1. (a) Let $\mathcal{F}$ be an algebra. Explain why $E[X|\mathcal{F}]$ can be interpreted as a random variable that is measurable with respect to the algebra $\mathcal{F}$. 

(b) Suppose $X$ is $\mathcal{F}$-measurable and $Y$ is any random variable. Show that

$$E[XY|\mathcal{F}] =XE[Y|\mathcal{F}]$$.

(c) Show the tower property of conditional expectation

$$E[E[X|\mathcal{F}]] = E[X]$$.

2. Consider a multiperiod securities model where all discounted security price processes are martingales under $Q$.

(a) Give the definition of a trading strategy being self-financing.

(b) Give the definition of an arbitrage opportunity associated with a trading strategy $H$.

(c) Given that a martingale pricing measure $Q$ exists, show that existence of $Q \Rightarrow$ non-existence of arbitrage opportunities.

3. This question tests your understanding of the concept of risk neutrality and martingale pricing approach.

(a) Under the actual probability measure $P$, the governing equation for the option price function $V(S,t)$ is given by

$$\frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S \frac{\partial^2 V}{\partial S^2} - \rho V = 0, \quad V(S,T) = h(S).$$

Here, $\rho$ is the expected rate of return of the underlying stock and $\rho_V$ is the expected rate of return of the option. Write down the discounted expectation formulation of $V(S,t)$ based on the Feynman-Kac representation formula (no derivation required). Give the financial interpretation of the representation formula.

(b) Give the financial interpretation of the following relation:

$$\frac{\rho_V - r}{\sigma_V} = \frac{\rho - r}{\sigma},$$

where $\sigma$ and $\sigma_V$ are the volatility of the underlying stock price and option price, respectively.
An investor is said to be risk neutral if he demands zero market price of risk on his risky investments. In the Black-Scholes option pricing equation, it appears that we price a derivative by assuming risk neutrality of the investor. It has been said that we simply use the mathematical convenience of risk neutrality. The absolute derivative price does depend on the risk preference of the investor (the actual market price of risk demanded by the investor). Give your comment on the above argument.

How does one link the martingale pricing approach with the risk neutrality concept in option pricing theory?

Let $F_t$ be the exchange rate process, $r_d$ and $r_f$ be the constant domestic and foreign interest rate, respectively. Suppose the dynamics of $F_t$ under the domestic risk neutral measure $Q_d$ is governed by

$$
\frac{dF_t}{F_t} = (r_d - r_f) dt + \sigma_F dZ_d.
$$

Find the dynamics of $1/F_t$ under (i) the domestic risk neutral measure $Q_d$, (ii) the foreign risk neutral measure $Q_f$.

The Radon-Nikodym derivative

$$
L_t = \left. \frac{dQ^*}{dQ} \right|_{\mathcal{F}_t} = \frac{e^{qt} S_t}{S_0} / e^{rt}, \quad t \in (0, T],
$$

effects the change of measure from the risk neutral measure $Q$ with the money market as the numeraire to $Q^*$ with the stock price as the numeraire. Here, $q$ is the continuous dividend yield. Let the dynamics under $Q$ be governed by

$$
\frac{dS_t}{S_t} = (r - q) dt + \sigma dZ_t^Q,
$$

where $Z_t^Q$ is $Q$-Brownian. Find the dynamics of $S_t$ under $Q^*$.

In the Black-Scholes formulation of riskless hedging, we consider the formation of a riskless hedged portfolio whose value $\Pi_t$ is given by

$$
\Pi_t = -V_t + \Delta_t S_t,
$$

where $\Delta_t$ is the hedge ratio, $V_t$ is the time-$t$ value of the derivative, and $S_t$ is the asset price at time $t$.

(a) Black and Scholes compute the differential of $\Pi_t$ as

$$
d\Pi_t = -dV_t + \Delta_t dS_t
$$

without the inclusion of $S_t d\Delta_t$. Explain why they manage to arrive at the correct option pricing equation even the product rule of calculus:

$$
d(\Delta_t S_t) = \Delta_t dS_t + S_t d\Delta_t
$$

is not observed.
(b) Suppose the actual dynamics of $S_t$ is governed by

$$\frac{dS_t}{S_t} = \rho \, dt + \sigma \, dZ_t,$$

where $\rho$ is the expected rate of return and $\sigma$ is the volatility, explain the concept of riskless hedging principle by finding the appropriate hedge ratio $\Delta_t$ so that the portfolio is instantaneously riskless at all times.

