1. If outcome \( j \) occurs, then the gain is given by

\[
G_j = \sum_{i=1}^{m} r_{ij} x_i,
\]

where \( x_i = \frac{1}{1 + d_i} \) and \( r_{ij} = \begin{cases} 
  d_i & \text{if } j = i \\
  -1 & \text{if } j \neq i
\end{cases} \)

We then have

\[
G_j = r_{jj} x_j - \sum_{i=1, i \neq j}^{m} x_i
\]

\[
= (1 + r_{jj}) x_j - \sum_{i=1}^{m} x_i
\]

\[
= (1 + d_j) x_j - \sum_{i=1}^{m} x_i
\]

\[
= \frac{1}{1 - \sum_{i=1}^{m} \frac{1}{1 + d_i}} - \frac{\sum_{i=1}^{m} \frac{1}{1 + d_i}}{1 - \sum_{i=1}^{m} \frac{1}{1 + d_i}}
\]

\[
= 1, \quad \forall j = 1, 2, \cdots, m.
\]

Therefore, the betting game will always yield a gain of exactly 1.

2. Suppose we hold \((\alpha_1, \alpha_2, \alpha_3)\) units of the three securities, with \(\sum_{i=1}^{3} \alpha_i \leq 1, \alpha_i \geq 0\). In this problem, we can set \(\sum_{i=1}^{3} \alpha_i = 1\) since the random returns are greater than one under all states of world. Using log utility criterion, the growth factor is

\[
m = E[\ln R] = \frac{1}{2} \ln(4\alpha_1 + 2\alpha_2 + 3(1 - \alpha_1 - \alpha_2)) + \frac{1}{2} \ln(2\alpha_1 + 4\alpha_2 + 3(1 - \alpha_1 - \alpha_2))
\]

\[
= \frac{1}{2} \ln(3 + \alpha_1 - \alpha_2) + \frac{1}{2} \ln(3 - \alpha_1 + \alpha_2).
\]
Applying the first order condition
\[
\frac{\partial m}{\partial \alpha_1} = \frac{1}{2} \frac{1}{3 + \alpha_1 - \alpha_2} - \frac{1}{2} \frac{1}{3 - \alpha_1 + \alpha_2} = 0 \quad \Leftrightarrow \quad 3 - \alpha_1 + \alpha_2 = 3 + \alpha_1 - \alpha_2 \\
\Leftrightarrow \quad \alpha_1 = \alpha_2
\]
\[
\frac{\partial m}{\partial \alpha_2} = \frac{1}{2} \frac{(-1)}{3 + \alpha_1 - \alpha_2} + \frac{1}{2} \frac{1}{3 - \alpha_1 + \alpha_2} = 0 \quad \Leftrightarrow \quad 3 + \alpha_1 - \alpha_2 = 3 - \alpha_1 + \alpha_2 \\
\Leftrightarrow \quad \alpha_1 = \alpha_2
\]

Possible optimal strategies are \( \left( \frac{1}{2} \frac{1}{2} 0 \right) \) and \( \left( \frac{1}{3} \frac{1}{3} 1 \right) \).

In fact, for any portfolio choice with \( \alpha_2 = \alpha_1, \alpha_3 = 1 - 2\alpha_1, \alpha_1 \geq 0 \), the return is
\[
4\alpha_1 + 2\alpha_2 + 3\alpha_3 = 4\alpha_1 + 2\alpha_1 + 3(1 - 2\alpha_1) = 3 \\
or \quad 2\alpha_1 + 4\alpha_2 + 3\alpha_3 = 2\alpha_1 + 4\alpha_2 + 3(1 - 2\alpha_1) = 3.
\]

Therefore, the optimal strategies always yield a return of 3 for all \( \alpha_1 \).

3. (a) By log-utility criterion, we have
\[
m = E[\ln R] = \sum_{j=1}^{n} p_j \ln \left( \alpha_j r_j + 1 - \sum_{i=1}^{n} \alpha_i \right).
\]

The first order condition gives
\[
\frac{\partial m}{\partial \alpha_k} = \sum_{j=1}^{n} \frac{p_j(-1)}{\alpha_j r_j + 1 - \sum_{i=1}^{n} \alpha_i} + \frac{p_k}{\alpha_k r_k + 1 - \sum_{i=1}^{n} \alpha_i} = 0
\]
so that
\[
\frac{p_k r_k}{\alpha_k r_k + 1 - \sum_{i=1}^{n} \alpha_i} - \sum_{j=1}^{n} \frac{p_j}{\alpha_j r_j + 1 - \sum_{i=1}^{n} \alpha_i} = 0, \quad \forall k = 1, \cdots, n.
\]

(b) Assume \( \sum_{i=1}^{n} \frac{1}{r_i} = 1 \). Let \( \alpha_i = p_i, i = 1, \cdots, n \), so that \( \alpha_1 \geq 0 \), and \( \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} p_i = 1 \).

