MAFS 5030 — Quantitative Modeling of Derivative Securities

Topic 1 — Introduction to Derivative Instruments

1.1 Basic instruments: bonds, forward contracts and futures
1.2 Exotic swap products
1.3 Options: rational boundaries of option values
1.4 American options: Optimal early exercise policies
1.1 Basic derivative instruments: bonds, forward contracts, swaps and options

A bond is a *debt instrument* requiring the issuer to repay to the lender/investor the amount borrowed (par or face value) plus interests over a specified period of time.

Specify (i) the maturity date when the principal is repaid; (ii) the coupon payments over the life of the bond.
• The *coupon rate* offered by the bond issuer represents the *cost of raising capital*. It depends on the prevailing risk free interest rate and the creditworthiness of the bond issuer. It is also affected by the values of the embedded options in the bond, like conversion right in convertible bonds.

• Assume that the bond issuer does not *default* or *redeem* the bond prior to maturity date, an investor holding this bond until maturity is assured of a *known* cash flow pattern. This is why bond products are also called *Fixed Income Products/Derivatives*.

*Pricing of a bond*

Based on the current information of the interest rates (yield curve) and the embedded option provisions, find the upfront cash amount that the bond investor should pay at the current time so that the deal is fair to both counterparties. Also, potential default losses (probability of default, exposure and recovery value) should be taken into account in deriving the fair bond price.
Features in bond indenture

1. **Floating rate bond**

   The coupon rates are reset periodically according to some pre-determined financial benchmark, like LIBOR + spread, where LIBOR is the LONDON INTER-BANK OFFERED RATE.


3. Callable feature (callable bonds)

   The issuer has the *right to buy back* the bond at a specified price. Usually this call price falls with time, and often there is an initial call protection period wherein the bond cannot be called.
4. Put provision – grants the bondholder the right to sell back to the issuer at par value on designated dates.

5. Convertible bond – gives the bondholder the right to exchange the bond for a specified number of shares of the issuer’s firm.
   - Bondholders can take advantage of the future growth of the issuer’s company.
   - Issuer can raise capital at a lower cost.

6. Exchangeable bond – allows the bondholder to exchange the bond par for a specified number of common stocks of another corporation.
Short rate

Let $r(t)$ denote the short rate, which is in general stochastic. This is the interest rate that is applied over the next infinitesimal $\Delta t$ time interval $(t, t + \Delta t]$. The short rate is a mathematical construction, not a market reality.

Money market account: $M(t)$

You put $1$ at time $t$ and let it earn interest at the rate $r(t)$ continuously over the period $(t, T)$. The governing differential equation of $M(t)$ is given by

$$dM(t) = r(t)M(t)\,dt.$$
\[ \int_t^T \frac{dM(u)}{M(u)} = \int_t^T r(u) \, du \]

so that

\[ M(T) = M(t) e^{\int_t^T r(u) \, du}. \]

Here, \( e^{\int_t^T r(u) \, du} \) is seen to be the growth factor of the money market account over \([t, T]\). If \( r \) is constant, then

\[ \text{growth factor} = e^{r\tau}, \quad \tau = T - t. \]

If \( r(t) \) is stochastic, then \( M(T) \) is also stochastic.

The reciprocal of the growth factor \( e^{-\int_t^T r(u) \, du} \) is called the discount factor.
Discount bond price

\[ B(t, T) \]

\[ t \]

\[ T \]

\[ \tau = T - t = \text{time to bond’s maturity} \]

The price that an investor on the zero-coupon (discount) bond with unit par is willing to pay at time \( t \) if the bond promises to pay him back $1 at a later time (maturity date) \( T \).

This fair value is called the discount bond price \( B(t, T) \), which is given by the expectation of the discount factor based on current information: \( E_t \left[ e^{-\int_t^T r(u) \, du} \right] \). If \( r \) is constant, then \( B(t, T) = e^{-r\tau}, \tau = T - t \).
Forward contract 遠期合約

The buyer of the forward contract agrees to pay the delivery price $K$ dollars at future time $T$ to purchase a commodity whose value at time $T$ is $S_T$. The pricing question is how to set $K$?

How about

$$E[\exp(-rT)(S_T - K)] = 0$$

so that $K = E[S_T]$?

This is the expectation pricing approach, which cannot enforce a price. When the expectation calculation $E[S_T]$ is performed, the distribution of the asset price process comes into play.
Objective of the buyer:

To hedge against the price fluctuation of the underlying commodity.

- Intension of a purchase to be decided earlier, actual transaction to be done later.
- The forward contract needs to specify the delivery price, amount, quality, delivery date, means of delivery, etc.

Potential default of either party (counterparty risk): writer or buyer.
Terminal payoff from a forward contract

\[ S_T - K \]

\[ K - S_T \]

\( K \) = delivery price, \( S_T \) = asset price at maturity

Zero-sum game between the writer (short position) and buyer (long position).
Is the forward price related to the expected price of the commodity on the delivery date? Provided that the underlying asset can be held for hedging by the writer, then

\[
\text{forward price} = \text{spot price} + \text{cost of fund} + \text{storage cost} = \text{spot price} + \text{cost of carry}
\]

- Upfront cost (through borrowing) is required to acquire the underlying commodity at the spot price. Cost of fund is the interest costs accrued over the period of the forward contract.
- Cost of carry is the total cost incurred to acquire and hold the underlying asset, say, including the cost of fund and storage cost.
- Dividends paid to the holder of the asset are treated as negative contribution to the cost of carry.
Numerical example on arbitrage

- spot price of oil is US$19
- quoted 1-year forward price of oil is US$25
- 1-year US dollar interest rate is 5% pa
- storage cost of oil is 2% per annum, paid at maturity

Any arbitrage opportunity? Yes

Sell the forward and expect to receive US$25 one year later. Borrow $19 now to acquire oil, pay back $19(1+0.05) = $19.95 a year later. Also, one needs to spend $0.38 = $19 × 2% as the storage cost.

\[
\text{total cost of replication (dollar value at maturity)} = \text{spot price} + \text{cost of fund} + \text{storage cost} = $20.33 < $25 \text{ to be received.}
\]

Close out all positions by delivering the oil to honor the forward. At maturity of the forward contract, guaranteed riskless profit = $4.67.
Value and price of a forward contract

Let \( f(S, \tau) \) = value of forward, \( F(S, \tau) \) = forward price, 

\[
\begin{align*}
\tau & = \text{time to expiration}, \\
S & = \text{spot price of the underlying asset}.
\end{align*}
\]

Further, we let 

\( B(\tau) = \) value of an unit par discount bond with time to maturity \( \tau \)

- If the interest rate \( r \) is constant and interests are compounded continuously, then \( B(\tau) = e^{-r\tau} \).
- Assuming no dividend to be paid by the underlying asset and no storage cost.

We construct a “static” replication of the forward contract by a portfolio of the underlying asset and bond.
Portfolio A: long one forward and a discount bond with par value $K$

Portfolio B: one unit of the underlying asset

Both portfolios become one unit of asset at maturity. Let $\Pi_A(t)$ denote the value of Portfolio $A$ at time $t$. Note that $\Pi_A(T) = \Pi_B(T)$. By the "law of one price", we must have $\Pi_A(t) = \Pi_B(t)$. The forward value is given by

$$f = S - KB(\tau).$$

The forward price is defined to be the delivery price which makes $f = 0$, so $K = S/B(\tau)$. Hence, the forward price is given by

$$F(S, \tau) = S/B(\tau) = \text{spot price} + \text{cost of fund}.$$

*Suppose $\Pi_A(t) > \Pi_B(t)$, then an arbitrage can be taken by selling Portfolio $A$ and buying Portfolio $B$. An upfront positive cash flow is resulted at time $t$ but the portfolio values are offset at maturity $T$. Failure of law of one price implies the existence of an arbitrage opportunity.*
**Discrete dividend paying asset**

\[ D = \text{present value of all dividends received from holding the asset during the life of the forward contract.} \]

We modify Portfolio \( B \) to contain one unit of the asset plus borrowing of \( D \) dollars. The loan of \( D \) dollars will be repaid by the dividends received by holding the asset. We then have

\[ f + KB(\tau) = S - D \]

so that

\[ f = S - [D + KB(\tau)]. \]

Setting \( f = 0 \) to solve for \( K \), we obtain \( F = (S - D)/B(\tau) \).

The “net” asset value is reduced by the amount \( D \) due to the anticipation of the dividends. Unlike holding the asset, the holder of the forward will not receive the dividends. As a fair deal, he should pay a lower delivery price at forward’s maturity.
Cost of carry

Additional costs to hold the commodities, like storage, insurance, deterioration, etc. These can be considered as negative dividends. Treating $U$ as $-D$, we obtain

$$F = (S + U)e^{r\tau},$$

$U =$ present value of total cost incurred during the remaining life of the forward to hold the asset.

Suppose the costs are proportional and paid continuously, we have

$$F = Se^{(r+u)\tau},$$

where $u =$ cost per annum as a proportion of the spot price.

In general, $F = Se^{b\tau}$, where $b$ is the cost of carry per annum. Let $q$ denote the continuous dividend yield per annum paid by the asset. With both continuous holding cost and dividend yield, the cost of carry $b = r + u - q$. 
Forward price formula with discrete carrying costs

Suppose an asset has a holding cost of $c(k)$ per unit in period $k$, and the asset can be sold short. Suppose the initial spot price is $S$. The theoretical forward price $F$ is

$$F = \frac{S}{d(0, M)} + \sum_{k=1}^{M} \frac{c(k)}{d(k, M)},$$

where $d(0, k)d(k, M) = d(0, M)$. The market expectation of the discount factor $d(k, M)$ at time zero can be realized by the market observed bond prices. That is,

$$E_0[d(k, M)] = B(0, M)/B(0, k),$$

where $B(0, k)$ denotes the market observable time-$0$ price of a discount bond that matures at time $k$.

