_T, T)$ be the terminal payoff. We stipulate an exogenous discount rate ρ.

Formally, the value of asset, $F(x, t)$, is given by

$$F(x,t) = E_t^* \left[\int_t^T e^{-\rho(s-t)} \pi(x_s, s)\, ds + e^{-\rho(T-t)} \Omega(x_T, T) \right]$$

where E_t^* is the expectation based on the information as of time t.

Next, we derive the governing equation under the scenario of continuation, which mean either continue to wait (investment option) or not yet abandon the operation (abandonment option). We assume the following lognormal process for x

$$\frac{dx}{x} = \alpha\, dt + \sigma\, dZ.$$

Assuming no action is taken during the time interval dt. In dt later, value of asset becomes $F(x + dx, t + dt)$.

$$F(x,t) = \pi(x,t)\,dt + e^{-\rho\,dt} \underbrace{E_t[F(x+dx, t+dt)]}_{\substack{dx \text{ is a random increment} \\ \text{so we must take an expectation}}}$$

$$\begin{aligned} \text{RHS} \quad &\approx \quad \pi(x,t)\,dt + (1 - \rho\,dt)\Big[F(x,t) + F_t(x,t)\,dt + F_x(x,t)\alpha x\,dt \\ &\qquad\qquad + \frac{\sigma^2 x^2}{2} F_{xx}(x,t)\,dt\Big] \\ &= \quad F(x,t) + dt\left[\frac{\sigma^2}{2}x^2 F_{xx}(x,t) + \alpha x F_x(x,t) + F_t(x,t) - \rho F(x,t) \right. \\ &\qquad\qquad \left. + \pi(x,t)\right] \end{aligned}$$

so that we obtain

$$\frac{\sigma^2}{2}x^2 F_{xx} + \alpha x F_x - \rho F + F_t + \pi(x,t) = 0.$$

The *total* rate of return of asset $\mu = \alpha + \delta$, where $\delta =$ dividend yield. From the capital asset pricing model (CAPM)

$$\mu - r = \beta(\mu_m - r)$$

where

$$\beta = \frac{\text{cov}(R_x, R_m)}{\sigma_m^2} = \rho_{xm}\sigma/\sigma_m \quad \text{and} \quad \phi = \frac{\mu_m - r}{\sigma_m}.$$

Here, ϕ is the market price of risk, ρ_{xm} is the correlation coefficient between return on x and the whole market portfolio m. Hence,

$$\mu = r + \phi\sigma\rho_{xm}.$$

When $\rho_{xm} = 0, \mu = r$ so that $\alpha = r - \delta$.

Combined Ito process and Poisson jump process

$$dq = \begin{cases} 0 & \text{with prob } 1 - \lambda\, dt \\ u & \text{with prob } \lambda\, dt \end{cases} .$$

The event is a jump size u, which can itself be a random variable. Write

$$dx = a(x,t)\,dt + b(x,t)\,dZ + g(x,t)\,dq.$$

Expected value of change in $F(x,t)$

$$\begin{aligned} E_t[dF] &= \left[\frac{\partial F}{\partial t} + a(x,t)\frac{\partial F}{\partial x} + \frac{b^2(x,t)}{2}\frac{\partial^2 F}{\partial x^2}\right] dt \\ &+ \lambda E_u[[F(x+g(x,t)u,t) - F(x,t)]]\,dt. \end{aligned}$$

↑

expectation with respect to the random jump size u

The modified equation for the derivative value $F(x,t)$ with Poisson jump is

$$\begin{aligned} &\pi(x,t) + \frac{\sigma^2}{2}x^2 F_{xx} + \alpha x F_x - \rho F \\ &+ F_t + \lambda\{E_u[F(x+g(x,t)u,t] - F\} = 0. \end{aligned}$$

If we write $E_u[F(x + g(x,t)u, t)] = R(x,t)$, where $R(x,t)$ is the expected rate of project value after the occurrence of jump event, then we obtain

$$F_t + \frac{\sigma^2}{2}x^2 F_{xx} + \alpha x F_x - (\rho + \lambda)F + \pi(x,t) + \lambda R(x,t) = 0.$$

If the firm loses the opportunity to invest in the project after the occurrence of the jump event, then we have $R(x,t) = 0$ and obtain

$$F_t + \frac{\sigma^2}{2}x^2 F_{xx} + \alpha x F_x - (\rho + \lambda)F + \pi(x,t) = 0.$$

Optimal stopping

Here, the choice in any period is binary. One alternative corresponds to stopping the process to take the termination payoff, and the other entails continuation for one period. For example, the investor decides to continue operation or abandon the operation to receive the scrap value.

Let $\Omega(x)$ denote the termination payoff

$$F(x) = \max\left\{\Omega(x), \pi(x) + \frac{1}{1+\rho}E[F(x')|x]\right\}.$$

There will be a single cutoff x^*, with termination optimal on one side and continuation on the other.

With finite time horizon, the dynamic programming procedure dictates

$$F(x,t) = \max\left\{\Omega(x), \pi(x,t)\,dt + \frac{1}{1+\rho\,dt}E_t[F(x', t+dt)|x]\right\}.$$

In the continuation region, 2nd term $>$ 1st term. We obtain the governing equation for the value function F

$$\frac{\sigma^2}{2} F_{xx}(x,t) + \alpha F_x(x,t) + F_t(x,t) - \rho F(x,t) + \pi(x,t) = 0.$$

The "free boundary" $x^*(t)$ divides the (x,t) space into the continuation region and the stopping region. It is not known aprior, rather it is determined as part of the solution.

(i) *Value-matching condition*

$$F(x^*(t),t) = \Omega(x^*(t),t) \quad \text{for all} \quad t$$

(ii) *Smooth-pasting condition*

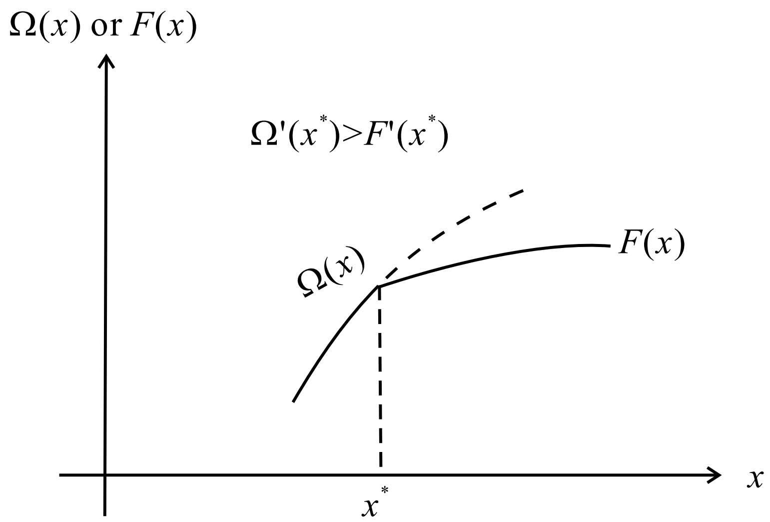
$$F_x(x^*(t),t) = \Omega_x(x^*(t),t) \quad \text{for all} \quad t.$$

The smooth pasting condition is added as an additional auxiliary condition to ensure that $x^*(t)$ is chosen such that the value function $F(x,t)$ is maximized.

