1. Suppose that it is impractical to use all the assets that are incorporated into a specified portfolio (such as a given efficient portfolio). One alternative is to find the portfolio, made up of a given set of $n$ stocks, that tracks the specified portfolio most closely — in the sense of minimizing the variance of the difference in returns.

Specifically, suppose that the target portfolio has (random) rate of return $r_M$. Suppose that there are $n$ assets with (random) rate of return $r_1, r_2, \ldots, r_n$. We wish to find the portfolio rate of return

$$r = \alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n$$

(with $\sum_{i=1}^{n} \alpha_i = 1$) minimizing $\text{var}(r - r_M)$.

(a) Find a set of equations for the $\alpha_i$’s.

(b) Although this portfolio tracks the desired portfolio most closely in terms of variance, it may sacrifice the mean. Hence a logical approach is to minimize the variance of the tracking error subject to achieving a given mean return. As the mean is varied, this results in a family of portfolios that are efficient in a new sense — say, tracking efficient. Find the equation for the $\alpha_i$’s that are tracking efficient.

2. Show that the tangency portfolio in the one-fund theorem is given by

$$w_t = \frac{\Omega^{-1}(\mu - r1)}{r - r_f}$$

where $b = 1^T \Omega^{-1} \mu$, $a = 1^T \Omega^{-1} 1$ and $r$ is the riskfree interest rate.

3. This problem illustrates an alternative derivation of the CAPM. Let $R = (R_1 \cdots R_N)^T$ denote the vector of the rates of return of $N$ risky assets and $R_t = w_t^T R$ denote the rate of return of the tangency portfolio. Explain why

$$\text{cov}(R, R_t) = \frac{E[R] - r1}{r - r_f}$$

and the variance of the tangency portfolio is given by

$$\sigma_t^2 = \frac{E[R_t] - r}{r - r_f}.$$ 

Combining the results, show that

$$E[R_t] - r = \beta_i (E[R_i] - r), \quad i = 1, 2, \ldots, N$$

where $\beta_i = \text{cov}(R_i, R_t) / \sigma_t^2$. 


4. Let \( w_0 \) be the portfolio (weights) of risky assets corresponding to the minimum-variance point in the feasible region. Let \( w_1 \) be any other portfolio on the efficient frontier. Define \( r_0 \) and \( r_1 \) to be the corresponding returns.

(a) There is a formula of the form \( \sigma_{01} = A \sigma_0^2 \). Find \( A \). [Hint: Consider the portfolios \( (1 - \alpha)w_0 + \alpha w_1 \), and consider small variations of the variance of such portfolios near \( \alpha = 0 \)]

(b) Corresponding to the portfolio \( w_1 \) there is a portfolio \( w_z \) on the minimum-variance set that has zero beta with respect to \( w_1 \); that is, \( \sigma_{1,z} = 0 \). This portfolio can be expressed as \( w_z = (1 - \alpha)w_0 + \alpha w_1 \). Find the proper value of \( \alpha \).

(c) Show the relation of the three portfolios on a diagram that includes the feasible region.

(d) If there is no risk-free asset, it can be shown that other assets can be priced according to the formula

\[
\bar{r}_i - \bar{r}_z = \beta_i(M)(\bar{r}_M - \bar{r}_z)
\]

where the subscript \( M \) denotes the market portfolio and \( \bar{r}_z \) is the expected rate of return on the portfolio that has zero beta with the market portfolio. Suppose that the expected returns on the market and the zero-beta portfolio are 15% and 9%, respectively. Suppose that a stock \( i \) has a correlation coefficient with the market of 0.5. Assume also that the standard deviation of the returns of the market and stock \( i \) are 15% and 5%, respectively. Find the expected return of stock \( i \).

5. Suppose an investor has utility function \( U \). There are \( n \) risky assets with rates of return \( r_i, i = 1, 2, \ldots, n \), and one risk-free asset with rate \( r_f \) return \( r_f \). The investor has initial wealth \( W_0 \). Suppose that the optimal portfolio for this investor has (random) payoff \( x^\tau \). Show that

\[
E[U'(x^\tau)(r_i - r_f)] = 0 \quad \text{for } i = 1, 2, \ldots, n.
\]

6. Suppose an investor uses the quadratic utility function \( U(x) = x - \frac{1}{2} cx^2 \). Suppose there are \( n \) risky assets and one risk-free asset with total return \( R \). Let \( R_M \) be the total return on the optimal portfolio of risky assets. Show that the expected return of any asset \( i \) is given by the formula

\[
\bar{R}_i - R = \beta_i(R_M - R)
\]

where \( \beta_i = \text{cov}(R_M, R_i)/\sigma_M^2 \). [Hint: Use Problem 5. Apply the result to \( R_M \) itself.]

7. In the asset-liability model, show the steps that lead to the formula for the efficient portfolios

\[
x^\tau = x^{min} + z^L + \tau z^*, \quad \tau \geq 0
\]

where

\[
x^{min} = \frac{1}{1^T \Omega^{-1}1} \Omega^{-1}1,
\]

\[
z^L = \Omega^{-1} \gamma - \frac{1^T \Omega^{-1} \gamma}{1^T \Omega^{-1}1} \Omega^{-1}1,
\]

\[
z^* = \Omega^{-1} \mu - \frac{1^T \Omega^{-1} \mu}{1^T \Omega^{-1}1} \Omega^{-1}1
\]

Here, \( \mu = (\mu_1 \cdots \mu_N)^T, \mu_i = E[R_i] \),

\( \gamma = (\gamma_1 \cdots \gamma_N)^T, \gamma_i = \frac{1}{f_0} \text{cov}(R_i, R_L) \).