The terms on the right hand side represent the future value at maturity of the total costs required for holding the underlying asset for hedging, where holding costs are visualized as negative dividends.
Proportional carrying charge

Forward contract is written at time 0 and there are $M$ periods until delivery. The carrying charge in period $k$ is $qS(k−1)$ to be paid at time $k$, where $q$ is a proportional constant. Show that the forward price is

$$F = \frac{S(0)/(1 − q)^M}{d(0, M)}.$$

- The above forward price formula becomes $F = Se^{(r+u)\tau}$ when we change from discrete compounding to continuous compounding, where $\frac{1}{d(0, M)}$ and $\frac{1}{(1−q)^M}$ are related to $e^{r\tau}$ and $e^{u\tau}$, respectively.

- We expect the forward price increases when the carrying charges become higher since the cost of setting up the static replicating portfolio is higher.
Borrow \( \alpha S(0) \) dollars at current time to buy \( \alpha \) units of assets and long one forward. Sell out \( q \) portion of asset at each period in order to pay for the carrying charge. After \( M \) period, \( \alpha \) units becomes \( \alpha(1 - q)^M \).

The goal is to make available one unit of the asset for delivery at maturity. We set \( \alpha(1 - q)^M \) to be one and obtain \( \alpha = \frac{1}{(1 - q)^M} \).

The portfolio of longing the forward with delivery price \( K \) and a bond with par \( K \) is equivalent to long \( \alpha \) units of the asset. This gives

\[
f + Kd(0, M) = \alpha S(0).\]

By setting \( f = 0 \) to obtain the forward price \( K \), we obtain

\[
K = \frac{S(0)}{(1 - q)^M} \bigg/ d(0, M).\]
**Futures contracts** 期貨合約

A futures contract is a legal agreement between a buyer (seller) and an established exchange or its clearing house in which the buyer (seller) agrees to take (make) delivery of a financial entity at a specified price at the end of a designated period of time. Usually the exchange specifies certain standardized features.

*Mark to market the account*

Pay or receive from the writer the change in the futures price through the margin account so that payment required on the maturity date is simply the spot price on that date.

*Credit risk is limited to one-day performance period*
Roles of the clearing house and margin account

- Minimize the **counterparty risk** through the margin account.

- Provide the **platform** for parties of a futures contract to unwind their position prior to the settlement date.

Margin requirements

Initial margin – paid at inception as a deposit for the contract.

Maintenance margin – minimum level before the investor is required to deposit additional margin.
Example (Margin)

Suppose that Mr. Chan takes a long position of one contract in corn (5,000 kilograms) for March delivery at a price of $2.10 (per kilogram). And suppose the broker requires margin of $800 with a maintenance margin of $600.

- The next day the price of this contract drops to $2.07. This represents a loss of $0.03 \times 5,000 = $150. The broker will take this amount from the margin account, leaving a balance of $650. The following day the price drops again to $2.05. This represents an additional loss of $100, which is again deducted from the margin account. As this point the margin account is $550, which is below the maintenance level.

- The broker calls Mr. Chan and tells him that he must deposit at least $250 in his margin account, or his position will be closed out.
Dynamic strategy that replicates the daily margin settlement in a futures contract

Consider an asset with price $\tilde{S}_T$ at time $T$. An investor who pays an amount $G_{t,T}$ that equals the futures price of the asset and together performs the dynamic strategy of long holding futures on successive dates is equivalent to pay the time-$t$ spot price of a security which has a payoff

$$\frac{\tilde{S}_T}{B_{t,t+1}B_{t+1,t+2}\cdots B_{T-1,T}}$$

at time $T$. Note that quantities with “tilde” at top indicate stochastic variables. Note that the payoff is seen to be some units of the underlying asset, where the number of units is $\frac{1}{B_{t,t+1}B_{t+1,t+2}\cdots B_{T-1,T}}$ (this is the same as the value of the money market account starting at $\$1$ and accumulating over the period $[t,T]$).
The dynamic strategy is presented as follows.

- We start with $\frac{1}{B_{t,t+1}}$ long futures contracts at time $t$. On day $\tau$, the investor long holds $\frac{1}{B_{t,t+1}...B_{\tau,\tau+1}}$ futures contracts. The gain/loss from the futures position on day $\tau$ earns/pays the overnight rate $\frac{1}{B_{\tau,\tau+1}}$.

- Also, invest $G_{t,T}$ in a one-day risk free bond and roll the cash position over on each day at the one-day rate. This is like “rolling over” in a money market account on a daily basis.

- The investment of $G_{t,T}$ is equivalent to the price paid to acquire the asset.
As an illustrative example, take $t = 0$ and $T = 3$.

1. Take $1/B_{0,1}$ long futures at $t = 0$;
   $1/B_{0,1}B_{1,2}$ long futures at $\tau = 1$;
   $1/B_{0,1}B_{1,2}B_{2,3}$ long futures at $\tau = 2$.

   Note that $B_{1,2}$ and $B_{2,3}$ are market observable bond prices at $\tau = 1$ and $\tau = 2$, respectively. Also, it causes nothing to change the holding of units of futures on successive dates.

2. Invest $G_{0,3}$ in one-day risk free bond and roll over the net cash position

<table>
<thead>
<tr>
<th>Time</th>
<th>Profits from futures</th>
<th>Bond position</th>
<th>Net position</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>—</td>
<td>$G_{0,3}$</td>
<td>$G_{0,3}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{B_{0,1}}(G_{1,3} - G_{0,3})$</td>
<td>$\frac{G_{0,3}}{B_{0,1}}$</td>
<td>$\frac{G_{1,3}}{B_{0,1}}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{B_{0,1}B_{1,2}}(G_{2,3} - G_{1,3})$</td>
<td>$\frac{G_{1,3}}{B_{0,1}B_{1,2}}$</td>
<td>$\frac{G_{2,3}}{B_{0,1}B_{1,2}}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{B_{0,1}B_{1,2}B_{2,3}}(G_{3,3} - G_{2,3})$</td>
<td>$\frac{G_{2,3}}{B_{0,1}B_{1,2}B_{2,3}}$</td>
<td>$\frac{G_{3,3}}{B_{0,1}B_{1,2}B_{2,3}} = \frac{S_3}{B_{0,1}B_{1,2}B_{2,3}}$</td>
</tr>
</tbody>
</table>

Note that $G_{3,3} = S_3$. 

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Consider an asset with a price $\tilde{S}_T$ at time $T$. The forward price of the asset, $F_{t,T}$, is the time-$t$ spot price of an asset which has a payoff $\tilde{S}_T/B_{t,T}$ at time $T$.

**Proof**

Change the daily settlement in a futures to have one single settlement at $T$ in a forward. This is done by long holding $1/B_{t,T}$ units of forward at time $t$ and invest $F_{t,T}$ on a discount bond at time $t$. The portfolio ends up to become $S_T/B_{t,T}$ at time $T$.

**Remark**

Since $B_{t,T}$ and $S_t$ are known, $F_{t,T}$ is the time-$t$ spot price of a security that pays $\frac{1}{B_{t,T}}$ units of the underlying asset. Hence, we have $F_{t,T} = \frac{S_t}{B_{t,T}}$. 
Pricing issues

We consider the discrete-time model and assume the existence of a risk neutral pricing measure $Q$. We have demonstrated the construction of the dynamic replication strategy. The fair time-$t$ price of $\frac{\tilde{S}_T}{B_{t,t+1} \cdots \tilde{B}_{T-1,T}}$ is the cost of setting up the replicating portfolio, which is $G_{t,T}$. Based on the risk neutral valuation principle, the time-$t$ price of a security is given by the $Q$-expectation of the discounted terminal payoff at $T$, where

$$G_{t,T} = E_Q \left[ \frac{\tilde{S}_T}{B_{t,t+1} \tilde{B}_{t+1,t+2} \cdots \tilde{B}_{T-1,T} B_{t,t+1} \tilde{B}_{t,t+1} \tilde{B}_{t+1,t+2} \cdots \tilde{B}_{T-1,T}} \right]$$

$$= E_Q[\tilde{S}_T].$$

The result remains valid for the continuous-time counterpart. Under the continuous-time model, the risk neutral valuation principle gives the time-$t$ price of a contingent claim as follows:

$$V_t = E_Q \left[ e^{-\int_t^T r(u) \, du} V_T \right].$$
Difference in futures price $G_t$ and forward price $F_t$

Difference in payment schedules may lead to difference in futures and forward prices since different interest rates are applied on intermediate payments. When the interest rates are deterministic, we have $G_{t,T} = F_{t,T}$. This is a sufficient condition for equality of the two prices. The necessary and sufficient condition is that the discount process and the underlying price process are uncorrelated under the risk neutral measure $Q$.