Proof by contradiction

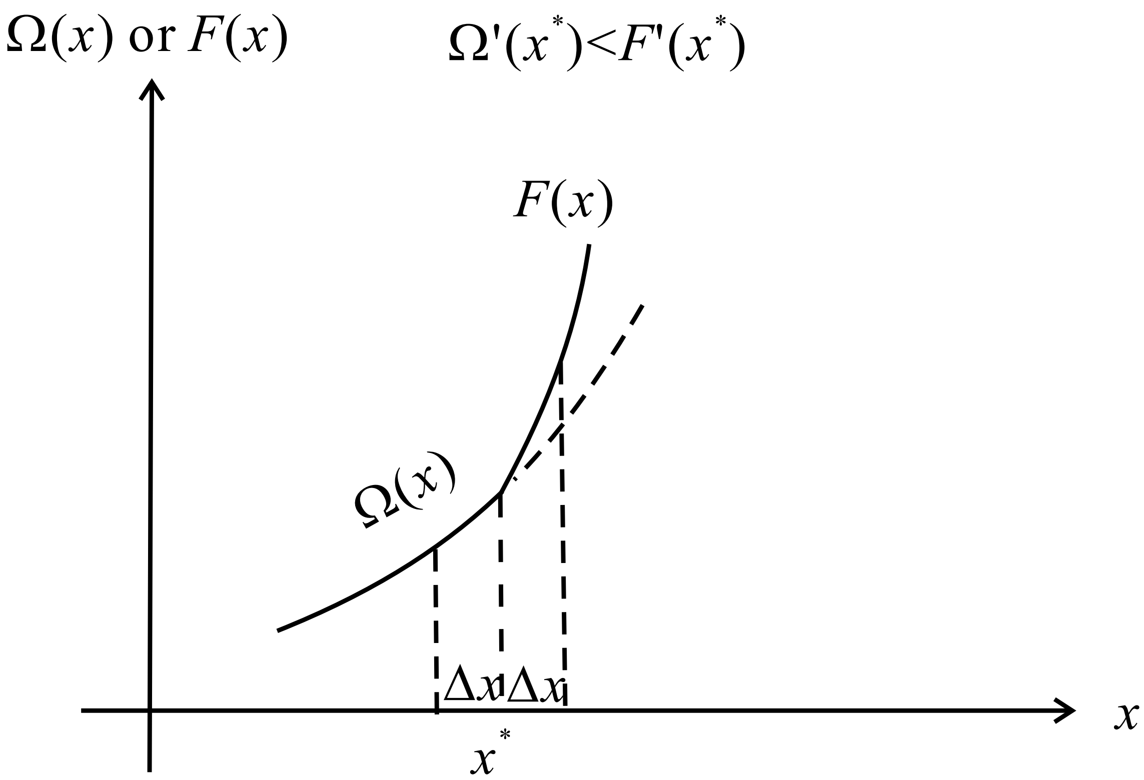
Case (a): upward-pointing kink

By continuity, $\Omega(x,t) > F(x,t)$ for $x > x^*$; termination rather than continuation would be optimal for such x. This is in contradiction to the definition of $x^*(t)$ as the threshold.



Case (b): downward-pointing kink

How to show that it is better to choose continuation at this given value of $x = x^*$? By waiting a little bit longer, we can observe the next step of x and choose positions on either side of the kink.



If $x^*(t)$ is the threshold, then $x^*(t)$ is a point of indifference between continuation and stopping. Suppose the project is continued for an extended time Δt, and follows the policy: further continuation if $\Delta x > 0$ and stopping if $\Delta x < 0$. The value function becomes

$$\pi(x^*(t), t)\Delta t + \frac{1}{1+\rho\Delta t}[pF(x^*(t)+\Delta x, t+\Delta t) + q\Omega(x^*(t)-\Delta x, t+\Delta t)]$$

where

$$\begin{aligned} p &= \text{ probability of upside move } = \frac{1}{2}\left[1 + \frac{a(x,t)\sqrt{\Delta t}}{b(x,t)}\right] \\ q &= 1 - p = \text{ probability of downside move.} \end{aligned}$$

The probability p can be found by equating the mean and variance of the continuous stochastic variable x at time $t + \Delta t$, given the value at t, and its discrete analog (binomial random walk).

By performing Taylor expansion up to Δx [recall that $\Delta t \sim \Delta x^2$], using the value-matching condition and the governing differential equation, we obtain

$$F(x^*(t),t) + \frac{1}{2}\underbrace{[F_x(x^*(t),t) - \Omega_x(x^*(t),t)]\Delta x}_{\text{positive from the assumption}}$$

An average of the two does better than the kink point itself. The delayed stopping at x^* gives a higher value. In other words, continuation for a short interval Δt is a better policy.

Contingent claims analysis

Suppose the profit flow depends on x, where x may be firm's output price

$$\frac{dx}{x} = \alpha\, dt + \sigma\, dZ.$$

Assume that the firm's output can itself be traded as an asset. The output is held by investors only if it provides a sufficiently high return μ, where $\mu = \alpha + \delta =$ expected total rate of return, $\delta =$ dividend yield, $\alpha =$ expected price appreciation.

We find the value $F(x, t)$ of a firm with profit flow $\pi(x, t)$ by replicating its return and risk characteristics using traded assets of known value.

Replicating portfolio: one dollar in riskless asset and n units of firm's output instantaneously for short time interval dt

initial cost $= 1+nx$; after time interval dt, the total return per dollar invested is

$$\frac{r + n(\alpha + \delta)x}{1 + nx}\, dt + \frac{\sigma n x}{1 + nx}\, dZ,$$

where $n\delta x\, dt$ is the dividend amount and $r\, dt$ is the interest return.

Consider the ownership of the firm over the same dt interval. This costs $F(x,t)$ to buy. The profit $\pi(x,t)\, dt$ can be treated as dividend. The random capital gain is

$$dF = \left[F_t(x,t) + \alpha x F_x(x,t) + \frac{\sigma^2 x^2}{2} F_{xx}(x,t)\right] dt + \sigma x F_x(x,t)\, dZ.$$

The total return per dollar invested is

$$\frac{\pi(x,t) + F_t(x,t) + \alpha x F_x(x,t) + \frac{\sigma^2 x^2}{2} F_{xx}(x,t)}{F(x,t)}\, dt + \frac{\sigma x F_x(x,t)}{F(x,t)}\, dZ.$$

To replicate the risk of owning the firm, we choose

$$\frac{nx}{1+nx} = \frac{xF_x(x,t)}{F(x,t)}.$$

In the market, two assets with identical risk must earn equal return. This choice would ensure

$$\frac{\pi(x,t) + F_t(x,t) + \alpha x F_x(x,t) + \frac{\sigma^2 x^2}{2} F_{xx}(x,t)}{F(x,t)} = \frac{r + n(\alpha+\delta)x}{1+nx}.$$

To eliminate $\dfrac{nx}{1+nx}$, we observe that the right hand side equals

$$r\left[1 - \frac{xF_x(x,t)}{F(x,t)}\right] + (\alpha + \delta)\frac{xF_x(x,t)}{F(x,t)}$$

so that

$$\frac{\sigma^2}{2}x^2 F_{xx}(x,t) + (r-\delta)xF_x(x,t) + F_t(x,t) - rF(x,t) + \pi(x,t) = 0.$$

Interestingly, α and ρ do not appear in the governing equation. Why?

Use of spanning set

Even if the risk in x is not directly traded in the market, it suffices to consider trading of some other asset whose stochastic fluctuations are perfectly correlated with the stochastic process for x. The price of the replicating spanning asset X follows

$$dX = A(x,t)X\,dt + B(x,t)X\,dZ.$$

- The coefficients $A(x,t)$ and $B(x,t)$ are functions of x, keeping with the notion that x summarizes all the information about the current state of the economy.

- The two Wiener process increments dZ must be the same if X is to track the stochastic fluctuations in x.

Let $D(x,t)$ denote the dividend yield of the replicating asset. One dollar invested in the replicating asset over $(t, t+dt)$ becomes

$$\underbrace{[D(x,t) + A(x,t)]}_{\mu_X(x,t)=r+\phi\rho_{xm}B(x,t)} \; dt + B(x,t)\, dZ.$$

Consider a portfolio that consists of the firm and n units of short position in the asset X. This costs $[F(x,t) - nX]$ dollars to buy. The total capital gain over dt interval:

$$\begin{aligned} & dF - n\, dX + \pi(x,t)\, dt - nD(x,t)X\, dt \\ = & \left[F_t + aF_x + \frac{b^2}{2}F_{xx} - nAX - nD(x,t)X + \pi(x,t)\right] dt \\ & + \; (bF_x - nBX)\, dZ. \end{aligned}$$

We choose $n = \dfrac{bF_x}{BX}$ to make the portfolio riskless. With this choice of n, we set the expected return on the portfolio to be $r(F - nX)\,dt$. The governing equation for $F(x,t)$:

$$\frac{b^2(x,t)}{2} F_{xx}(x,t) + \left\{ a(x,t) - \frac{b(x,t)}{B(x,t)} [\mu_X(x,t) - r] \right\} F_x(x,t)$$
$$- \; rF(x,t) + F_t(x,t) + \pi(x,t) = 0.$$

We require not only that the stochastic component of the returns on x and X obey the same probability law, but each and every path (realization) of one process is replicated by the other. This is implicit in the assumption that the same dZ is used in both stochastic terms of dx and dX.