Hence, this is a possible strategy. Now substitute into part (a), we have
\[
\frac{p_k r_k}{p_k r_k + 1 - \sum_{i=1}^{n} p_i} - \sum_{j=1}^{n} \frac{p_j}{p_j r_j + 1 - \sum_{i=1}^{n} \alpha_i} = 1 - \sum_{j=1}^{n} \frac{1}{r_j} = 0.
\]

Therefore, this is indeed an optimal trading strategy.
(c) Suppose $p_n r_n < p_i r_i$ for all $i = 1, \cdots, n - 1$. Furthermore, suppose $\alpha_n = 0$ is a part of an optimal solution. Write $1 - \sum_{i=1}^{n} \alpha_i = L$. From the result in part (a), we put $k = n$ and $\alpha_n = 0$ so that

$$\frac{p_n r_n}{L} = \sum_{j=1}^{n} \frac{p_j}{\alpha_j r_j + L}.$$ 

Substituting back for other case $k \neq n$, we obtain

$$\frac{p_k r_k}{\alpha_k r_k + L} = \frac{p_n r_n}{L}.$$ 

Rearranging the terms, we have

$$\alpha_k = L \left( \frac{p_k r_k - p_n r_n}{p_n r_n r_k} \right).$$

It is desirable to express $L$ in terms of $\theta = \sum_{k=1}^{n} \frac{1}{r_k}$. We observe

$$L = 1 - \sum_{k=1}^{n} \alpha_k = 1 - L \sum_{k=1}^{n} \left( \frac{p_k}{p_n r_n} - \frac{1}{r_k} \right) = 1 - L \left( \frac{1}{p_n r_n} - \theta \right)$$

so that

$$L = \frac{1}{p_n r_n + 1 - \theta}.$$ 

The solution $\alpha_k$ can be expressed as

$$\alpha_k = \frac{p_n r_n}{1 + p_n r_n (1 - \theta)} \frac{p_k r_k - p_n r_n}{p_n r_n r_k} = \frac{p_k r_k - p_n r_n}{r_k + p_n r_n r_k (1 - \theta)}.$$ 

However, this solution is valid provided $0 \leq L \leq 1$. That is,

$$\frac{1}{p_n r_n} + 1 - \theta \geq 0 \iff \theta \leq 1 + \frac{1}{p_n r_n}$$

and

$$\frac{1}{p_n r_n} + 1 - \theta \geq 1 \iff \theta \leq \frac{1}{p_n r_n} < 1 + \frac{1}{p_n r_n}.$$ 

This also implies $\alpha_k \geq 0$ since $p_k r_k - p_n r_n \geq 0$. Hence, we deduce the following necessary condition: $\sum_{k=1}^{n} \frac{1}{r_k} \leq \frac{1}{p_n r_n}$ for the existence of the solution. If this condition fails, then the optimal solution must lie on the boundary. We may set $\alpha_{n-1} = 0$ with $p_{n-1} r_{n-1} < p_i r_i$ for $i = 1, \cdots, n - 2$, and repeat the above procedure.
4. Suppose at \( t_k \) the stock \( S_{\ell(k)} \) splits. Recall that the weights of the stock in the index are the same. Let \( w_k \) be the weight of the stocks in the index after the \( k^{\text{th}} \) split so that \( w_k = w_{k-1} + \varepsilon_k \). The index value is given by

\[
I_{t_k} = w_{k-1} \sum_{i=1}^{30} S_i(t_k) = I_{t_k} = (w_{k-1} + \varepsilon_k) \sum_{i=1}^{30} S_i(t_k),
\]

with

\[
S_i(t_k) = \begin{cases} 
S_i(t_k) & \text{if } i \neq \ell(k) \\
2S_i(t_k) & \text{if } i = \ell(k) 
\end{cases}.
\]

We obtain

\[
w_{k-1} \sum_{i=1}^{30} S_i(t_k) + w_{k-1}S_{\ell(k)}(t_k) = w_{k-1} \sum_{i=1}^{30} S_i(t_k) + \varepsilon_k \sum_{i=1}^{30} S_i(t_k)
\]

so that

\[
\varepsilon_k = w_{k-1} \frac{S_{\ell(k)}(t_k)}{\sum_{i=1}^{30} S_i(t_k)}.
\]

The extra proportion, \( \frac{\varepsilon_k}{w_{k-1}} \), is the proportion of the stock price from the sum, thus it is between 0 and 1.

Now in the index fund, when the stock splits, the fund holds \( w_0 + \varepsilon_0 \) units of the stock, down from \( 2w_0 \) as in Mr. D Jones’ drawer. For the other stock it goes up from \( w_0 \) to \( w_0 + \varepsilon_0 \). This follows the dictum of “buy low sell high” because the splitting stock is near $100 while the other are lower. In the long run, this strategy will yield a higher return than that of a comparable static portfolio.