- When physical holding of the underlying index (say, snow fall amount) for hedging is infeasible, then the buyer sets

  \[
  \text{forward price} = E_P[S_T],
  \]

  where $P$ is the subjective probability measure of the buyer.
• When the physical holding of the asset is subject to daily settlement through the margin requirement (dynamic rebalancing)

\[ \text{futures price} = \mathbb{E}_Q[S_T], \]

where \( Q \) is the risk neutral measure that uses the money market account as the numeraire. As a remark, \( \frac{1}{B_{t,t+1}B_{t+1,t+2} \cdots B_{T-1,T}} \) is seen as the money market account rolling daily over the period \([t, T]\). The asset \( \frac{\tilde{S}_T}{B_{t,t+1}B_{t+1,t+2} \cdots B_{T-1,T}} \) when normalized by the money market account becomes \( \tilde{S}_T \).
By the risk neutral valuation principle, we have

\[ B(t, T) = EQ \left[ e^{-\int_t^T r(u)du} \right] \quad \text{and} \quad S_t = EQ \left[ e^{-\int_t^T r(u)du} S_T \right] \]

so that

\[
G_t - F_t = EQ [S_T] - \frac{S_t}{B(t, T)} = \frac{EQ[S_T]EQ \left[ e^{-\int_t^T r(u)du} \right] - EQ \left[ e^{-\int_t^T r(u)du} S_T \right]}{B(t, T)} = -\frac{\text{cov}_Q \left[ e^{-\int_t^T r(u)du}, S_T \right]}{B(t, T)}.
\]

The spread between \(G_t\) and \(F_t\) becomes zero when the discount process and the price process of the underlying asset are uncorrelated under the risk neutral measure \(Q\). In the special case where interest rates are deterministic, we have equality of \(G_t\) and \(F_t\).
Currency forward

The underlying is the exchange rate $X$, which is the domestic currency price of one unit of foreign currency.

\[
\begin{align*}
    r_d &= \text{constant domestic interest rate} \\
    r_f &= \text{constant foreign interest rate}
\end{align*}
\]

Portfolio $A$: Hold one currency forward with delivery price $K$ and a domestic bond of par $K$ maturing on the delivery date of forward.

Portfolio $B$: Hold a foreign bond of unit par maturing on the delivery date of forward.

Holding of the domestic and foreign bonds allow the bonds to earn the interest rate in the respective currency.
Remark

The underlying asset of a foreign currency forward is one unit of foreign currency (in the form of market account) that pays dividend yield $r_f$.

Let $\Pi_A(t)$ and $\Pi_A(T)$ denote the value of Portfolio $A$ at time $t$ and $T$, respectively. On the delivery date, the holder of the currency forward has to pay $K$ domestic dollars to buy one unit of foreign currency. Hence, $\Pi_A(T) = \Pi_B(T)$, where $T$ is the delivery date.

Using the law of one price, $\Pi_A(t) = \Pi_B(t)$ must be observed at the current time $t$.

- Failure of the law of one price leads to existence of arbitrage opportunity (an important financial economic concept to be discussed in Topic 2)
Interest rate parity relation

Note that

\[ B_d(\tau) = e^{-r_d \tau}, \quad B_f(\tau) = e^{-r_f \tau}, \]

where \( \tau = T - t \) is the time to expiry. Let \( f \) be the value of the currency forward in domestic currency,

\[ f + KB_d(\tau) = XB_f(\tau), \]

where \( XB_f(\tau) \) is the value of the foreign bond in domestic currency.

By setting \( f = 0 \), the forward price of the currency forward is

\[ K = \frac{XB_f(\tau)}{B_d(\tau)} = Xe^{(r_d-r_f)\tau}. \]

We may recognize \( r_d \) as the cost of fund and \( r_f \) as the dividend yield. This result is the well known Interest Rate Parity Relation.
Consider a 6-month forward contract. The exchange rate over each one-month period is preset to assume some constant value.

The holder can exercise parts of the notional at any time during the life of the forward, but she has to exercise the whole notional by the maturity date of the currency forward.

1. As the optimal policy is independent of the notional, the holder chooses either no action or exercise the full notional (partial exercise is non-optimal). This is called the bang-bang strategy.

2. How does the seller set the predetermined exchange rates so that the value of this flexible notional currency forward is zero at initiation? One needs to calculate the early exercise premium and takes note of the time value of the forward prices at different exercise points.
**Bond forward**

The underlying asset is a zero-coupon bond of maturity $T_2$ with the settlement date $T_1$, where $t < T_1 < T_2$.

The holder pays the delivery price $F$ of the bond forward on the forward maturity date $T_1$ to receive a bond with par value $P$ on the maturity date $T_2$. 

*Holder’s cashflows*
Bond forward price in terms of traded bond prices

Let $B_t(T)$ denote the traded price of unit par discount bond at current time $t$ with maturity date $T$.

Present value of the net cashflows

$$ = -FB_t(T_1) + PB_t(T_2).$$

To determine the forward price $F$, we set the above value zero and obtain

$$F = PB_t(T_2)/B_t(T_1).$$

Here, $PB_t(T_2)$ can be visualized as the spot price of the discount bond. The forward price is given in terms of the known market bond prices observed at time $t$ with maturity dates $T_1$ and $T_2$. 
Forward on a coupon-paying bond

The underlying is a coupon-paying bond with maturity date $T_B$.

Note that the bond is a traded security whose value changes with respect to time.

Let $T_F$ be the delivery date of the bond forward, where $T_F < T_B$. Let $t_i$ be the coupon payment date of the bond on which deterministic coupon $c_i$ is paid. Let $t$ be the current time, where $t < T_F < T_B$. Some of the coupons have been paid at earlier times. Let $F$ be the forward price, the amount paid by the forward contract holder at time $T_F$ to buy the bond.
At $T_B$, the bondholder receives par plus the last coupon.
Based on the forward price formula: \( F = \frac{S-D}{B(\tau)} \), we deduce that

\[
F = \frac{\text{spot price of bond}}{B_t(T_F)} - \frac{c_4 B_t(t_4)}{B_t(T_F)} - \frac{c_5 B_t(t_5)}{B_t(T_F)}.
\]

Let \( P \) be the par value of the bond. After receiving the bond at \( T_F \), the bond forward holder is entitled to receive \( c_6, c_7 \) and \( P \) once he has received the underlying bond. By considering the cash flows after \( T_F \), he pays \( F \) at \( T_F \) and receives \( c_6 \) at \( t_6 \), \( c_7 + P \) at \( T_B \).

**Present value of cash flows at time \( t \)**

\[
= -FB_t(T_F) + c_6 B_t(t_6) + c_7 B_t(T_B) + PB_t(T_B).
\]

Hence, the bond forward price is given by

\[
F = \frac{c_6 B_t(t_6) + c_7 B_t(T_B) + PB_t(T_B)}{B_t(T_F)}.
\]

**Remark**

Equating the two expressions gives the spot price of the bond in terms of the cash flows.
Example — Bond forward

- A 10-year bond is currently selling for $920.

- Currently, hold a forward contract on this bond that has a delivery date in 1 year and a delivery price of $940.

- The bond pays coupons of $80 every 6 months, with one due 6 months from now and another just before maturity of the forward.

- The current interest rates for 6 months and 1 year (compounded semi-annually) are 7% and 8%, respectively (annual rates compounded every 6 months).

- What is the current value of the forward?
Let $d(0, k)$ denote the discount factor over the $(0, k)$ semi-annual period. We have $d(0, 2) = \frac{1}{(1.04)^2}$ and $d(0, 1) = \frac{1}{1.035}$. Consider the future value of the cash flows associated with holding the bond one year later and payment of $F_0$ under the forward contract. The current forward price of the bond

$$F_0 = \frac{\text{spot price}}{d(0, 2)} - \frac{c(1)d(0, 1)}{d(0, 2)} - \frac{c(2)d(0, 2)}{d(0, 2)}$$

$$= 920(1.04)^2 - \frac{80(1.04)^2}{1.035} - \frac{80(1.04)^2}{(1.04)^2} = 831.47.$$ 

The difference in the forward prices is discounted to the present value. The current value of the forward contract $= \frac{831.47 - 940}{(1.04)^2} = -100.34$. 


**Implied forward interest rate**

The forward price of a forward on a discount bond should be related to the implied forward interest rate $R(t; T_1, T_2)$. The implied forward rate is the interest rate over $[T_1, T_2]$ as implied by time-$t$ discount bond prices. The bond forward buyer pays $F$ at $T_1$ and receives $P$ at $T_2$ and she is expected to earn $R(t; T_1, T_2)$ over $[T_1, T_2]$, so

$$F[1 + R(t; T_1, T_2)(T_2 - T_1)] = P.$$  

Together with

$$F = PB_t(T_2)/B_t(T_1),$$

we obtain

$$R(t; T_1, T_2) = \frac{1}{T_2 - T_1} \left[ \frac{B_t(T_1)}{B_t(T_2)} - 1 \right].$$
Forward rate agreement (FRA)

The FRA is an agreement between two counterparties to exchange floating and fixed interest payments on the future settlement date $T_2$.

- The floating rate will be the LIBOR $L[T_1, T_2]$ as observed on the future reset date $T_1$.

Question

Should the fixed rate be set equal to the implied forward rate over the same period (determined based on traded bond prices as observed today)?
Determination of the forward price of LIBOR

\[ L[T_1, T_2] = \text{LIBOR to be observed at future time } T_1 \]

for the accrual period \([T_1, T_2]\)

\[ K = \text{fixed rate} \]

\[ N = \text{notional of the FRA} \]

Cash flow of the fixed rate receiver

\[ NK(T_2 - T_1) \]

\[ NL(T_1, T_2) (T_2 - T_1) \]
Replication argument

Adding $N$ to both parties at time $T_2$, the cash flows of the fixed rate payer can be replicated by

(i) long holding of the $T_2$-maturity zero coupon bond with par $N[1 + K(T_2 - T_1)]$.

(ii) short holding of the $T_1$-maturity zero coupon bond with par $N$.

It is assumed that the par amount $N$ collected at $T_1$ will be put in a deposit account that earns the floating LIBOR $L[T_1, T_2]$.
Cash flow of the fixed rate receiver

- Collect \( N + NK(T_2 - T_1) \) from \( T_2 \)-maturity bond
- Collect \( N \) at \( T_1 \) from \( T_1 \)-maturity bond;
- Invest in bank account earning \( L[T_1, T_2] \) rate of interest
Value of the portfolio of bonds that replicate the cash flow of the fixed rate receiver at the current time

\[ = N\{[1 + K(T_2 - T_1)]B_t(T_2) - B_t(T_1)\}. \]

We find \( K \) such that the above value is zero.

\[ K = \frac{1}{T_2 - T_1} \left[ \frac{B_t(T_1)}{B_t(T_2)} - 1 \right]. \]

The fair fixed rate \( K \) is seen to be the forward price of the LIBOR rate \( L[T_1, T_2] \) over the time period \([T_1, T_2]\).
1.2 Exotic swap products

**Interest rate swaps**

The two parties agree to exchange periodic interest payments.