If the two assets x and X have the same market price of risk

$$\frac{a - r}{b} = \frac{\mu_X - r}{B},$$

then the coefficient of $F_x(x,t)$ can be simplified as

$$a - \frac{b}{B}(\mu_X - r) = r.$$

A contingent claims related to hitting a barrier

We would like to find the fair value of a contingent claims that pays \$1 when the process Y_t hits a fixed barrier level Y_2. We assume that the investor is risk neutral meaning that he demands zero market price of risk. In this case, $\rho = r$ since

$$\frac{\rho - r}{\sigma} = \text{market price of risk } = 0.$$

Suppose the random process Y_t is governed by

$$\frac{dY_t}{Y_t} = \alpha\, dt + \sigma\, dZ.$$

We would like to compute

$$E[e^{-rT}],$$

where T is the (random) first passage time that the random process reaches a fixed level Y_2 starting from the general position Y.

Let $f(Y) = E[e^{-rT}]$. Applying dynamic programming like recursive argument

$$f(Y) = e^{-r\,dt}E[f(Y + dY)].$$

Using Ito's lemma,

$$\begin{aligned} f(Y) &= [1 - r\,dt + o(dt)]\left[f(Y) + \alpha Y f'(Y)\,dt + \frac{\sigma^2}{2}Y^2 f''(Y)\,dt + o(dt)\right] \\ &= f(Y) + dt\left[\frac{\sigma^2}{2}Y^2 f''(Y) + \alpha Y f'(Y) - r f(Y)\right] + o(dt). \end{aligned}$$

Taking $dt \to 0$, we obtain

$$\frac{\sigma^2}{2} Y^2 f''(Y) + \alpha Y f'(Y) - r f(Y) = 0$$

with general solution

$$f(Y) = A_1 Y^{\beta_1} + A_2 Y^{\beta_2}, \quad \beta_1 > 0, \beta_2 < 0,$$

where β_1 and β_2 are the roots of

$$\frac{\sigma^2}{2} \beta(\beta - 1) + \alpha\beta - r = 0.$$

As $Y \to Y_2, e^{-rT} \to 1$ and so $f(Y_2) = 1$. When Y is very small, T is very large and $e^{-rT} \to 0$, so $f(0) = 0$. We then obtain

$$f(Y) = \left(\frac{Y}{Y_2} \right)^{\beta_1}.$$

Present values associated with Geometric Brownian motion

Consider a contingent claims which pays the profit flow $a + bY_t$ until the upper barrier Y_2 is hit. The fair value of this claims is given by

$$g(Y) = E\int_0^T e^{-rt}(a + bY_t)\,dt, \qquad Y_0 = Y.$$

Using the usual dynamic programming argument

$$g(Y) = a + bY_t + e^{-r\,dt}E_t[g(Y + dY)].$$

The governing equation for $g(Y)$ is found to be

$$\frac{\sigma^2}{2}Y^2 g''(Y) + \alpha Y g'(Y) - rg(Y) + a + bY = 0,$$

with boundary conditions:

$$g(Y_2) = 0 \quad \text{and} \quad g(0) = \int_0^\infty a e^{-rt}\,dt = \frac{a}{r}.$$

The general solution is

$$g(Y) = \beta_1 Y^{\beta_1} + \beta_2 Y^{\beta_2} + \frac{a}{r} + \frac{Y}{r - \alpha}.$$

Solving for the arbitrary constants, we obtain

$$g(Y) = \frac{a}{r} + \frac{Y}{r - \alpha} - \frac{Y_2}{r - \alpha}\left(\frac{Y}{Y_2}\right)^{\beta_1}.$$

When there is no boundary, we can compute the present value of a perpetual dividend flow *without* using the ODE, but we do need to assume $\alpha < r$. Recall that $E[Y_t] = Y_0 e^{\alpha t}$ so that

$$E\left[\int_0^\infty e^{-rt}(a + bY_t)\,dt\right] = \int_0^\infty e^{-rt}(a + bE[Y_t])\,dt = \frac{a}{r} + \frac{bY_0}{r - \alpha}.$$

Remark

Under the risk neutral valuation framework, the discount rate is taken to be r and $\alpha = r - \delta$ so that $r - \alpha = \delta$, where $\delta > 0$ is the dividend yield.

Value of a project and the decision to invest

Assuming that the firm's investment project, once completed, will produce a fixed flow of output forever.

Let P denote the price of one unit of output that would be produced by the project

$$\frac{dP}{P} = \alpha\, dt + \sigma\, dZ,$$

that is, expected value of P grows at the trend rate α.

- Assume that the quantity of output from the project is one unit per year. The holder of the project will receive profit flow $P\, dt$ over dt interval. Also, we assume the satisfaction of the *spanning* requirement.

Let δ be the dividend yield of the output product. The drift rate under the risk neutral valuation framework is $r - \delta$.

Governing equation: $\frac{\sigma^2}{2} P^2 \frac{d^2V}{dP^2} + (r - \delta) P \frac{dV}{dP} - rV + P = 0.$

Solution: $V(P) = B_1 P^{\beta_1} + B_2 P^{\beta_2} + P/\delta$

where $\beta_1 > 0$ and $\beta_2 < 0$ are the roots of the characteristic equation:

$$\frac{\sigma^2 \beta(\beta - 1)}{2} + (r - \delta)\beta - r = 0.$$

Since $V(0) = 0$ and $\beta_2 < 0$, we can eliminate P^{β_2} by choosing $B_2 = 0$. The other term $B_1 P^{\beta_1}$ is attributable to "speculative bubbles" as $P \to \infty$. It represents the extra value above the fundamental value P/δ. People might value the asset above its fundamentals if they expected to be able to resell it later at a sufficient capital gain.

When β_1 is a root of the fundamental quadratic, the expected rate of growth of an asset that is always valued at P^{β_1} can be found via Ito's lemma as follows:

$$\begin{aligned}
\frac{dP^{\beta_1}}{P^{\beta_1}} &= \left[\beta_1 P^{\beta_1 - 1}\, dP + \frac{\beta_1(\beta_1 - 1)}{2} P^{\beta_1 - 2}\sigma^2 P^2\, dt\right] \Big/ P^{\beta_1} \\
&= \left[\beta_1 \alpha + \frac{\beta_1(\beta_1 - 1)}{2}\sigma^2\right] dt + \beta_1 \sigma\, dZ, \quad \text{since } \frac{dP}{P} = \alpha\, dt + \sigma\, dZ.
\end{aligned}$$

Recall that $\mu - \alpha = \delta$ and $\frac{\beta_1(\beta_1 - 1)}{2}\sigma^2 + (r - \delta)\beta_1 - r = 0$ so that

$$\begin{aligned}
\frac{dP^{\beta_1}}{P^{\beta_1}} &= [r + (\mu - r)\beta_1]\, dt + \beta_1 \sigma\, dZ \\
&= [r + \phi \beta_1 \rho_{Pm} \sigma]\, dt + \beta_1 \sigma\, dZ.
\end{aligned}$$

Hence, the expected rate of growth of P^{β_1} is found to be $r + \phi \beta_1 \rho_{Pm} \sigma$?

Risk-adjusted rate of return of P^{β_1}

- Standard deviation of the return on P^{β_1} is exactly β_1 times that of P.
- Covariance of P^{β_1} with the market portfolio also becomes β_1 times that of P with the market portfolio.
- The correlation coefficient between P^{β_1} and the market portfolio is simply equal to ρ_{Pm}.

Hence, the risk-adjusted rate of return for P^{β_1} is $r + \phi\rho_{Pm}\beta_1\sigma$. The risk-adjusted rate of return for P^{β_1} equals the expected rate of growth of P^{β_1} when β_1 is a root of the fundamental quadratic.

In subsequent discussion, we may rule out the "speculative bubbles" term since there is no excess expected rate of growth beyond the risk-adjusted rate of return for holding such asset. We keep only the fundamental component of value of the project, arising from the profit flow, namely, $V(P) = P/\delta$.

Value of the option to invest, $F(P)$

Equation: $\dfrac{\sigma^2}{2}P^2\dfrac{d^2F}{dP^2}+(r-\delta)P\dfrac{dF}{dP}-rF=0.$

Solution: $F(P)=A_1P^{\beta_1}$. To determine A_1 and P^*, we apply

(i) value-matching condition:

$F(P^*)=V_0(P^*)-I=P^*/\delta-I, I=$ cost of investment

(ii) Smooth-pasting condition: $F'(P^*)=V_0'(P^*)$.