(c) How do you modify the above riskless hedging approach if the price of the derivative is dependent on some stochastic index that is not tradeable? Demonstrate how to form a riskless hedged portfolio under such scenario.

(d) Let $I_t$ denote the value process of the stochastic index whose dynamics is governed by

$$\frac{dI_t}{I_t} = \rho_I \, dt + \sigma_I \, dZ_t.$$

Let $\lambda_I$ denote the market price of risk of the stochastic index $I$. Derive the governing equation for the value of the derivative $V(I, t)$ in terms of $\rho_I, \lambda_I$, and $\sigma_I$.

7. A forward start call option is a call option that comes into existence at some future time $T_1$ and expires at $T_2$, where $T_2 > T_1$. The strike price is set equal to the asset price at $T_1$ such that the call option is at-the-money at the future option’s initiation time $T_1$.

<table>
<thead>
<tr>
<th>current time</th>
<th>forward starting date</th>
<th>option’s expiration date</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$T_1$</td>
<td>$T_2$</td>
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</table>

Let $S$ denote the asset price at the current time $t$ and the dynamics of the asset price process $S_u, u \geq t, under the martingale measure $Q$ is governed by

$$\frac{dS_u}{S_u} = (r - q) \, du + \sigma \, dZ_u, \quad u \geq t,$$

where $q$ is the dividend yield, $r$ is the riskless interest rate, and $\sigma$ is the volatility. We define $c(S, \tau; X)$ to be the price function of a call option with current asset price $S$, time to expiry $\tau$, and strike price $X$. Find the value of the forward start call option in terms of $r, q, T_1, T_2, t$, and the call price function $c(S, \tau; X)$.

**Hint:** Use the following relations:

(i) For $t < T_1 < T_2$,

$$e^{-r(T_2-t)}E_Q[(S_{T_2} - S_{T_1})1_{\{S_{T_2} \geq S_{T_1}\}}]F_t] = e^{-r(T_1-t)}E_Q[e^{-r(T_2-T_1)}E_Q[(S_{T_2} - S_{T_1})1_{\{S_{T_2} \geq S_{T_1}\}}]F_{T_1}]F_t]:
(ii) For $t < T_1$,

$$E_Q[S_{T_1} | \mathcal{F}_t] = e^{(r-q)(T_1-t)} S.$$  

8. Let $F_{S\backslash U}$ denote the Singaporean currency price of one unit of US currency and $F_{H\backslash S}$ denote the Hong Kong currency price of one unit of Singaporean currency. Suppose we assume $F_{S\backslash U}$ to be governed by the following dynamics under the risk neutral measure $Q_S$ in the Singaporean currency world:

$$dF_{S\backslash U} = (r_{SGD} - r_{USD}) dt + \sigma_{F_{S\backslash U}} dZ^S_{F_{S\backslash U}},$$

where $r_{SGD}$ and $r_{USD}$ are the Singaporean and US riskless interest rates, respectively. Similar Geometric Brownian process assumption is made for other exchange rate processes. The digital quanto option pays one US dollar at maturity if $F_{S\backslash U}$ is above $\alpha F_{H\backslash U}$ for some constant value $\alpha$. Find the value of the digital quanto option in Hong Kong dollar in terms of the exchange rates, the riskless interest rates of the different currency worlds and volatility values. [6]

9. (a) Consider the Merton risky debt model where the value of the risky debt $V(A, \tau)$ can be viewed as a contingent claim on the firm asset value $A$. The equity value or shareholders’ stake is the firm value less the debt liability. Show that the equity value is equal to the call value with terminal payoff: $\max(A - F, 0)$, where $F$ is the par value of the risky debt. [3]

(b) In their pioneering papers on option pricing theory, both Black-Scholes and Merton made the error of taking the firm value volatility $\sigma_A$ and equity volatility $\sigma_E$ to be the same. Find the relation between $\sigma_E$ and $\sigma_A$ using Ito’s lemma, where under a martingale measure $Q$, the dynamics of $A_t$ and $E_t$ are given by

$$\frac{dA_t}{A_t} = r dt + \sigma_A dZ^Q_t \quad \text{and} \quad \frac{dE_t}{E_t} = r dt + \sigma_E dZ^Q_t.$$ [3]

— End —