- The interest payments exchanged are calculated based on some predetermined dollar principal, called the notional amount.

- One party is the fixed-rate payer and the other party is the floating-rate payer. The floating interest rate is based on some reference rate (the most common index is the LONDON INTERBANK OFFERED RATE, LIBOR).
Example

Notional amount = $50 million
fixed rate = 10%
floating rate = 6-month LIBOR

Tenor = 3 years, semi-annual payments

<table>
<thead>
<tr>
<th>6-month period</th>
<th>Cash flows</th>
<th>floating rate bond</th>
<th>fixed rate bond</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Net (float – fix)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-50</td>
<td>50</td>
</tr>
<tr>
<td>1</td>
<td>LIBOR₁/2 × 50 – 2.5</td>
<td>LIBOR₁/2 × 50</td>
<td>-2.5</td>
</tr>
<tr>
<td>2</td>
<td>LIBOR₂/2 × 50 – 2.5</td>
<td>LIBOR₂/2 × 50</td>
<td>-2.5</td>
</tr>
<tr>
<td>3</td>
<td>LIBOR₃/2 × 50 – 2.5</td>
<td>LIBOR₃/2 × 50</td>
<td>-2.5</td>
</tr>
<tr>
<td>4</td>
<td>LIBOR₄/2 × 50 – 2.5</td>
<td>LIBOR₄/2 × 50</td>
<td>-2.5</td>
</tr>
<tr>
<td>5</td>
<td>LIBOR₅/2 × 50 – 2.5</td>
<td>LIBOR₅/2 × 50</td>
<td>-2.5</td>
</tr>
<tr>
<td>6</td>
<td>LIBOR₆/2 × 50 – 2.5</td>
<td>LIBOR₆/2 × 50</td>
<td>-2.5</td>
</tr>
</tbody>
</table>

- One interest rate swap contract can effectively establish a payoff equivalent to a package of forward contracts.
A swap can be interpreted as a package of cash market instruments – a portfolio of forward rate agreements.

- Buy $50 million par of a 3-year floating rate bond that pays 6-month LIBOR semi-annually.
- Finance the purchase by borrowing $50 million for 3 years at 10% interest rate paid semi-annually.

The fixed-rate payer has a cash market position equivalent to a long position in a floating-rate bond and a short position in a fixed rate bond (borrowing through issuance of a fixed rate bond). Both bonds are assumed to be par bonds, where the present value is equal to the par value.
Application to asset/liability management

- Holding a 5-year term commercial loans of $50 million with a fixed interest rate of 10%, that is, interest of $2.5 million received semi-annually and par received at the end of 5 years.

- To fund its loan portfolio, the bank issues 6-month certificates of deposit with floating interest rate of LIBOR + 40 bps (100 bps = 1%). This may be a response to the market demand from investors for floating rate certificates of deposit.

Risk 6-month LIBOR may be 9.6% or greater.

Possible strategy Swap the fixed rate income into a floating rate cash stream to hedge against uncertainty in floating rate.
Choice of swap for the bank

- Every six months, the bank will pay 8.45% (annualized rate).
- Every six month, the bank will receive LIBOR.

**Outcome**

<table>
<thead>
<tr>
<th>To be received</th>
<th>10.00% + 6-month LIBOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>To be paid</td>
<td>8.45% + 0.4% + 6-month LIBOR</td>
</tr>
<tr>
<td>spread income</td>
<td>1.15% or 115 basis points</td>
</tr>
</tbody>
</table>

The bank faces potential default risk of loans.
Life insurance company's position

- Has committed to pay a 9% rate for the next 5 years on a guaranteed investment contract (GIC) of amount $50 million.
- Can invest $50 million in an attractive 5-year floating-rate instrument with floating interest rate of 6-month LIBOR +160 bps.

Risk 6-month LIBOR may fall to 7.4%.

Possible strategy Swap the floating rate income into a fixed rate cash stream.
Choice of swap for the insurance company

- Every six months, the insurance company will pay LIBOR.
- Every six months, the insurance company will receive 8.45%.

**Outcome**

<table>
<thead>
<tr>
<th>To be received</th>
<th>8.45% + 1.6% + 6-month LIBOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>To be paid</td>
<td>9.00% + 6-month LIBOR</td>
</tr>
<tr>
<td>Spread income</td>
<td>1.05% or 105 basis points</td>
</tr>
</tbody>
</table>

Diagram:

- Floating rate instrument
  - LIBOR + 1.6%
  - GIC
    - 9%
- Insurance company
  - LIBOR
  - 8.45%
- Swap counterparty
  - 55
Valuation of interest rate swaps

- When a swap is entered into, it typically has zero value.
- Valuation involves finding the fixed swap rate $K$ such that the fixed and floating legs have equal value at inception.
- Consider a swap with payment dates $t_1, t_2, \cdots, t_n$ (tenor structure) set in the term of the swap. $L_{i-1}$ is the LIBOR observed at $t_{i-1}$ but payment is made at $t_i$. Write $\delta_i \approx t_i - t_{i-1}$ as the accrual period over $[t_{i-1}, t_i]$. Note that $\delta_i$ is in general not exactly the same as $t_i - t_{i-1}$ since some form of day count convention is used to compute $\delta_i$ (see below).
- The fixed payment at $t_i$ is $KN\delta_i$ while the floating payment at $t_i$ is $L_{i-1}N\delta_i$, $i = 1, 2, \cdots n$. Here, $N$ is the notional.
**Day count convention**

For the 30/360 day count convention of the time period \((D_1, D_2]\) with \(D_1\) excluded but \(D_2\) included, the year fraction is

\[
\frac{\max(30 - d_1, 0) + \min(d_2, 30) + 360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1)}{360}
\]

where \(d_i, m_i\) and \(y_i\) represent the day, month and year of date \(D_i, i = 1, 2\).

For example, the year fraction between *Feb 27, 2006* and *July 31, 2008*

\[
= \frac{30 - 27 + 30 + 360 \times (2008 - 2006) + 30 \times (7 - 2 - 1)}{360}
\]

\[
= \frac{33}{360} + 2 + \frac{4}{12}.
\]
Replication of cash flows

- The fixed payment at $t_i$ is $K N \delta_i$. The fixed payments are packages of discount bonds with par $K N \delta_i$ at maturity date $T_i$, $i = 1, 2, \cdots, n$.

- To replicate the floating leg payments at current time $t, t < T_0$, we long $T_0$-maturity discount bond with par $N$ and short $T_n$-maturity discount bond with par $N$. The $N$ dollars collected at $T_0$ can generate the floating leg payments $L_{i-1} N \delta_i$ at all $T_i$ by rolling over $N$ dollars in a deposit bank account earning interest rate $L_{i-1}$ over $[t_{i-1}, t_i]$, $i = 1, 2, \cdots, n$. Note that we always have $N$ dollars at the beginning of each accrual period since the interests earned are used to honor the floating leg payments. The remaining $N$ dollars at $T_n$ is used to pay the par of the $T_n$-maturity bond.

- Let $B(t, T)$ be the time-$t$ value of the discount bond with maturity $t$. 
- Sum of present value of the floating leg payments

\[= N[B(t, T_0) - B(t, T_n)];\]

sum of present value of fixed leg payments

\[= NK \sum_{i=1}^{n} \delta_i B(t, T_i).\]

The swap rate \(K\) is given by equating the present values of the two sets of payments:

\[K = \frac{B(t, T_0) - B(t, T_n)}{\sum_{i=1}^{n} \delta_i B(t, T_i)}.\]

The interest rate swap reduces to a FRA when \(n = 1\). As a check, we obtain

\[K = \frac{B(t, T_0) - B(t, T_1)}{(T_1 - T_0)B(t, T_1)}.\]
Asset swap

- Combination of a defaultable bond with an interest rate swap.

  \( B \) pays the notional amount upfront to acquire the asset swap package.

1. A fixed defaultable coupon bond issued by \( C \) with coupon \( \bar{c} \) payable on coupon dates.

2. A fixed-for-floating swap.

\[ \text{LIBOR} + s^A \quad \bar{c} \]

\( \text{defaultable bond } C \)
The asset swap spread $s^A$ is adjusted to ensure that the asset swap package has an initial value equal to the par value of the defaultable bond.

Remarks

1. Asset swap transactions are driven by the desire to replace the fixed coupons by floating coupons. Asset swaps are more liquid than the underlying defaultable bond.

2. An asset swaption gives $B$ the right to enter an asset swap package at some future date $T$ at a predetermined asset swap spread $s^A$. 
**Default free bond**

\[ C(t) = \text{time-}t \text{ price of default-free bond with fixed-coupon } \bar{c} \]
\[ B(t, T) = \text{time-}t \text{ price of default-free zero-coupon bond with unit par} \]

**Defaultable bond**

\[ \overline{C}(t) = \text{time-}t \text{ price of defaultable bond with fixed-coupon } \bar{c} \]

The difference \( C(t) - \overline{C}(t) \) represents the credit risk premium of the defaultable bond. Investors pay a less amount of \( C(t) - \overline{C}(t) \) due to potential losses arising from bond default.
**Forward swap rate**

\[ s(t) = \begin{cases} \text{forward swap rate at time } t \text{ of a standard fixed-for-floating} \\ \text{over the tenor } [t_n, t_{n+1}, \ldots, t_N] \end{cases} \]

\[ = \frac{B(t, t_n) - B(t, t_N)}{A(t; t_n, t_N)}, \quad t \leq t_n \]

where \( A(t; t_n, t_N) = \sum_{i=n+1}^{N} \alpha_i B(t, t_i) \) = value of the annuity payment stream paying \( \alpha_i \) on each date \( t_i \). The first swap payment starts on \( t_{n+1} \) (based on the accrual period \([t_n, t_{n+1}]\)) and the last payment date is \( t_N \).