5. Given \( \frac{dS_i}{S_i} = \mu_i dt + \sigma_i dZ_i \), \( \mu = 15\% \), \( \sigma_i = 40\% \), \( \sigma_{ij} = 0.08 \) for \( i \neq j \), and \( dZ_i dZ_j = \rho_{ij} dt \), \( \rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = 0.08/0.16 = 0.5 \), we have

expected growth rate \( = E \left[ \ln \frac{V(t)}{V(0)} \right] \)

\[
= \sum_{i=1}^{n} \frac{1}{n} \mu_i t - \frac{1}{2} \sum_{i,j=1}^{n} \frac{1}{n^2} \rho_{ij} \sigma_i \sigma_j t
\]

\[
= 0.15t - \frac{1}{2} \sum_{i,j=1}^{n} \frac{1}{n^2} 0.08t - \frac{1}{2} \sum_{i=1}^{n} \frac{1}{n^2} 0.16t
\]

\[
= 0.15t - \frac{1}{2} \frac{n^2 - n}{n^2} 0.08t - \frac{1}{2} \frac{n}{n^2} 0.16t
\]

\[
= 0.15t - 0.04 \left( \frac{1}{n} \right) t - \frac{0.08t}{n}
\]

\[
= 0.11t - \frac{0.04t}{n},
\]
where $V(t) = \sum_{i=1}^{n} \frac{1}{n} S_i(t)$. Also, we consider

\[
\text{variance} = \text{var} \left[ \ln \frac{V(t)}{V(0)} \right] = \int_{0}^{t} \left[ E \left( \sum_{i=1}^{n} \frac{1}{n} \sigma_i dZ_i \right) \right] dt
\]

\[
= \int_{0}^{t} \sum_{i,j=1}^{n} \frac{1}{n^2} \sigma_{ij} dt
\]

\[
= \frac{t}{n^2} \left[ \sum_{i,j=1}^{n} 0.08 + \sum_{i=1}^{n} (0.4)^2 \right] = \frac{t}{n^2} [0.08(n^2 - n) + 0.16n]
\]

\[
= 0.08t + \frac{0.08}{n}.
\]

6. The log-optimal strategy maximizes $E \ln r_1$, where $X_k = r_k \cdots r_1 X_0$. Recall that $\vec{w}_i \cdot \vec{R}_i = r_i$ and observe that

\[
X_k^B = \prod_{i=1}^{k} \vec{w}_i \cdot \vec{R}_i \quad \text{and} \quad X_k^A = \prod_{i=1}^{k} \vec{w}^* \cdot \vec{R}_i.
\]

From the lemma in the lecture note, assuming that the rates of return in successive periods are independent, we have

\[
E \left[ X_k^B \right] = E \left[ \prod_{i=1}^{k} r_i \cdots r_1 X_0 \right] = \prod_{i=1}^{k} E \left[ r_i \right] = \prod_{i=1}^{k} E \left[ \vec{w}_i \cdot \vec{R} \right] \leq 1.
\]

On the other hand, since $A$ invests using the log-optimal strategy, the log-optimal portfolio also satisfies

\[
E[\ln X_k^A] \geq E[\ln X_k^B] \quad \text{so that} \quad E \left[ \frac{X_k^A}{X_k^B} \right] \geq 0.
\]

By the Jensen inequality, we obtain

\[
\ln \left[ \frac{X_k^A}{X_k^B} \right] \geq E \left[ \ln \frac{X_k^A}{X_k^B} \right] \geq 0 \quad \text{so that} \quad E \left[ \frac{X_k^A}{X_k^B} \right] \geq 1.
\]

Both inequalities argue in favor of the log-optimal strategy.

7. Consider $U(x) = \frac{x^\gamma}{\gamma}$ for $\gamma \leq 1$, maximizing $U(x)$ is equivalent to maximizing $\frac{x^\gamma - 1}{\gamma}$. Taking $\gamma \to 0$, we have $\lim_{\gamma \to 0} \frac{x^\gamma \ln x}{\gamma} = \ln x$. This class includes the log-utility.
For the log-optimal strategy, it has been shown that maximizing $\ln X_k$ is equivalent of maximizing $\ln X_1$. For the power utility, assuming $R_i$'s are independent, and identically distributed, we have

$$EU(X_k) = \frac{1}{\gamma} E X_k^\gamma = \frac{1}{\gamma} E \left[ \left( \prod_{i=1}^k R_i^\gamma \right) X_0^\gamma \right] = \frac{1}{\gamma} \left( \prod_{i=1}^k E R_i^\gamma \right) X_0^\gamma = \frac{1}{\gamma} (E R_1^\gamma) X_0^\gamma.$$

Thus maximizing $EU(X_k)$ can be achieved by maximizing $E R_1^\gamma$. Furthermore,

$$\frac{1}{\gamma} E R_1^\gamma X_0^\gamma = E \frac{1}{\gamma} (R_1 X_0)^\gamma = E \frac{1}{\gamma} X_1^\gamma = EU(X_1),$$

hence the result.