We then obtain

$$A_1{P^*}^{\beta_1}=\frac{P^*}{\delta}-I \quad \text{and} \quad \beta_1A_1{P^*}^{\beta_1-1}=1/\delta.$$

Upon solving

$$P^*=\frac{\beta_1}{\beta_1-1}\delta I \quad \text{and} \quad A_1=(\beta_1-1)^{\beta_1-1}I^{-(\beta_1-1)}/(\delta\beta_1)^{\beta_1}.$$

Operating costs and temporary suspension

- Assume that operation of the project entails a flow cost C, but the operation can be temporarily and costlessly suspended when P falls below C, and costlessly resumed later if P rises above C.

- The project flow from the project

$$\pi(P) = \max(P - C, 0).$$

Governing equation: $\dfrac{\sigma^2}{2}P^2\dfrac{d^2V}{dP^2} + (r - \delta)P\dfrac{dV}{dP} - rV + \pi(P) = 0.$

We have taken implicitly that the operating threshold and suspending threshold are at $P = C$ under the scenario of zero cost. Is it justifiable?

(i) $P < C, \pi(P) = 0$

$$V_1(P) = K_1 P^{\beta_1} + K_2 P^{\beta_2}$$

(ii) $P > C, \pi(P) = P - C$

$$V_2(P) = B_1 P^{\beta_1} + B_2 P^{\beta_2} + P/\delta - C/r.$$

Interpretation When $P < C$, operation is suspended and the project yields no current profit flow. There is a positive probability that the price process will at some future time move into $P > C$. $V_1(P)$ is just the present value of such future flows.

1. $K_1 P^{\beta_1} + K_2 P^{\beta_2}$ represents the value of option to resume operations in the future; it tends to 0 as $P \to 0$, making $K_2 = 0$.

2. $B_1 P^{\beta_1} + B_2 P^{\beta_2}$ is the value of future suspension option; it tends to 0 as $P \to \infty$, making $B_1 = 0$.

We then have

$$V(P) = \begin{cases} K_1 P^{\beta_1} & \text{if } P < C \\ B_2 P^{\beta_2} + P/\delta - C/r & \text{if } P > C \end{cases}.$$

Value matching and smooth pasting conditions at $P = C$

$$K_1 C^{\beta_1} = B_2 C^{\beta_2} + P/\delta - C/r$$
$$\beta_1 K_1 C^{\beta_1 - 1} = \beta_2 B_2 C^{\beta_2 - 1} + 1/\delta.$$

Upon solving, we obtain

$$K_1 = \frac{C^{1-\beta_1}}{\beta_1 - \beta_2}\left(\frac{\beta_2}{r} - \frac{\beta_2 - 1}{\delta}\right) \quad \text{and} \quad B_2 = \frac{C^{1-\beta_2}}{\beta_1 - \beta_2}\left(\frac{\beta_1}{r} - \frac{\beta_1 - 1}{\delta}\right).$$

Finite costs on suspension and restarting

A. Dixit, "Entry and exist decisions under uncertainty", *Journal of Political Economy*, vol. 97 (1989) p.620–638.

Example: Underground mine operation

If the operation is suspended, a sunk cost and an ongoing fixed cost must be incurred to prevent the mine from flooding with water; and an additional sunk cost must be incurred to actually reopen it.

- Since restarting is costly, there will be an option value of keeping the operation alive. Abandonment will be optimal only at a sufficiently high threshold level of operation losses.

⋆ Restarting is costly, but not quite as costly as the new instrument. Also, the cost of restarting may increase with the duration of the suspension.

Model The firm must incur a lump-sum I to invest in the project and a lump-sum cost E to abandon it.

Remark E may be negative if a portion of the investment can be recouped upon exist; obviously, $|E| < I$.

Hysteresis — failure of an effect to reverse itself as its underlying cause is reversed.

Two threshold prices: $P_H > P_L$.

Idle film	optimal to remain idle if $P < P_H$ and invest as soon as P reaches P_H
Active film	optimal to remain active as long as $P > P_L$ and abandon if P falls to P_L.

When $P_L < P < P_H$, the optimal policy is to continue with the status quo.

Valuation of the two options

$$\begin{aligned} V_0(P) &= \text{value of the option to invest} \\ &\quad (= \text{ value of an idle firm}) \\ V_1(P) &= \text{value of an active film} \\ &= \text{entitlement to the profit from operation} \\ &\quad + \text{ option to abandon.} \end{aligned}$$

Idle firm: $\dfrac{\sigma^2}{2}P^2V_0''(P) + (r-\delta)PV_0'(P) - rV_0(P) = 0$

Solution: $V_0(P) = A_1P^{\beta_1}, \quad \beta_1 > 0, \quad 0 \leq P \leq P_H.$

Active film: $\dfrac{\sigma^2}{2}P^2V_1''(P) + (r-\delta)PV_1'(P) - rV_1(P) + P - c = 0$

Solution: $V_1(P) = B_2P^{\beta_2} + P/\delta - c/r, \quad \beta_2 < 0, P \geq P_L.$

Value matching and smooth pasting conditions

$$V_0(P_H) = V_1(P_H) - I, \quad V_0'(P_H) = V_1'(P_H);$$
$$V_1(P_L) = V_0(P_L) - E, \quad V_1'(P_L) = V_0'(P_L).$$

Equations for P_H, P_L, A_1 and B_2 are

$$\begin{cases} -A_1 P_H^{\beta_1} + B_2 P_H^{\beta_2} + P_H/\delta - c/r = I \\ -\beta_1 A_1 P_H^{\beta_1 - 1} + \beta_2 B_2 P_H^{\beta_2 - 1} + 1/\delta = 0 \\ -A_1 P_L^{\beta_1} + B_2 P_L^{\beta_2} + P_L/\delta - c/r = -E \\ -\beta_1 A_1 P_L^{\beta_1 - 1} + \beta_2 B_2 P_L^{\beta_2 - 1} + 1/\delta = 0 \end{cases}.$$

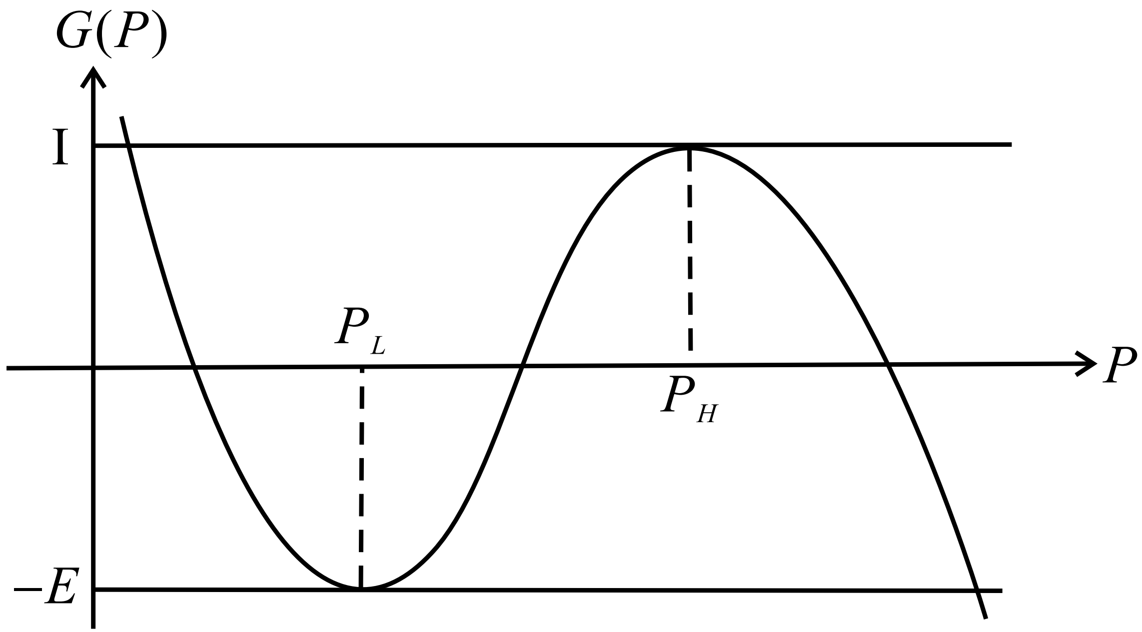
Define

$$\begin{aligned} G(P) &= V_1(P) - V_0(P) \\ &= -A_1 P^{\beta_1} + B_2 P^{\beta_2} + P/\delta - c/r, \quad 0 < P < \infty. \end{aligned}$$

Over $(P_L, P_H), G(P)$ is interpreted as the firm's incremental value of becoming active.