The forward swap rate is market observable. It may occur that the market swap rate may not agree exactly with the above theoretical formula based on traded bond prices.
Asset swap packages

An asset swap package consists of a defaultable coupon bond $\overline{C}$ with coupon $\overline{c}$ and an interest rate swap. The bond’s coupon is swapped into LIBOR plus the asset swap rate $s^A$.

Payoff streams to the buyer of the asset swap package

<table>
<thead>
<tr>
<th>time</th>
<th>defaultable bond</th>
<th>interest rate swap</th>
<th>net</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0^\dagger$</td>
<td>$-\overline{C}(0)$</td>
<td>$-1 + \overline{C}(0)$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$t = t_i$</td>
<td>$\overline{c}^*$</td>
<td>$-\overline{c} + L_{i-1} + s^A$</td>
<td>$L_{i-1} + s^A + (\overline{c}^* - \overline{c})$</td>
</tr>
<tr>
<td>$t = t_N$</td>
<td>$(1 + \overline{c})^*$</td>
<td>$-\overline{c} + L_{N-1} + s^A$</td>
<td>$1^* + L_{N-1} + s^A + (\overline{c}^* - \overline{c})$</td>
</tr>
<tr>
<td>default recovery</td>
<td>unaffected recovery</td>
<td></td>
<td>recovery</td>
</tr>
</tbody>
</table>

$^\dagger$ denotes payment contingent on survival.

$^\dagger$ The value of the interest rate swap at $t = 0$ is not zero. The sum of the values of the interest rate swap and defaultable bond is equal to par at $t = 0$. 
The additional asset spread $s^A$ above LIBOR serves as the compensation for bearing the potential loss upon default.

$s(0) = \text{fixed-for-floating swap rate (market quote)}$

$A(0) = \text{value of an annuity paying at$1$per annum over the same tenor as the interest rate swap (calculated based on observable default free bond prices)}$

The value of asset swap package is set at par at $t = 0$, so that

$$\overline{C}(0) + A(0)s(0) + A(0)s^A(0) - A(0)c = 1.$$

The present value of the floating coupons is given by $A(0)s(0)$. The swap continues even after default so that $A(0)$ appears in all terms associated with the swap arrangement.
Solving for $s^A(0)$

$$s^A(0) = \frac{1}{A(0)}[1 - \bar{C}(0)] + \bar{c} - s(0).$$

Rearranging the terms,

$$\bar{C}(0) + A(0)s^A(0) = [1 - A(0)s(0)] + A(0)\bar{c} \equiv C(0)$$

default-free bond

where the right-hand side gives the value of a default-free bond with coupon $\bar{c}$. Note that $1 - A(0)s(0)$ is the present value of receiving $\$1$ at maturity $t_N$. We obtain

$$s^A(0) = \frac{1}{A(0)}[C(0) - \bar{C}(0)].$$

The difference in the bond prices is equal to the present value of annuity stream at the rate $s^A(0)$.

- One may relate $s^A(0)$ to the premium in a credit default swap.
**Alternative proof**

A combination of the non-defaultable counterpart (bond with coupon rate $\overline{c}$) plus an interest rate swap (whose floating leg is LIBOR while the fixed leg is $\overline{c}$) becomes a par floater. Hence, the new asset package should also be sold at par.

\[
\begin{align*}
C(0) &= \frac{1 - A(0)s(0)}{1 - A(0)\overline{c}} + A(0)\overline{c} \\
\text{while } A(0)s(0) - A(0)\overline{c} &\text{ gives the value of the interest rate swap.}
\end{align*}
\]
In other words, we obtain

\[
\text{value of interest rate swap } + \text{price of nondefaultable bond } = \left[ A(0)s(0) - A(0)c \right] + [1 - A(0)s(0) + A(0)c] = 1.
\]

- The two interest swaps with floating leg at \(\text{LIBOR} + s^A(0)\) and \(\text{LIBOR}\), respectively, differ in values by \(s^A(0)A(0)\).

- Let \(V_{\text{swap}\_L+s^A}\) denote the value of the swap at \(t = 0\) whose floating rate is set at \(\text{LIBOR} + s^A(0)\). Both asset swap packages are sold at par. We then have

\[
1 = \overline{C}(0) + V_{\text{swap}\_L+s^A} = C(0) + V_{\text{swap}\_L}\cdot
\]

Hence, the difference in \(C(0)\) and \(\overline{C}(0)\) is the present value of the annuity stream at the rate \(s^A(0)\), that is,

\[
C(0) - \overline{C}(0) = V_{\text{swap}\_L+s^A} - V_{\text{swap}\_L} = s^A(0)A(0).
\]
IBM/World Bank – first currency swap structured in 1981

- IBM had existing debts in DM and Swiss francs. This had created a FX exposure since IBM had to convert USD into DM and Swiss Francs regularly to make the coupon payments. Due to a depreciation of the DM and Swiss franc against the dollar, IBM could realize a large foreign exchange gain, but only if it could eliminate its DM and Swiss franc liabilities and “lock in” the gain and remove any future exposure.

- The World Bank was raising most of its funds in DM (interest rate = 12%) and Swiss francs (interest rate = 8%). It did not borrow in dollars, for which the interest rate cost was about 17%. Though it wanted to lend out in DM and Swiss francs, the bank was concerned that saturation in the bond markets could make it difficult to borrow more in these two currencies at a favorable rate. Its objective, as always, was to raise cheap funds.
World Bank/IBM Currency Swap, 1981

FX Market

Bond Market

Borrow $290m
Pay interest and repay debt in $, from swap payments by IBM

Convert $ into DM and SFr

WORLD BANK

SWAP
Pay in DM and SFr out of proceeds from loans to customers
Pay in $

IBM

Lend in DM and SFr

Repayment of debts to World Bank in DM and SFr

Bank's customers

Existing DM and SFr loans

Pay interest and repay debt in DM and SFr from swap payments

$ income from trading activities
• IBM was willing to take on dollar liabilities and made dollar payments (periodic coupons and principal at maturity) to the World Bank since it could generate dollar income from normal trading activities.

• The World Bank could borrow dollars, convert them into DM and SFr in FX market, and through the swap take on payment obligations in DM and SFr.

1. The foreign exchange gain on dollar appreciation is realized by IBM through the negotiation of a favorable swap rate in the swap contract.

2. The swap payments by the World Bank to IBM were scheduled so as to allow IBM to meet its debt obligations in DM and SFr.
Under the currency swap

- IBM pays regular US coupons and US principal at maturity.

- World Bank pays regular DM and SFr coupons together with DM and SFr principal at maturity.

Now IBM converted its DM and SFr liabilities into USD, and the World Bank effectively raised hard currencies at a cheap rate. Both parties achieved their objectives!
1.3 Options: Rational boundaries of option values

Financial options

- A *call* (or *put*) option is a contract which gives the holder the *right* to buy (or sell) a prescribed asset, known as the *underlying asset*, by a certain date (*expiration date*) for a predetermined price (commonly called the *strike price* or *exercise price*).

- The option is said to be *exercised* when the holder chooses to buy or sell the asset.

- If the option can only be exercised on the expiration date, then the option is called a *European* option.

- If the exercise is allowed at any time prior to the expiration date, then it is called an *American* option.
**Terminal payoff**

- Let $S_T$ denote the asset price at maturity date $T$ and $X$ be the strike price.

- The terminal payoff from the long position (holder’s position) of a European call is then
  \[
  \max(S_T - X, 0).
  \]

- The terminal payoff from the long position in a European put can be shown to be
  \[
  \max(X - S_T, 0),
  \]
  since the put will be exercised at expiry only if $S_T < X$, whereby the asset worth $S_T$ is sold at a higher price of $X$. 
call payoff

\[ \max(S_T - X, 0) \]

put payoff

\[ \max(X - S_T, 0) \]
Questions and observations

What should be the fair option premium (usually called option price or option value) so that the deal is fair to both writer and holder?

What should be the optimal strategy to exercise prior the expiration date for an American option?

At least, the option price is easily seen to depend on the strike price, time to expiry and current asset price. The less obvious factors for the pricing models are the prevailing interest rate and the degree of randomness of the asset price, commonly called the volatility.
Hedging

- If the writer of a call does not simultaneously own any amount of the underlying asset, then he is said to be in a *naked position*.

- Suppose the call writer owns some amount of the underlying asset, the loss in the short position of the call when asset price rises can be compensated by the gain in the long position of the underlying asset.

- This strategy is called *hedging*, where the risk in a portfolio is monitored by taking opposite directions in two securities which are highly negatively correlated.

- In a *perfect hedge* situation, the *hedger* combines a risky option and the corresponding underlying asset in an appropriate proportion to form a riskless portfolio.
Swaptions – Product nature

- The buyer of a swaption has the right to enter into an interest rate swap by some specified date. The swaption also specifies the maturity date of the swap. The buyer of the swaption either pays the premium upfront or the premium is structured into the swap rate.

- The buyer can be the fixed-rate receiver or the fixed-rate payer. The writer becomes the counterparty to the swap if the buyer exercises. The strike rate indicates the fixed rate that will be swapped versus the floating rate.

- Suppose the buyer of the swaption is the fixed rate payer in the underlying swap, she chooses to exercise the swaption when the prevailing swap rate is higher than the strike rate. This is because the swaption buyer would pay a lower fixed rate in the interest rate swap under the swaption contract when compared with the higher prevailing fixed rate in a newly negotiated interest rate swap.
• To the buyer, the exercise of the swaption is the sale of the fixed rate bond at the price of the floating rate bond.

