1. Assume that the following two-index model describes returns

\[ R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + e_i \]

Assume that the following three portfolios are observed.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Expected Return</th>
<th>( b_{i1} )</th>
<th>( b_{i2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>12.0</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>B</td>
<td>13.4</td>
<td>3</td>
<td>0.2</td>
</tr>
<tr>
<td>C</td>
<td>12.0</td>
<td>3</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

Find the equation of the plane that must describe equilibrium returns.

2. Someone who believes that the collection of all stocks satisfies a single-factor model with the market portfolio serving as the factor gives you information on three stocks which make up a portfolio. (See Table.) In addition, you know that the market portfolio has an expected rate of return of 12% and a standard deviation of 18%. The risk-free rate is 5%.

(a) What is the portfolio’s expected rate of return?

(b) Assuming the factor model is accurate, what is the standard deviation of this rate of return?

<table>
<thead>
<tr>
<th>Simple Portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stock</strong></td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td>C</td>
</tr>
</tbody>
</table>

3. Two stocks are believed to satisfy the two-factor model

\[ r_1 = a_1 + 2f_1 + f_2 \]
\[ r_2 = a_2 + 3f_1 + 4f_2. \]

In addition, there is a risk-free asset with a rate of return of 10%. It is known that \( r_1 = 15\% \), \( r_2 = 20\% \). What are the values of \( \lambda_0, \lambda_1 \) and \( \lambda_2 \) for this model?

4. A general model for information about expected returns can be expressed in vector-matrix form as

\[ p = F + e. \]
In the model $\mathbf{P}$ is an $m \times n$ matrix, $\mathbf{v}$ is an $n$-dimensional vector, and $\mathbf{p}$ and $\mathbf{e}$ are $m$-dimensional vectors. The vector $\mathbf{p}$ is a set of observation values and $\mathbf{e}$ is a vector of errors having zero mean. The error vector has a covariance matrix $\mathbf{Q}$. The best (minimum-variance) estimate of $\mathbf{v}$ is

$$
\hat{\mathbf{v}} = (\mathbf{P}^T \mathbf{Q}^{-1} \mathbf{P})^{-1} \mathbf{P}^T \mathbf{Q}^{-1} \mathbf{p}.
$$

(a) Suppose there is a single asset and just one measurement of the form $\mathbf{p} = \mathbf{v} + \mathbf{e}$. Show that according to the above equation we have $\hat{\mathbf{v}} = \mathbf{p}$.

(b) Suppose there are two uncorrelated measurements with values $p_1$ and $p_2$, having variances $\sigma_1^2$ and $\sigma_2^2$. Show that

$$
\hat{\mathbf{v}} = \left( \frac{p_1}{\sigma_1^2} + \frac{p_2}{\sigma_2^2} \right) \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{-1}.
$$

5. Consider the $N$-factor index model

$$
\mathbf{r} = \mathbf{a} + B\mathbf{f} + \mathbf{e}.
$$

With stochastic residuals and under the assumption of absence of asymptotic arbitrage opportunities, it can be shown that there exist risk premiums $\lambda_0$ and $\mathbf{\lambda}$ such that

$$
\mathbf{r} = \mathbf{a} = \lambda_0 \mathbf{1} + B\mathbf{\lambda} + \mathbf{v}.
$$

Assume that the risk premiums are known and suppose we write

$$
\mathbf{a} \approx \lambda_0 \mathbf{1} + B\mathbf{\lambda},
$$

show that the arbitrage model can be simplified into one that involves only one positive factor risk premium.

Hint Suppose we perform the transformation

$$
\hat{\mathbf{f}} = \mathbf{T}^T \mathbf{f}
$$

where $\mathbf{T}$ is an orthogonal matrix. The arbitrage model can be expressed as

$$
\mathbf{a} \approx \lambda_0 \mathbf{1} + \hat{\mathbf{B}} \mathbf{T}^T \mathbf{\lambda}, \quad \text{where} \quad \hat{\mathbf{B}} = \mathbf{B} \mathbf{T}^T.
$$

It suffices to show that $\mathbf{T}^T \mathbf{\lambda}$ can be reduced to a vector with only one non-zero component by an appropriate choice of $\mathbf{T}$. One possible choice is

$$
\mathbf{T} = (\mathbf{\lambda}(\mathbf{\lambda}^T \mathbf{\lambda})^{-1/2}, \mathbf{A})
$$

where $\mathbf{A}$ is a $N \times (N - 1)$ matrix of mutually orthogonal unitary columns all being orthogonal to $\mathbf{\lambda}$. Also, one has to verify that

$$
E[\mathbf{e} \hat{\mathbf{f}}^T] = 0 \quad \text{and} \quad E[\hat{\mathbf{f}} \hat{\mathbf{f}}^T] = \mathbf{I}.
$$