Value-matching conditions and smooth-pasting conditions become

$$G(P_H) = I, \quad G(P_L) = -E$$
$$G'(P_H) = 0, \quad G'(P_L) = 0$$



Governing equation: $\frac{\sigma^2}{2}G''(P) + (r-\delta)G'(P) - rG(P) + P - c = 0.$

Evaluating at $P = P_H$,

$$-rI + P_H - c = -\frac{\sigma^2}{2}G''(P_H) > 0$$

so that $P_H > c + rI$. We use the intuitive result that maximum value of $G(P)$ is achieved at $P = P_H$.

Similarly, evaluation at $P = P_L$ gives $P_L < c - rE$

We may deduce that when $I = E = 0$, we have

$$P_H = P_L = c.$$

Static expectation (Marshallian concept)

Compare the rate of return on the investment $(P - c)/I$ and that on disinvestment $(c - P)/E$ to the riskfree interest rate r.

The Marshallian thresholds are $c + rI$ and $c - rE$, which are less than and greater than the thresholds P_H and P_L, respectively.

When inactive (active) firms take into account the uncertainty over future prices, they are more reluctant to invest (more reluctant to abandon).

R & D investment with stochastic innovation

- Invest in a research project, facing no potential competitors.

- Technological and economic uncertainty.

  discovery occurs randomly ↑ value of new tech is stochastic ↑

- Firm's discount rate (exogenously specified) is the risk free interest rate r (firm is taken to be risk neutral).

- Invest by setting up a research unit of irrecoverable cost K; incurs a flow cost of c per unit time throughout the period of research.

- Abandonment requires sunk cost L; the same set-up cost K must be incurred again if the project is resumed later.

- Firm achieves the discovery according to a Poisson distribution with hazard rate $h > 0$.

(a) Probability of success in the next dt interval is $h\,dt$, conditional on no success up till now;

(b) Density function for the duration of research (or random time of discovery) is he^{-ht}.

- Discovery is single-step resulting in the creation of a marketable product (ignores stages from initial breakthrough to mass production of marketable good).

- When the firm exercises its option to invest in research (fixed investment cost plus continuous operating cost), it unfolds the next level of possibility, that of making the discovery itself (though the discovery time occurs randomly.)

- Parameters considered: sunk costs in investment and abandonment, degree of uncertainties in discovery and patent value, π. We assume that π evolves as

$$\frac{d\pi}{\pi} = \mu \, dt + \sigma \, dZ.$$

Valuation of options

$$\begin{aligned} V_0(\pi) &= \text{value of inactive firm} \\ &= \text{value of the call option to invest at a later date} \\ V_1(\pi) &= \text{value of active firm} \\ &= \text{sum of expected benefits of research,} \\ &\quad\ \text{negative flow costs and put option to abandon the project.} \end{aligned}$$

Optimal investment strategy leads to a pair of trigger points:

π_H for investment and π_L for abandonment, with $\pi_L < \pi_H$.

When $\pi_L < \pi < \pi_H$, the optimal policy is to continue with the status quo. This is called *hysteresis*.

Over a time interval dt, the return on the investment option $V_0(\pi)$, given by $rV_0(\pi)\,dt$, is equal to its expected rate of capital appreciation $E[dV_0(\pi)]$. In continuation region:

$$rV_0(\pi)\,dt = E[dV_0(\pi)]$$

Since $E[dV_0] = \mu\pi V_0'(\pi)\,dt + \frac{\sigma^2}{2}\pi^2 V_0''(\pi)\,dt$ so that the Bellman equation becomes

$$\frac{\sigma^2}{2}\pi^2 V_0''(\pi) + \mu\pi V_0'(\pi) - rV_0(\pi) = 0, \quad 0 < \pi < \pi_H.$$

Since the option to invest is almost worthless as $\pi \to 0$, so $V_0(\pi) \to 0$ as $\pi \to 0$. We obtain

$$V_0(\pi) = B\pi^{\beta_0}$$

where $B > 0$ and $\beta_0 = \frac{1}{2}\left\{1 - \frac{2\mu}{\sigma^2} + \sqrt{\left(1 - \frac{2\mu}{\sigma^2}\right)^2 + \frac{8r}{\sigma^2}}\right\} > 1,\ \mu < r.$

Bellman equation for $V_1(\pi)$

From $rV_1(\pi)\,dt = h\pi\,dt - c\,dt - hV_1(\pi)\,dt + E[dV_1(\pi)]$, we obtain

$$\frac{\sigma^2}{2}\pi^2 V_1''(\pi) + \mu\pi V_1'(\pi) - (r+h)V_1(\pi) + h\pi - c = 0, \quad \pi_L < \pi < \infty.$$

Interpretation of $-hV_1(\pi)$: With probability $h\,dt$, the discovery is made and the continuation value $V_1(\pi)$ is lost within $[t, t+dt]$.

When $\pi \to \infty$, the value of the option to shut down is small; $V_1(\pi)$ then tends to the simple NPV of the research project.

$$V_1(\pi) = A\pi^{-\alpha_1} + \frac{h\pi}{r+h-\mu} - \frac{c}{r+h}, \quad A > 0$$

and

$$\alpha_1 = \frac{1}{2}\left\{\frac{2\mu}{\sigma^2} - 1 + \sqrt{\left(1 - \frac{2\mu}{\sigma^2}\right)^2 + \frac{8(r+h)}{\sigma^2}}\right\} > 0.$$

Value matching and smooth-pasting conditions

1. At the upper trigger point π_H at which the firm commences research;

$$V_0(\pi_H) = V_1(\pi_H) - K$$
$$V_0'(\pi_H) = V_1'(\pi_H)$$

2. At the lower trigger point π_L at which research is abandoned;

$$V_1(\pi_L) = V_0(\pi_L) - L$$
$$V_1'(\pi_L) = V_0'(\pi_L).$$

These four conditions lead to four non-linear algebraic equation for A, B, π_H and π_L.

$$
\left\{
\begin{array}{l}
A\pi_H^{-\alpha_1} + \frac{h\pi_H}{r+h-\mu} - \frac{c}{r+h} = B\pi_H^{\beta_0} + K \\
-A\alpha_1\pi_H^{-\alpha_1-1} + \frac{h}{r+h-\mu} = B\beta_0\pi_H^{\beta_0-1} \\
A\pi_L^{-\alpha_1} + \frac{h\pi_L}{r+h-\mu} - \frac{c}{r+h} = B\pi_L^{\beta_0} - L \\
-A\alpha_1\pi_L^{-\alpha_1-1} + \frac{h}{r+h-\mu} = B\beta_0\pi_L^{\beta_0-1}
\end{array}
\right. .
$$

Static expectation (Marshallian investment theory)

Ignores the option value (with $\sigma = 0$ and $\mu = 0$). Compare the rate of return on the investment $\dfrac{h\pi_{MH} - c}{K}$ to the interest rate r. The Marshallian investment point is given by

$$
\pi_{MH} = \frac{c + rK}{h}.
$$

The Marshallian abandonment point π_{ML} is given by

$$\frac{c - h\pi_{ML}}{L} = r \quad \text{or} \quad \pi_{ML} = \frac{c - rL}{h}.$$

Incremental value of becoming active

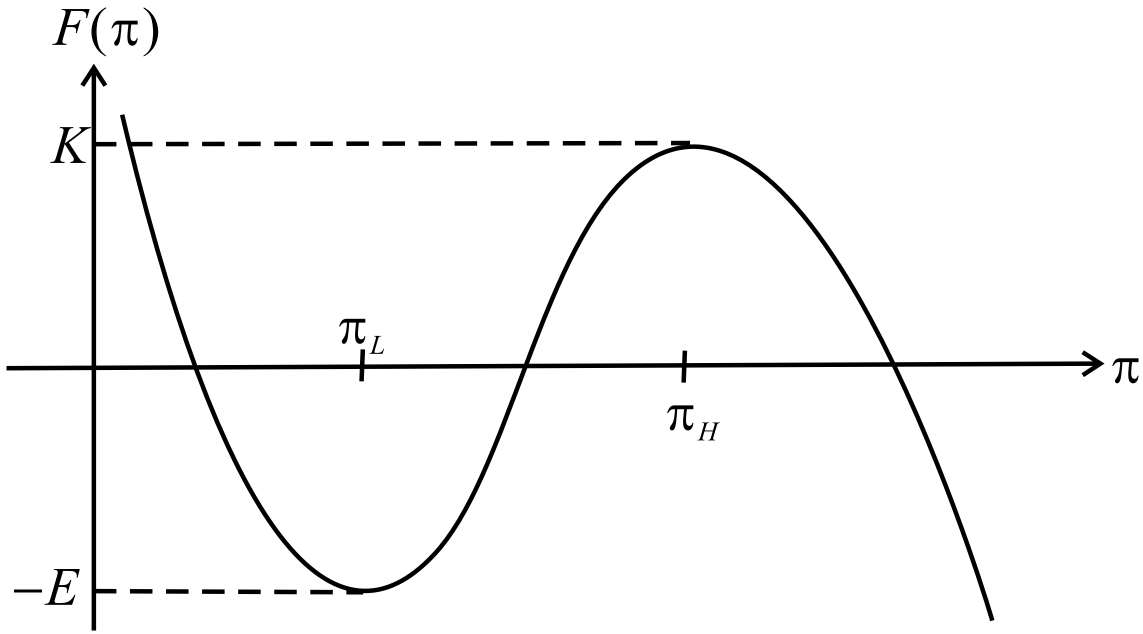
$$\begin{aligned} F(\pi) &= V_1(\pi) - V_0(\pi) \\ &= A\pi^{-\alpha_1} - B\pi^{\beta_0} + \frac{h\pi}{r + h - \mu} - \frac{c}{r + h}. \end{aligned}$$