• The value of the floating rate bond equals the par at initiation of the swap, so it may be viewed as fixed value (strike price of the swaption). This swaption is thus seen as a put swaption.

• Note that the value of the fixed rate bond with coupon rate equals to the prevailing swap rate would be equal to the par value. Obviously, the value of the fixed rate bond with coupon rate same as the preset fixed rate in the swaption (lower than the prevailing swap rate) would be below par.
Uses of swaptions

Used to hedge a portfolio strategy that involves the use of an interest rate swap while the cash flow of the underlying asset or liability is uncertain.

Uncertainties come from (i) callability, eg, a callable bond or prepayment of mortgage loans, (ii) exposure to default risk.

Example

Consider a Savings & Loans Association entering into a 4-year swap in which it agrees to pay 9% fixed and receive LIBOR.

- The fixed rate payments come from a portfolio of mortgage pass-through securities with a coupon rate of 9%. One year later, mortgage rates decline, resulting in large prepayments.
- The purchase of a call swaption with a strike rate of 9% would be useful to offset the original swap position.
Due to decline in the interest rate, large prepayments are resulted in the mortgage pass-through securities. The source of 9% fixed payment dissipates. The swaption is in-the-money since the interest rate declines, so does the swap rate.
By exercising the call swaption, the Savings & Loans Association receives a fixed rate of 9%. The risk of potential decline in interest rates is hedged via the purchase of a swaption.

- Treating the fixed rate bond as the underlying asset in the swaption and the floating rate bond as the fixed par value, the “pay-float” swaption is visualized as a call swaption since the holder pays the fixed strike to receive the underlying asset.
Management of callable debts

Three years ago, XYZ issued 15-year fixed rate callable debt with a coupon rate of 12%.

Strategy

The bond issuer XYZ sells a two-year fixed rate receiver option on a 10-year swap that gives the holder the right but not the obligation to receive the fixed rate of 12%.
Call monetization

By selling the swaption today, the company has committed itself to paying a 12% coupon for the remaining life of the original bond.

- The swaption was sold in exchange for an upfront swaption premium received at date 0. The monetization of the callable right is realized via the swaption premium received.
Call-Monetization cash flow: Swaption expiration date

*Interest rate (swap rate) ≥ 12%*

- Counterparty does not exercise the swaption
- \( XYZ \) earns the full proceed of the swaption premium
Interest rate (swap rate) < 12%

- Counterparty exercises the swaption

- XYZ calls the bond. Once the old bond is retired, XYZ issues a new floating rate bond that pays floating rate LIBOR (funding rate depends on the creditworthiness of XYZ at that time).
Rational boundaries for option values

- Mathematical properties of the option values as functions of the strike price \( X \), asset price \( S \) and time to expiry \( \tau \) are derived. We do not specify the probability distribution of the movement of the asset price so that the *fair* option value cannot be derived. We study the impact of dividends on these rational boundaries.

- The optimal early exercise policy of American options on a non-dividend paying asset can be inferred from the analysis of these bounds on option values.

- The relations between put and call prices (called the *put-call parity relations*) are also deduced. These relations are distribution free.
Non-negativity of option prices

All option prices are non-negative, that is,

\[ C \geq 0, \quad P \geq 0, \quad c \geq 0, \quad p \geq 0, \]

as derived from the non-negativity of the payoff structure of option contracts.

If the price of an option were negative, this would mean an option buyer receives cash up-front. He is guaranteed to have a non-negative terminal payoff. In this way, he can always lock in a riskless profit.
Intrinsic values of American options

- $\max(S-X, 0)$ and $\max(X-S, 0)$ are commonly called the intrinsic value of a call and a put, respectively.

- Since American options can be exercised at any time before expiration, their values must be worth at least their intrinsic values, that is,

  \begin{align*}
  C(S, \tau; X) & \geq \max(S - X, 0) \\
  P(S, \tau; X) & \geq \max(X - S, 0).
  \end{align*}

- Suppose $C$ is less than $S-X$ when $S \geq X$, then an arbitrageur can lock in a riskless profit by borrowing $C + X$ dollars to purchase the call and exercise it immediately to receive the asset worth $S$. The riskless profit would be $S - X - C > 0$. 
American options are worth at least their European counterparts

An American option confers all the rights of its European counterpart plus the privilege of early exercise. The additional privilege cannot have negative value.

\[
C(S, \tau; X) \geq c(S, \tau; X) \\
P(S, \tau; X) \geq p(S, \tau; X).
\]

- The European put value can be below the intrinsic value \(X - S\) at sufficiently low asset value. In this case, the European put is almost sure to expire in-the-money. However, the par is received only at maturity for the European put.

- The value of a European call on a dividend paying asset can be below the intrinsic value \(S - X\) at sufficiently high asset value.
Values of options with different dates of expiration

Consider two American options with different times to expiration $\tau_2$ and $\tau_1$ ($\tau_2 > \tau_1$), the one with the longer time to expiration must be worth at least that of the shorter-lived counterpart since the longer-lived option has the additional right to exercise between the two expiration dates.

\[ C(S, \tau_2; X) > C(S, \tau_1; X), \quad \tau_2 > \tau_1, \]
\[ P(S, \tau_2; X) > P(S, \tau_1; X), \quad \tau_2 > \tau_1. \]

The above argument cannot be applied to European options due to lack of the early exercise privilege.
Values of options with different strike prices

\[ c(S, \tau; X_2) < c(S, \tau; X_1), \quad X_1 < X_2, \]
\[ C(S, \tau; X_2) < C(S, \tau; X_1), \quad X_1 < X_2. \]

and

\[ p(S, \tau; X_2) > p(S, \tau; X_1), \quad X_1 < X_2, \]
\[ P(S, \tau; X_2) > P(S, \tau; X_1), \quad X_1 < X_2. \]

Values of options at varying asset value levels

\[ c(S_2, \tau; X) > c(S_1, \tau; X), \quad S_2 > S_1, \]
\[ C(S_2, \tau; X) > C(S_1, \tau; X), \quad S_2 > S_1; \]

and

\[ p(S_2, \tau; X) < p(S_1, \tau; X), \quad S_2 > S_1, \]
\[ P(S_2, \tau; X) < P(S_1, \tau; X), \quad S_2 > S_1. \]
Upper bounds on call and put values

- A call option is said to be a perpetual call if its date of expiration is infinitely far away. The asset itself can be considered as an American perpetual call with zero strike price plus additional privileges such as voting rights and receipt of dividends, so we deduce that \( S \geq C(S, \infty; 0) \).

\[
S \geq C(S, \infty; 0) \geq C(S, \tau; X) \geq c(S, \tau; X).
\]

- The price of an American put equals the strike value when the asset value is zero; otherwise, it is bounded above by the strike price.

\[
X \geq P(S, \tau; X) \geq p(S, \tau; X).
\]
Lower bounds on the values of call options on a non-dividend paying asset

Portfolio $A$ consists of a European call on a non-dividend paying asset plus a discount bond with a par value of $X$ whose date of maturity coincides with the expiration date of the call. Portfolio $B$ contains one unit of the underlying asset.

<table>
<thead>
<tr>
<th>Asset value at expiry</th>
<th>$S_T &lt; X$</th>
<th>$S_T \geq X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio $A$</td>
<td>$X$</td>
<td>$(S_T - X) + X = S_T$</td>
</tr>
<tr>
<td>Portfolio $B$</td>
<td>$S_T$</td>
<td>$S_T$</td>
</tr>
<tr>
<td>Result of comparison</td>
<td>$V_A &gt; V_B$</td>
<td>$V_A = V_B$</td>
</tr>
</tbody>
</table>

The present value of Portfolio $A$ (dominant portfolio) must be equal to or greater than that of Portfolio $B$ (dominated portfolio). If otherwise, an arbitrage opportunity can be secured by buying Portfolio $A$ and selling Portfolio $B$. 
Write $B(\tau)$ as the price of the unit-par discount bond with time to expiry $\tau$. Then

$$c(S, \tau; X) + XB(\tau) \geq S.$$ 

Together with the non-negativity property of option value,

$$c(S, \tau; X) \geq \max(S - XB(\tau), 0).$$

The upper and lower bounds of the value of a European call on a non-dividend paying asset are given by (see Figure)

$$S \geq c(S, \tau; X) \geq \max(S - XB(\tau), 0).$$
\[ c(S, \tau; X) \]

\[ c = S \]

\[ c = S - XB(\tau) \]

call price
Convexity properties of the option price functions

The call prices are convex functions of the strike price. Write \( X_2 = \lambda X_3 + (1 - \lambda) X_1 \) where \( 0 \leq \lambda \leq 1, X_1 \leq X_2 \leq X_3 \).

\[
\begin{align*}
c(S, \tau; X_2) & \leq \lambda c(S, \tau; X_3) + (1 - \lambda)c(S, \tau; X_1), \\
C(S, \tau; X_2) & \leq \lambda C(S, \tau; X_3) + (1 - \lambda)C(S, \tau; X_1).
\end{align*}
\]
Consider the payoffs of the following two portfolios at expiry. Portfolio $C$ contains $\lambda$ units of call with strike price $X_3$ and $(1-\lambda)$ units of call with strike price $X_1$, and Portfolio $D$ contains one call with strike price $X_2$.

Since $V_C \geq V_D$ for all possible values of $S_T$, Portfolio $C$ is dominant over Portfolio $D$. Therefore, the present value of Portfolio $C$ must be equal to or greater than that of Portfolio $D$.

