1. This exercise is related to the *Dictionary Order*. Consider the choice set
\[ B = \{(x, y) : x \in [0, \infty) \text{ and } y \in [0, \infty)\}. \]
Consider the following preference relation:
\[(x_1, y_1) \in B \text{ and } (x_2, y_2) \in B \]
\[(x_1, y_1) \succeq (x_2, y_2) \text{ if and only if } \]
\[ [x_1 > x_2] \text{ or } [x_1 = x_2 \text{ and } y_1 \geq y_2]. \]
Show that \( \succeq \) satisfies the three axioms of Reflexivity, Comparability and Transitivity.

2. Recall the “Order Preserving” Axiom:
For any \( x, y \in B \), where \( x \succ y \) and \( \alpha, \beta \in [0, 1] \),
\[ [\alpha x + (1 - \alpha)y] \succ [\beta x + (1 - \beta)y] \text{ if } \alpha > \beta. \]
Show that the above Dictionary Order satisfies this Axiom.

3. It is known that the Dictionary Order does not satisfy the “Intermediate Value” Axiom. Show that the function
\[ U(x, y) = \ln(x + y) \]
cannot be an utility function representing the Dictionary Order.
*Hint*: A utility function \( U : B \to \mathbb{R} \) satisfies
(i) \( x \succ y \) if and only if \( U(x) > U(y) \).
(ii) \( x \sim y \) if and only if \( U(x) = U(y) \).

4. Suppose an investor has exponential utility function \( U(x) = -e^{-ax} \) and an initial wealth \( W \).
The investor is faced with an opportunity to invest an amount \( w \leq W \) and obtain a random payoff \( x \). Show that his evaluation of this incremental investment is independent of \( W \). Try to develop your argument from the first principle.

5. Consider the exponential utility function
\[ u(x; \gamma) = \begin{cases} 
\frac{e^{\gamma x}}{\gamma}, & x \in \mathbb{R} \text{ for } \gamma \in (-\infty, 0) \\
x, & x \in \mathbb{R} \text{ for } \gamma = 0
\end{cases} , \]
show that its inverse function is given by

(i) \( \gamma \in (-\infty, 0) \)

\[
\begin{align*}
  u^{-1}(y; \gamma) = \begin{cases} 
    \frac{1}{\gamma} \ln y, & y < 0 \\
    \infty & y \geq 0
  \end{cases}
\end{align*}
\]

(ii) \( \gamma = 0 \)

\[
  u^{-1}(y; \gamma) = y \quad y \in \mathbb{R}.
\]

Assume \( X > 0 \) is a bounded positive random variable. Suppose we choose \( \gamma \neq 0 \) but small, by taking the Taylor expansion of \( u^{-1}(E[u(X); \gamma]) \), show that

\[
  u^{-1}(E[u(X); \gamma]; \gamma) = E[X] + \frac{\gamma}{2} \text{var}[X] + \cdots.
\]

Give the financial interpretation of the above result (recall that \( \gamma < 0 \)).

6. Suppose an individual faces with an uncertain loss \( X \) in the next period. He is willing to pay an amount \( \rho \) as risk premium so that the future loss becomes a deterministic quantity \( E[X] \).

Let \( u(x) \) denote his utility function. The payment of the insurance premium is to guarantee

\[
  E[u(x)] \leq u(E[X] - \rho).
\]

Since \( u(E[X] - \rho) \) is a monotonically decreasing function with respect to \( \rho \), the maximum amount of premium paid by the individual is given by

\[
  E[u(X)] = u(E[X] - \rho).
\]

Show that

\[
  \rho(X) = - \frac{\text{var}(X) u''(E[X])}{2} + \frac{\rho^2}{u'(E[X])} \int_0^1 \alpha u''(E[X] - (1 - \alpha)\rho) \, d\alpha
\]

\[
  - E \left[ \frac{r(X, E[X])}{u'(E[X])} (X - E[X])^2 \right].
\]

**Hint:** The Taylor series expansion of \( u(x) \) up to the second order term is given by

\[
  u(x) = u(x_0) + u'(x_0)(x - x_0) + \frac{u''(x_0)}{2}(x - x_0)^2 + r(x, x_0)(x - x_0)^2
\]

where

\[
  r(x, x_0) = \int_0^1 \alpha [u''(\alpha x_0 + (1 - \alpha)x) - u''(x_0)] \, d\alpha.
\]