Governing equation:

$$\frac{\sigma^2\pi^2}{2} F''(\pi) + \mu\pi F'(\pi) - rF(\pi) - hV_1(\pi) + h(\pi) - c = 0.$$

Value-matching and smooth-pasting conditions

$$\begin{aligned} &F(\pi_H) = K, \quad F(\pi_L) = -L, \\ &F'(\pi_H) = 0, \quad F'(\pi_L) = 0. \end{aligned}$$



Signs of $F''(\pi)$ at π_H and π_L are

$$F''(\pi_H) < 0 \quad \text{and} \quad F''(\pi_L) > 0.$$

Proposition 1 $\pi_H > \pi_{MH}$

Proof

Evaluate the governing equation for $F(\pi)$ at $\pi = \pi_H$.

$$h\pi_H - c = rF(\pi_H) - \mu\pi_H F'(\pi_H) - \frac{\sigma^2}{2}\pi_H^2 F''(\pi_H) + hV_1(\pi_H)$$

and note that $F(\pi_H) = K, F'(\pi_M) = 0$ and $F''(\pi_H) < 0$. We obtain $h\pi_H - c > rK$ or $\pi_H > \dfrac{c + rK}{h} = \pi_{MH}$.

The option effect due to sunk costs is augmented by the additional term $hV_1(\pi_H)$, thus further raises the level of π_H.

Proposition 2 π_L may be larger or smaller than π_{ML}

Proof

Now, we evaluate the equation for $F(\pi)$ at $\pi = \pi_L$. Note that $F(\pi_L) = -L, F'(\pi_L) = 0, F''(\pi_L) > 0$; but $V_1(\pi_L)$ can be positive or negative depending on the relative magnitudes of $A\pi^{-\alpha_1}$ and $\dfrac{h\pi}{r+h-\mu} - \dfrac{c}{r+h}$.

$$\boxed{h\pi_L - c = -rL - \frac{\sigma^2}{2}\pi_L^2 F''(\pi_L) + hV_1(\pi_L)}$$

(i) When $F''(\pi_L)$ is large, the sunk cost effect dominates. This leads to $\pi_L < \pi_{ML}$.

(ii) When $V_1(\pi_L)$ is large and positive, the discovery effect dominates. In this case, $\pi_L > \pi_{ML}$.

Remarks

1. Increases in σ^2 (economic uncertainty), sunk costs K and L tend to widen the hysteresis interval (π_L, π_H).

2. An increase in h increases both π_L and π_H. The firm chooses to invest at a higher critical value π_H if the rate of discovery is higher [higher $V_1(\pi; h)$ with higher h value]. On the other hand, the firm becomes less reluctant to abandon when h gets smaller.

3. With zero sunk costs, the hysteresis effects are eliminated. With $K = L = 0$, the four equations from value-matching and smooth-pasting conditions reduce to two equations and $\pi_L = \pi_H (= \pi_0$, say).

Solving for A and B, we obtain

$$A = \frac{-h\pi_0^{\alpha_1}}{\beta_0+\alpha_1}\left\{(\beta_0-1)\frac{\pi_0}{r+h-\mu} - \beta_0\frac{c}{h(r+h)}\right\}$$
$$B = \frac{h\pi_0^{-\beta_0}}{\beta_0+\alpha_1}\left\{(\alpha_1+1)\frac{\pi_0}{r+h-\mu} - \frac{\alpha_1}{h}\frac{1}{r+h}\right\}.$$

Now, $V_0(\pi) = B(\pi_0)\pi^{\beta_0}$; we find π_0 such that $V_0(\pi)$ is maximized. From

$$\frac{dB}{d\pi_0} = \frac{h\pi_0^{-\beta_0-1}}{\beta_0+\alpha_1}\left\{(1-\beta_0)(\alpha_1+1)\frac{\pi_0}{r+h-\mu} + \frac{\beta_0\alpha_1 c}{h(r+h)}\right\} = 0,$$

we obtain

$$\pi_0 = \frac{c(r+h-\mu)}{h(r+h)}\frac{\beta_0\alpha_1}{(\beta_0-1)(\alpha_1+1)}.$$

One can deduce that π_0 is undefined when $h = 0$. This is obvious since investor will not start the research phase when the probability of discovery is zero.

4. As $h \to \infty$, the value of the option to abandon research

$$A\pi^{-\alpha_1} \to 0.$$

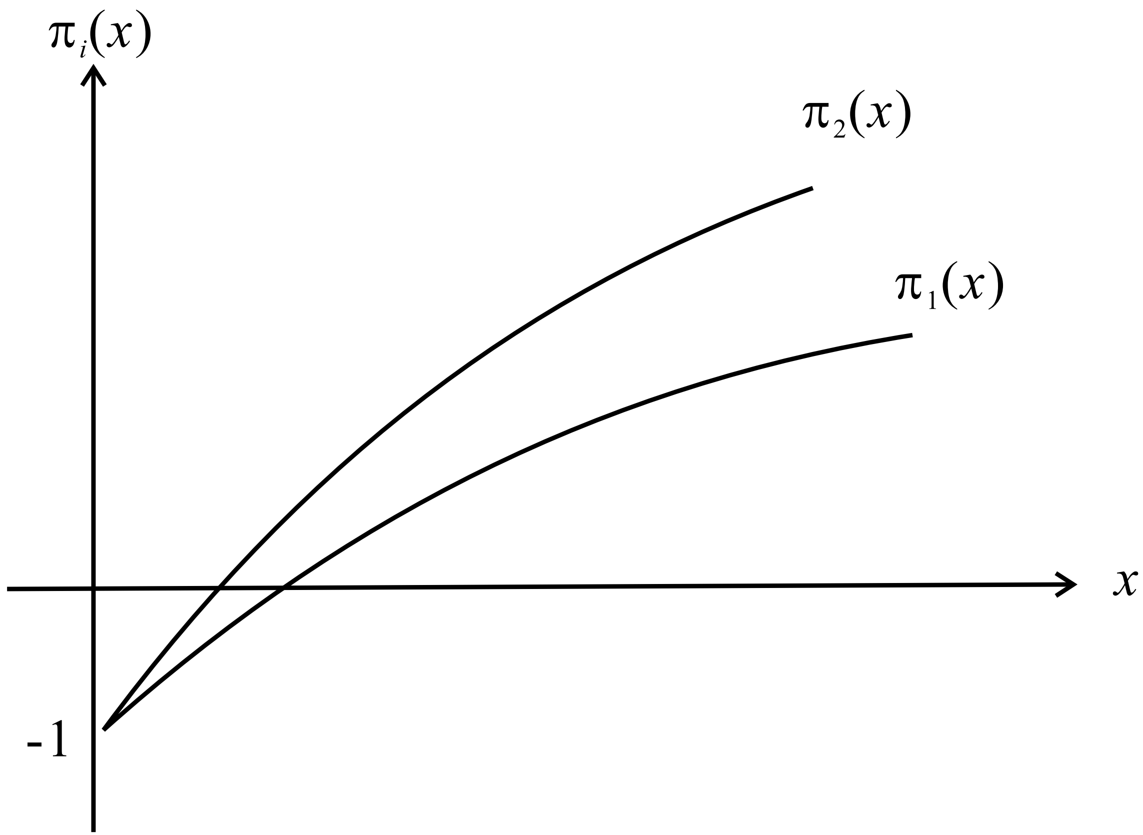
   This is because discovery becomes instantaneous, occurring as soon as the investment takes place, abandonment of the uncompleted research project is no longer a realistic possibility.