- The drop in the European call value for one dollar increase in the strike price should be less than one dollar. The loss in the terminal payoff of the call due to the increase in the strike price is realized only when the call expires in-the-money. More precisely, we have $\left| \frac{\partial c}{\partial X} \right| \leq B(\tau)$. The factor $B(\tau)$ appears since the potential loss of paying extra one dollar in the strike price occurs at maturity so its present value is $B(\tau)$. 
Payoff at expiry of Portfolios $C$ and $D$.

<table>
<thead>
<tr>
<th>Asset value at expiry</th>
<th>$S_T \leq X_1$</th>
<th>$X_1 \leq S_T \leq X_2$</th>
<th>$X_2 \leq S_T \leq X_3$</th>
<th>$X_3 \leq S_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio $C$</td>
<td>0</td>
<td>$(1 - \lambda)(S_T - X_1)$</td>
<td>$(1 - \lambda)(S_T - X_1)$</td>
<td>$\lambda(S_T - X_3) + (1 - \lambda)(S_T - X_1)$</td>
</tr>
<tr>
<td>Portfolio $D$</td>
<td>0</td>
<td>0</td>
<td>$S_T - X_2$</td>
<td>$S_T - X_2$</td>
</tr>
<tr>
<td>Result of comparison</td>
<td>$V_C = V_D$</td>
<td>$V_C \geq V_D$</td>
<td>$V_C \geq V_D^*$</td>
<td>$V_C = V_D$</td>
</tr>
</tbody>
</table>

* Recall $X_2 = \lambda X_3 + (1 - \lambda)X_1$, and observe

\[
(1 - \lambda)(S_T - X_1) \geq S_T - X_2 \\
\Rightarrow X_2 - (1 - \lambda)X_1 \geq \lambda S_T \\
\Rightarrow X_3 \geq S_T.
\]
There is no factor involving $\tau$, so the result also holds even when
the calls in the two portfolios are exercised prematurely. Hence,
the convexity property also holds for American calls.

By changing the call options in the above two portfolios to the
Corresponding put options, it can be shown that European and
American put prices are also convex functions of the strike price.

By using the linear homogeneity property of the call and put
option functions with respect to the asset price and strike price,
where for the case of an European call option:

$$c(\lambda S, \tau, \lambda X) = \lambda c(S, \tau, X), \quad \lambda > 0,$$

one can show that the call and put prices (both European and
American) are convex functions of the asset price. That is,

$$c(\lambda S_1 + (1 - \lambda) S_2, \tau; X) \leq \lambda c(S_1, \tau, X) + (1 - \lambda) c(S_2, \tau; X), \quad \lambda > 0.$$
Impact of dividends on the asset price

- When an asset pays a certain amount of dividend, we can use no arbitrage argument to show that the asset price is expected to fall by the same amount (assuming there exist no other factors affecting the income proceeds, like taxation and transaction costs).

- Suppose the asset price falls by an amount less than the dividend, an arbitrageur can lock in a riskless profit by borrowing money to buy the asset right before the dividend date, selling the asset right after the dividend payment and returning the loan.
It is assumed that the deterministic dividend amount $D_i$ is paid at time $t_i$, $i = 1, 2, \cdots, n$. The current time is $t$ and write $\tau_i = t_i - t$, $i = 1, 2, \cdots, n$. The sum of the present value of the dividends is

$$D = D_1 e^{-r\tau_1} + \cdots + D_n e^{-r\tau_n}.$$  

The dividend stream may be visualized as a portfolio of bonds with par value $D_i$ maturing at $t_i$, $i = 1, 2, \cdots, n$.

Weakness in the assumption

One may query whether the asset can honor the deterministic dividend payments when the asset value becomes very low.
Put-call parity relations

For a pair of European put and call options on the same underlying asset and with the same expiration date and strike price, we have

\[ p = c - S + D + X B(\tau). \]

When the underlying asset is non-dividend paying, we set \( D = 0 \).

The first portfolio involves long holding of a European call, a cash amount of \( D + X B(\tau) \) and short selling of one unit of the asset. The second portfolio contains only one European put. The cash amount \( D \) in the first portfolio is used to compensate the dividends due to the short position of the asset. At expiry, both portfolios are worth \( \max(X - S_T, 0) \).

Since both options are European, they cannot be exercised prior to expiry. Hence, both portfolios have the same value throughout the life of the options.
Impact of dividends on the lower bound on a European call value and the early exercise policy of an American call option

- Portfolio $B$ is modified to contain one unit of the underlying asset and liabilities of $D$ dollars of cash (in the form of a portfolio of bonds as specified earlier). At expiry, the value of Portfolio $B$ will always become $S_T$ since the loan of $D$ will be paid back during the life of the option using the dividends received.

- Since $V_A \geq V_B$ at expiry, hence the present value of Portfolio $A$ must be at least as much as that of Portfolio $B$. Together with the non-negativity property of option values, we obtain

$$c(S, \tau; X, D) \geq \max(S - XB(\tau) - D, 0).$$
• When $S$ is sufficiently large, the European call almost behaves like a forward whose value is $S - D - XB(\tau)$. Using the put-call parity relation: $c = p + S - D - XB(\tau)$, we obtain

$$p(S, \tau; X, D) \to 0 \text{ as } S \to \infty.$$ 

• Since the call price is lowered due to the dividends of the underlying asset, it may be possible that the call price becomes less than the intrinsic value $S - X$ when the lumped dividend $D$ is deep enough.

• A necessary condition on $D$ such that $c(S, \tau; X, D)$ may fall below the intrinsic value $S - X$ is given by

$$S - X > S - XB(\tau) - D \text{ or } D > X[1 - B(\tau)].$$ 

If $D$ does not satisfy the above condition, it is never optimal to exercise the dividend-paying American call prematurely.
When $D > X[1 - B(\tau)]$, the lower bound $S - D - XB(\tau)$ becomes less than the intrinsic value $S - X$. As $S \to \infty$, the European call price curve falls below the intrinsic value line.
**Bounds on puts**

The bounds for American and European puts can be shown to be

\[ P(S, \tau; X, D) \geq p(S, \tau; X, D) \geq \max(XB(\tau) + D - S, 0). \]

- When \( XB(\tau) + D < X \iff D < X[1 - B(\tau)] \), the lower bound \( XB(\tau) + D - S \) may become less than the intrinsic value \( X - S \) when the put is sufficiently deep in-the-money (corresponding to low value for \( S \)). It becomes sub-optimal for the holder of an American put option to continue holding the put option when the put value falls below the intrinsic value, the American put should be exercised prematurely.

- The presence of dividends makes the early exercise of an American put option less likely since the holder loses the future dividends when the asset is sold upon exercising the put.
Lower and upper bounds on the difference of the prices of American call and put options

- The parity relation cannot be applied to American options due to their early exercise feature.

First, we assume the underlying asset to be non-dividend paying. From the put-call parity relation, since $P > p$ and $C = c$, and putting $D = 0$, we have

$$C - P < S - XB(\tau),$$

giving the upper bound on $C - P$. 
Consider the following two portfolios: one contains a European call plus cash of amount $X$, and the other contains an American put together with one unit of underlying asset.

If there were no early exercise of the American put prior to maturity, the terminal value of the first portfolio is always higher than that of the second portfolio. If the American put is exercised prior to maturity, the second portfolio’s value becomes $X$, which is always less than $c + X$. The first portfolio dominates over the second portfolio, so we have

$$c + X > P + S.$$  

Further, since $c = C$ when the asset does not pay dividends, the lower bound on $C - P$ is given by

$$S - X < C - P.$$
Combining the two bounds, the difference of the American call and put option values on a non-dividend paying asset is bounded by

$$S - X < C - P < S - XB(\tau).$$

- The right side inequality: $C - P < S - XB(\tau)$ also holds for options on a dividend paying asset since dividends decrease call value and increase put value.

- The left side inequality has to be modified as: $S - D - X < C - P$.

- Combining the results, the difference of the American call and put option values on a dividend paying asset is bounded by

$$S - D - X < C - P < S - XB(\tau).$$
1.4 American options: Optimal early exercise policies

*Non-dividend paying asset*

- At any moment when an American call is exercised, its value immediately becomes $\max(S - X, 0)$. The exercise value is less than $\max(S - XB(\tau), 0)$, the lower bound of the call value if the American call remains alive. The act of exercising prior to expiry causes a decline in value of the American call.

- Since the early exercise privilege is forfeited, the American and European call values should be the same when the underlying asset does not pay any dividend within the life of the American call option.

- For an American put, it may become optimal to exercise prematurely when $S$ falls to sufficiently low value. Since the gain in time value of strike is zero when the interest rate is zero, an American put is never exercised prematurely under such scenario.
Dividend paying asset

- When the underlying asset pays dividends, the early exercise of an American call prior to expiry may become optimal when (i) \( S \) is very high and (ii) the dividends are sizable. Under these circumstances, it then becomes more attractive for the investor to acquire the asset rather than holding the option.
  - When \( S \) is high, the chance of regret of early exercise is low; equivalently, the insurance value of holding the call is lower.
  - When the dividends are sizable, it is more attractive to hold the asset directly instead of holding the call.

- For an American put, when \( D \) is sufficiently high, it may become non-optimal to exercise prematurely even at very low value of \( S \) (even when the put is very deep-in-the-money).
American call on an asset with discrete dividends

- Since the holder of an American call on an asset with discrete dividends will not receive any dividend in between dividend times, so within these periods, it is never optimal to exercise the American call.

- It may be optimal to exercise the American call *immediately before* the asset goes ex-dividend. What are the necessary and sufficient conditions for optimal early exercise?
One-dividend model – Amount of $D$ is paid out at $t_d$

- If the American call is exercised at $t_d^-$, the call value becomes $S_d^- - X$. The asset price drops to $S_d^+ = S_d^- - D$ right after the dividend payout.

- It behaves like an ordinary European option for $t > t_d^+$. This is because when there is no further dividend, it becomes always non-optimal to exercise the American call.