Compound real option value

"Adoption of certain multi-stage technology projects: a real options approach," by L.H.R. Alvarez and R. Stenbacka, *Journal of Mathematical Economics*, vol. 35 (2001) p.71–97.

To characterize the optimal timing of adopting an incumbent technology, incorporating as an embedded option a technologically uncertain prospect of opportunities for updating the technology to future superior versions.

- market uncertainty
- technological uncertainty —– Poisson process with respect to the arrival date of the improved version of the technology.



1. The profit flow in stage i $\pi_i(x)$ is strictly increasing and continuous in $x, i = 1, 2$.

2. $-\infty < \pi_i(0) \leq 0$ (boundedness from below)

The underlying diffusion process $\{X(t); t \geq 0\}$ is defined on a filtered probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$, evolves independent of the arrival time T on R_+ according to

$$dX(t) = \mu(X(t))\, dt + \sigma(X(t))\, dW(t), \quad X(0) = x.$$

Boundedness condition

$$E_x \int_0^\infty e^{-rs} |\pi_i(X(s))|\, ds < \infty, \quad i = 1, 2,$$

for all $x \in R_+$.

Upgrading the incumbent technology represents a real option, the value of which is

$$V_1(x,\lambda) = E_x\left[\int_0^T e^{-rs}\pi_1(X(s))\,ds + \int_T^\infty e^{-rs}\pi_2(X(s))\,ds - c_2 e^{-rT}\right],$$

where c_2 is the sunk cost incurred at the moment T of updating the technology.

As T and X are assumed to be independent, we may write

$$\begin{aligned} V_1(x,\lambda) &= E_x\int_0^\infty e^{-rs}\pi_2(X(s))\,ds \\ &\quad -E_x\int_0^\infty e^{-(r+\lambda)s}\Delta\pi(X(s))\,ds - \frac{\lambda c_2}{r+\lambda}. \end{aligned}$$

Note that

$$\lim_{\lambda\to 0+} V_1(x,\lambda) = V_1(x,0) = E_x\int_0^\infty e^{-rs}\pi_1(X(s))\,ds$$

$$\lim_{\lambda\to\infty} V_1(x,\lambda) = V_1(x,\infty) = E_x\int_0^\infty e^{-rs}[\pi_2(X(s)) - rc_2]\,ds.$$

Further properties on $V_1(x,\lambda)$

- $\dfrac{\partial V_1}{\partial \lambda}(x,\lambda) = E_x \int_0^{\infty} e^{-(r+\lambda)s} s \left[\Delta\pi(X(s)) - rc_2\right] ds$

  If $\pi_2(x) - rc_2 \geq \pi_1(x)$ for all $x \in R_+$, then $V_1(x,\lambda)$ is an increasing function of λ.

- $E_x \int_0^{\infty} e^{-rs}\pi_1(X(s))\, ds - \dfrac{\lambda c_2}{r+\lambda} \leq V_1(x,\lambda) \leq E_x \int_0^{\infty} e^{-rs}\pi_2(X(s))\, ds - \dfrac{\lambda c_2}{r+\lambda}$

Optimal adoption timings

Adopting of the existing technology is a compound real option with a value

$$V_0(x,\lambda) = \sup_{\tau} E_x[e^{-r\tau}\{V_1(x(\tau),\lambda) - c_1\}]$$

which includes the upgrading opportunity embedded in $V_1(x,\lambda)$.

“Investment under uncertainty: The case of replacement investment decisions” by D.C. Mauer & Steven H. Ott., JFQA, vol. 30 (Dec. 1995) p.581-605.

- Analyze the determinants of sequential replacement investment decisions with maintenance and operation cost uncertainty and realistic tax effects.

- Determine the best time to replace a deteriorating asset with a new one (say, computers, aircraft, etc) that will produce the same product or service.

In reality, the aggregate annual rate of replacement investment typically exceeds expansion (new) investment by a wide margin.

The model allows for

(i) stochastic maintenance and operation cost,

(ii) salvage value that fluctuates with that cost

(iii) tax effects include depreciation tax shields, investment tax credit on the purchase price of a new asset, taxation of the capital gain/loss on the sale of replaced asset

(iv) uncertainty about the arrival of a technological innovation.

Goal: Solve for the replacement cycle that minimizes the present expected cost of a chain of stochastically equivalent assets.

Replacement decision

A firm operates an asset that produces a fixed level of output for a given maintenance and operation cost, C. The before-tax cost C evolves according to

$$\frac{dC}{C} = \alpha\,dt + \sigma\,dW, \qquad \alpha > 0,$$

where C is a measure of the deterioration of the asset over time. With $\alpha > 0$, the asset is expected to deteriorate. All replacement assets are stochastically equivalent and they have the same initial cost $C_N > 0$.

If the asset is purchased at price P at $t = 0$, then the remaining book value is

$$P(1-\phi)e^{-\delta t}$$

where $P(1-\phi)$ is the net purchase price of the asset and ϕ is the investment tax credit and δ is the rate of depreciation.

Try to link the tax depreciation of the asset to the economic depreciation (as characterized by C_t) of the asset. Time scale: Use the expected first passage time $E[\widetilde{t}]$ from C_N to C_t as a proxy for t

$$E[\widetilde{t}] = \frac{1}{Z} \ln \frac{C_t}{C_N}, \quad Z = \alpha - \frac{\sigma^2}{2} > 0.$$

Here, C_t indicates the current cost.

The depreciation tax shield of the asset over $[t, t + dt]$ is

$$\tau\delta P(1 - \phi)e^{-\delta E(\widetilde{t})}\, dt = \tau\delta P(1 - \phi)\left(\frac{C_t}{C_N}\right)^{-\delta/Z} dt,$$

where τ is the corporate tax rate.

At some critical level of C_t, say $\overline{C}$, the firm will discontinue operation of the asset, sell it in the secondary market, and replace it with a stochastically equivalent asset with initial cost C_N.

Formulation of optimal replacement policy

The firm's problem is to minimize the after-tax costs of operating the asset by determining $\overline{C}$, defined as the optimal replacement policy.

$$
\begin{aligned}
& \text{Discounted expected value of after-tax costs} \\
= \; & V(C) \\
= \; & \min_{\overline{C}} E\left[\int_0^\infty e^{-\mu t}\left\{C_t(1-\tau) - \tau\delta P(1-\phi)\left(\frac{C_t}{C_N}\right)^{-\delta/Z}\right\} dt \middle| C_0 = C\right]
\end{aligned}
$$

where μ is the risk-adjusted discounted rate for cost. Here, C_0 is the state of the asset at time zero (in general not the same as the new asset C_N).

Governing equation

Assume that the risk of cost dW is spanned by traded assets. The governing equation for $V(C)$ is

$$\frac{\sigma^2}{2}C^2V_{CC} + \alpha^*CV_C + C(1-\tau) - \tau\delta P(1-\phi)\left(\frac{C}{C_N}\right)^{-\delta/Z} = rV$$

where $\alpha^* = \alpha - \eta\rho\sigma$ is the risk-adjusted rate of cost, η is the market price of risk and ρ is the instantaneous correlation between cost and the systematic pricing factor.

Without the spanning assumption, we need to use actual μ instead of r as the discount factor and α instead of $\alpha - \eta\rho\sigma$ as the expected rate of cost.

Note that Z must also reflect the risk-adjusted drift rate so that it is changed to $\alpha^* - \dfrac{\sigma^2}{2}$ to ensure that the valuation of the depreciation tax shield of the asset is consistent with the risk-adjusted stochastic process of cost.

General solution

$$V(C) = K_1 C^{\beta_+} + K_2 C^{\beta_-} + \frac{C(1-\tau)}{r-\alpha^*} + \frac{\theta C^{\xi}}{\Psi(\xi)},$$

where β_+ and β_- are the roots of $\dfrac{\sigma^2}{2}\beta(\beta-1) + \alpha^*\beta - r = 0$,

$$\theta = -\tau\delta P(1-\phi) C_N^{\frac{\delta}{Z}}, \quad \xi = -\delta/Z, \quad \Psi(\xi) = r - \alpha^*\xi - \frac{\sigma^2}{2}\xi(\xi-1).$$

Solve for K_1, K_2 and $\overline{C}$ by imposing 3 auxiliary conditions.