- The lower bound of the one-dividend American call value at $t_d^+$ is the same lower bond for a European call, which is given by $S_d^+ - X e^{-r(T-t_d^+)}$, where $T - t_d^+$ is the time to expiry.

- By virtue of the continuity of the call value across the dividend date, the lower bound $B$ for the call value at time $t_d^-$ should also be equal to $B = S_d^+ - X e^{-r(T-t_d)} = (S_d^- - D) - X e^{-r(T-t_d)}$. 

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By comparing the call value (continuation) with the early exercise proceed: $S_d^- - X$, we deduce

(i) If $S_d^- - X \leq B$,

$$S_d^- - X \leq (S_d^- - D) - X e^{-r(T-t_d)}$$

or

$$D \leq X[1 - e^{-r(T-t_d)}]$$

it is never optimal to exercise since exercising leads to a drop in call value.

(ii) When the discrete dividend $D$ is sufficiently deep such that

$$D > X[1 - e^{-r(T-t_d)}],$$

it may become optimal to exercise at $t_d^-$ when the asset price $S_d^-$ is above some threshold value $S_d^*$. 
Let \( C_d(S_d^-, T - t_d^-) \) denote the American call price at \( t_d^- \), where time to expiry is \( T - t_d^- \). Suppose the American call stays alive, by continuity of call value across the dividend date, we have

\[ C_d(S_d^-, T - t_d^-) = c(S_d^+, T - t_d^+) = c(S_d^- - D, T - t_d^+) . \]

The optimal exercise price \( S_d^* \) is the solution to

\[ c(S_d^- - D, T - t_d^+) = S_d^- - X . \]

When \( D > X[1 - e^{-r(T-t_d^0)}] \), then

\[ C_d(S_d^-, T - t_d^-) = \begin{cases} c(S_d^- - D, T - t_d^+) & \text{when } S_d^- < S_d^* \\ S_d^- - X & \text{when } S_d^- \geq S_d^* \end{cases} . \]

Thus, \( S_d^* \) depends on \( D \), which decreases when \( D \) increases. This is because the price curve of \( c(S_d^- - D, T - t_d^+) \) is lowered and it cuts the intrinsic value line \( \ell_1 \) at a lower value of \( S_d^* \). Financially, the holder chooses to receive the asset even at a lower asset value when the dividend is deeper.
Determination of $S_d^*$ (potential early exercise at $t_d^-$ when $D$ is sufficiently deep)

The European call price function $V = c(S_d^--D,t_d^+)$ falls below the exercise payoff line $\ell_1 : E = S_d^- - X$ when $\ell_1$ lies to the left of the lower bound value line $\ell_2 : B = S_d^- - D - X e^{-r(T-t_d)}$. Here, $S_d^*$ is the value of $S_d^-$ at which the European call price curve cuts the exercise payoff line $\ell_1$. 

\[ c(S_d^--D,t_d^+) \]

\[ \ell_1 \]

\[ \ell_2 \]

\[ S_d^- \]

\[ X \]

\[ S_d^* \]

\[ X e^{-r(T-t_d)} + D \]
Summary of early exercise policies of American calls

- With no dividend, the decision of early exercise of an American option (call or put) depends on the competition between the time value of $X$ and the loss of insurance value associated with the holding of the option.
- Early exercise of non-dividend paying American call is non-optimal since this leads to the loss of insurance value of the call plus the loss of time value of $X$.
- For an American call on a discrete dividend paying asset, it may become optimal to exercise at time right before the ex-dividend time, provided that the dividend amount is sizable and the call is sufficiently deep in-the-money. The critical asset price is a decreasing function of the size of dividend. Early exercise of the American call at a lower asset price level leads to a greater loss of insurance value but the loss is offset by the more sizable dividend received.
Continuous dividend model

Under constant dividend yield $q$, the dividend amount received during $(t, t + dt)$ from holding one unit of asset is $qS_t dt$. Also, $e^{-q(T-t)}$ unit of the asset at time $t$ will become one unit at time $T$ through the accumulation of the dividends into purchase of the asset.

Why do we consider dividend yield model?

- It is considered as a continuous approximation to the discrete dividends model. Otherwise, pricing under the discrete $n$-dividend model requires the joint distribution of asset prices at all dividend dates: $S_{td_1}, S_{td_2}, \ldots, S_{td_n}$.

- The foreign money market account, which serves as the underlying asset in exchange options, earns the foreign interest rate $r_f$ as dividend yield.
When the underlying asset pays dividend yield \( q \), the lower bound of a European call \( c(S, \tau; X, q) \) becomes \( \max(Se^{-q\tau} - Xe^{-r\tau}, 0) \). As \( S \) becomes sufficiently high, \( Se^{-q\tau} - Xe^{-r\tau} \) becomes less than \( S - X \). Actually, as \( S \to \infty \), the value of the European call tends to that of the forward, where

\[
c(S, \tau; X, q) \to Se^{-q\tau} - Xe^{-r\tau}.
\]
Smooth pasting condition at $S^*(\tau)$ under continuous dividend yield model

Value matching at $S^*(\tau)$: $C(S^*(\tau), \tau) = S^*(\tau) - X$.

Smooth pasting at $S^*(\tau)$: \[ \frac{\partial C}{\partial S}(S^*(\tau), \tau) = 1. \]

$S^*(\tau)$ can be visualized as the lowest asset price at which the American call does not depend on the time to expiry. That is,
\[ \frac{\partial C}{\partial \tau} = 0 \quad \text{at} \quad S = S^*(\tau). \]

Find the total derivative of the value matching condition with respect to $\tau$:
\[ \frac{d}{d\tau} [C(S^*(\tau), \tau)] = \frac{\partial C}{\partial \tau}(S^*(\tau), \tau) + \frac{\partial C}{\partial S}(S^*(\tau), \tau) \frac{dS^*(\tau)}{d\tau} = \frac{dS^*(\tau)}{d\tau} \]
so that $\frac{\partial C}{\partial S}(S^*(\tau), \tau) = 1$. The smooth pasting condition can also be derived from the optimality of early exercise (first order derivative condition).
American call on a continuous dividend yield paying asset

The option price curve of a longer-lived American call will be above that of its shorter-lived counterpart for all values of $S$. The upper price curve cuts the intrinsic value tangentially at a higher critical asset value $S^*(\tau)$. Hence, $S^*(\tau)$ for an American call is an increasing function of $\tau$. 
Properties of optimal early exercise boundary $S^*(\tau)$ of an American call under continuous dividend yield model

\[ S^*(0) = X \max(1, \frac{r}{q}), \quad S^*_\infty = \frac{\mu + 1}{\mu + - 1} X, \]
\[ 0 < \mu_+ = - (r - q - \frac{\sigma^2}{2}) + \sqrt{(r - q - \frac{\sigma^2}{2})^2 + 2\sigma^2 r} \]

\(\text{\textbf{X}} = \text{strike price}, \quad r = \text{riskfree interest rate}, \quad q = \text{constant dividend yield}, \quad \sigma = \text{volatility of asset price}\)
Stopping region = \{ (S, \tau) : S \geq S^*(\tau) \}, inside which the American call should be optimally exercised. When \( S < S^*(\tau) \), it is optimal for the holder to continue holding the American call option.

1. \( S^*(\tau) \) is monotonically increasing with respect to \( \tau \) with

\[
S^*(0^+) = X \max \left( 1, \frac{r}{q} \right) \quad \text{and} \quad S^*_\infty = \frac{\mu_+}{\mu_+ - 1} X.
\]

The determination of \( S^*_\infty \) requires a pricing model of the perpetual American call option.

2. \( S^*(\tau) \) is a continuous function of \( \tau \) when the asset price process is continuous.

3. \( S^*(\tau) \geq X \) for \( \tau \geq 0 \)

Suppose \( S^*(\tau) < X \), then the early exercise proceed \( S^*(\tau) - X \) becomes negative. This must be ruled out.
One-dividend paying model for an American put

- A single dividend $D$ is paid at $t_d$.

- Never exercise immediately prior to the dividend payment for time $t < t_d$, interest income $= X[e^{r(t_d-t)} - 1]$. We would like to find $t_s$ such that the holder of the American put is indifferent to the time value of strike and the dividend. That is,

$$X[e^{r(t_d-t_s)} - 1] = D$$

$$t_s = t_d - \frac{\ln\left(1 + \frac{D}{X}\right)}{r}.$$

- When $t < t_s$, early exercise is optimal when $S$ falls below some critical asset price $S^*(t)$. 


The behavior of the optimal exercise boundary $S^*(t)$ as a function of $t$ for a one-dividend American put option. Note that $S^*(T) = X$ since the underlying asset is non-dividend paying after $t_d$. 
In summary, the optimal exercise boundary $S^*(t)$ of the one-dividend American put model exhibits the following behavior.

(i) When $t < t_s$, $S^*(t)$ first increases then decreases smoothly with increasing $t$ until it drops to the zero value at $t_s$. When the calendar time is well before $t_s$, $S^*(t)$ is increasing as it follows a general trend of increasing monotonically in time. However, when the calendar time comes closer to $t_s$, $S^*(t)$ decreases in time in order to adopt the trend that $S^*(t)$ falls to zero when $t$ reaches $t_s$.

(ii) $S^*(t)$ stays at the zero value in the time interval $[t_s, t_d]$.

(iii) When $t \in (t_d, T]$, $S^*(t)$ is a monotonically increasing function of $t$ with $S^*(T) = X$. 
Here, $t_d < t_1 < t_2 < T$. The put price curve at time $t_1$ intersects the intrinsic value line tangentially at $S^*(t_1)$. We observe: $S^*(t_1) < S^*(t_2) < X$. At longer time to expiry, the American put has to be deeper in the money in order to induce optimal early exercise.