Boundary conditions

1. The expected discounted value of after-tax costs at the instant before replacement must be equal to the expected discounted value of after-tax costs the instant after replacement plus the cost of a new asset $P(1-\phi)$ minus the after-tax salvage value of the old asset.

$$V(\overline{C}) = V(C_N) + P(1-\phi) - \left[S(\overline{C}) - \underbrace{\tau \left\{ S(\overline{C}) - P(1-\phi)\left(\frac{\overline{C}}{C_N}\right)^{-\delta/Z} \right\}}_{\text{tax effect from the capital gain/loss on the sale}} \right].$$

Here, $S(C)$ is the sale price of the asset and $S'(C) < 0$.

2. C_t is "reflected" if the maintenance and operation costs fall to C_N. It is not possible to have an old machine with a lower maintenance cost than a stochastically equivalent new machine. This leads to the following reflecting boundary condition

$$\left.\frac{\partial V}{\partial C}\right|_{C=C_N} = 0.$$

3. Smooth-pasting condition

$$\left.\frac{\partial V}{\partial C}\right|_{C=\overline{C}} = \frac{\delta}{Z}\tau P(1-\phi)C_N^{\delta/Z}(\overline{C})^{-\frac{\delta}{Z}-1} - S'(\overline{C})(1-\tau).$$

Expected replacement cycle (mean time between replacements)

- This is given by the mean time it takes for C_t to reach $\overline{C}$, conditional on having started at C_N.

$$\overline{T} = \frac{\ln \overline{C} - \ln C_N}{\alpha - \frac{\sigma^2}{2}} - \underbrace{\frac{1}{2}\frac{\sigma^2}{\left(\alpha - \frac{\sigma^2}{2}\right)^2}\left[1 - \left(\frac{\overline{C}}{C_N}\right)^{1-\frac{2\alpha}{\sigma^2}}\right]}_{\substack{\text{adjustment due to} \\ \text{reflecting barrier}}}$$

Note that the actual drift rate α is used instead in the calculation of the actual time.

parmeters	σ	ρ	P	ϕ	τ	δ
$\bar{C}$	increasing	decreasing	increasing	decreasing	increasing	increasing
$\bar{T}$	increasing	decreasing	increasing	decreasing	increasing	increasing

Technological breakthrough

$$\begin{aligned} V^0(C; C_N^0) &= \text{value function before the potential breakthrough} \\ &\quad \text{(expected present value of after-tax costs)} \\ V^1(C; C_N^1) &= \text{value function based on new } C_N \text{ after the breakthrough.} \end{aligned}$$

Breakthrough only impacts on the initial cost, and this leads to $C_N^0 > C_N^1$, and the arrival of the breakthrough follows a Poisson process with a constant intensity $\lambda > 0$. The expected change in the value function is

$$\lambda[V^1(C; C_N^1) - V^0(C; C_N^0)]\, dt.$$

What happens after the occurrence of a breakthrough?

- The value function changes since it now rationally anticipates that at the next replacement, there will be a lower initial cost. Let $V^I(C; C_N^0)$ denote the intermediate value, showing dependence on C_N^0. This is because the firm does not immediately replace the existing asset.

- At the first time that the firm replaces the equipment, it may then take advantage of the lower initial maintenance and operation cost. The value function then becomes $V^1(C; C_N^1)$.

The forms of solution for $V^I(C)$ and $V^1(C)$ are identical to the earlier solution for $V(C)$. However, $V^I(C)$ and $V^1(C)$ are linked by

$$\begin{aligned} V^I(\overline{C}_I) &= V^1(C_N^1) + P(1-\phi) \\ &- \left\{ S(\overline{C}_I) - \tau \left[S(\overline{C}_I) - P(1-\phi) \left(\frac{\overline{C}_I}{C_N^0} \right)^{-\delta/Z} \right] \right\}. \end{aligned}$$

This is because upon the first replacement after the technological breakthrough, the replacement asset has initial cost C_N^1.

Formulation of the value function $V^0(C; C_N^0)$ prior to technological change

$$V^0(C; C_N^0) = \begin{cases} V^{01}(C; C_N^0) & \text{for} \quad C_N^0 \le C \le \overline{C}_I \\ V^{02}(C; C_N^0) & \text{for} \quad \overline{C}_I \le C < \infty \end{cases}.$$

Why we need to consider the two separate ranges? This is because when $C \ge \overline{C}_I$, upon the arrival of a technological change, the firm will replace the asset immediately.

- When $C_N^0 \le C \le \overline{C}_I$, the change in the value function conditional on the occurrence of a technological change is $V^I(C) - V^{01}(C)$.

- When $\overline{C}_I \le C < \infty$, the corresponding change in value function is $V^I(\overline{C}_I) - V^{02}(C)$.

Be careful, we need to distinguish whether

$$\overline{C}_0 < \overline{C}_I \quad \text{or} \qquad \text{otherwise.}$$

- If $\overline{C}_0 \geq \overline{C}_I$, then both V^{01} and V^{02} must be used to determine $\overline{C}_0$, since it is possible for a technological breakthrough to occur when $\overline{C}_I \leq C_0 \leq \overline{C}_0$.

- If $\overline{C}_0 < \overline{C}_I$, as any change in the initial cost will always occur in $C_N^0 \leq C < \overline{C}_I$, so it is only necessary to consider V^{01} when determining $\overline{C}_0$.

Problem one

$$\frac{\sigma^2}{2}C^2V^{01}_{CC}+\alpha^* CV^{01}_C+C(1-\tau)-\tau\delta P(1-\phi)\left(\frac{C}{C^0_N}\right)^{-\delta/Z}$$
$$+\ \lambda[V^I-V^{01}]=rV^{01},\quad C^0_N\le C<\overline{C}_I.$$

Problem two

$$\frac{\sigma^2}{2}C^2V^{02}_{CC}+\alpha^* CV^{02}_C+C(1-\tau)-\tau\delta P(1-\phi)\left(\frac{C}{C^0_N}\right)^{-\delta/Z}$$
$$+\ \lambda[V^I(\overline{C}_I)-V^{02}]=rV^{02},\quad \overline{C}_I<C<\infty.$$

Boundary conditions:

$$\left.\frac{\partial V^{01}}{\partial C}\right|_{C=C_N^0} = 0, \quad V^{01}(\overline{C}_I) = V^{02}(\overline{C}_I), \quad \left.\frac{\partial V^{01}}{\partial C}\right|_{C=\overline{C}_I} = \left.\frac{\partial V^{02}}{\partial C}\right|_{C=\overline{C}_I},$$

$$\begin{aligned} V^{02}(\overline{C}_0) &= V^{01}(C_N^0) + P(1-\phi) \\ &\quad - \left[S(\overline{C}_0) - \tau \left\{ S(\overline{C}_0) - P(1-\phi) \left(\frac{\overline{C}_0}{C_N^0} \right)^{-\delta/Z} \right\} \right] \end{aligned}$$

and

$$\left.\frac{\partial V^{02}}{\partial C}\right|_{C=\overline{C}_0} = \frac{\delta}{Z} \tau P(1-\phi)(C_N^0)^{\delta/Z} (\overline{C}_0)^{-\delta/Z-1} - S'(\overline{C}_0)(1-\tau).$$

Let $V^{01}_{(1)}(C^0_N)$ = present value of costs for $\overline{C}_0 < \overline{C}_I$

and $V^{01}_{(2)}(C^0_N)$ = present value of costs for $\overline{C}_0 \geq \overline{C}_I$.

Then $\overline{C}_0 = \arg\min \left[V^{01}_{(1)}(C^0_N), V^{01}_{(2)}(C^0_N)\right]$.

λ	0.0	0.1	0.2	0.3	0.4	0.5
$\overline{C}_0$	2.736	3.041	3.440	3.919	4.456	5.029
$\overline{T}_0$	6.704	7.432	8.283	9.182	10.067	10.902

$C^0_N = 1$
$C^1_N = 0.8$

Firm hangs on a deteriorating asset longer as λ increases, hoping that technological uncertainty is resolved. The magnitude of this effect is quite impressive.

Stochastic tax model

Let $V^0(C;T^0)$ represent the firm's value function for some initial tax policy $T^0 = \{\tau_0, \phi_0, \delta_0\}$ and $V^1(C;T^1)$ be for other potential tax policy $T^1 = \{\tau_1, \phi_1, \delta_1\}$.

Assume a tax policy change follows a Poisson process.

The steps to formulate the stochastic tax model are similar to those in the technological change model.