MAFS 5030 - Quantitative Modeling of Derivative Securities

Topic 4 – Extended option models

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4.1 Continuous dividend yield models

Let \( q \) denote the constant continuous dividend yield, that is, the holder receives dividend of amount equal to \( qS \, dt \) within the interval \( dt \). The asset price dynamics is assumed to follow the Geometric Brownian motion

\[
\frac{dS}{S} = \rho \, dt + \sigma \, dZ.
\]

We form a riskless hedging portfolio by short selling one unit of the European call and long holding \( \Delta \) units of the underlying asset. The differential change of the portfolio value \( \Pi \) is given by

\[
d\Pi = -dc + \Delta \, dS + q\Delta S \, dt
= \left( -\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + q\Delta S \right) dt + \left( \Delta - \frac{\partial c}{\partial S} \right) dS.
\]
The last term $q \Delta S \, dt$ is the wealth added to the portfolio due to the dividend payment received. By choosing $\Delta = \frac{\partial c}{\partial S}$, we obtain a riskless hedge for the portfolio. The hedged portfolio should earn the riskless interest rate.

We then have

$$d\Pi = \left( -\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + qS \frac{\partial c}{\partial S} \right) \, dt = r \left( -c + S \frac{\partial c}{\partial S} \right) \, dt,$$

which leads to

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + (r - q)S \frac{\partial c}{\partial S} - rc, \quad \tau = T - t, \quad 0 < S < \infty, \quad \tau > 0.$$
Martingale pricing approach

Suppose all the dividend yields received are used to purchase additional units of asset, then the wealth process of holding one unit of the underlying asset initially is given by

\[ \hat{S}_t = e^{qt} S_t, \]

where \( e^{qt} \) represents the growth factor in the number of units. The wealth process \( \hat{S}_t \) follows

\[ \frac{d\hat{S}_t}{\hat{S}_t} = (\rho + q) \, dt + \sigma \, dZ_t^P. \]

We would like to find the equivalent risk neutral measure \( Q \) under which the discounted wealth process \( \hat{S}_t^* \) is \( Q \)-martingale. We choose \( \gamma(t) \) in the Radon-Nikodym derivative to be

\[ \gamma(t) = \rho + q - r. \]
Now $Z^Q_t$ is a $Q$-Brownian motion and

$$dZ^Q_t = dZ^P_t + \frac{\rho + q - r}{\sigma} dt.$$ 

Also, $\hat{S}^*_t$ becomes $Q$-martingale since

$$\frac{d\hat{S}^*_t}{\hat{S}^*_t} = \sigma dZ^Q_t.$$ 

The asset price $S_t$ under the equivalent risk neutral measure $Q$ becomes

$$\frac{dS_t}{S_t} = (r - q) dt + \sigma dZ^Q_t.$$ 

Hence, the risk neutral drift rate of $S_t$ is $r - q$.

**Analogy with the foreign currency options**

The continuous yield model is also applicable to options on foreign currencies where the continuous dividend yield can be considered as the yield due to the interest earned by the foreign currency at the foreign interest rate $r_f$. 
Call and put price formulas

The price of a European call option on a continuous dividend paying asset can be obtained by changing $S$ to $Se^{-q\tau}$ in the price formula since the asset price $S$ will be depleted at the rate $q$ due to payment of dividend yield. Suppose we let $\hat{S} = Se^{-q\tau}$, then the option pricing equation becomes

$$\frac{\partial c}{\partial \tau} = \sigma^2 \hat{S}^2 \frac{\partial^2 c}{\partial \hat{S}^2} + r\hat{S} \frac{\partial c}{\partial \hat{S}} - rc.$$ 

The European call price formula with continuous dividend yield $q$ is

$$c(S, \tau) = Se^{-q\tau} N(\hat{d}_1) - Xe^{-r\tau} N(\hat{d}_2),$$

where

$$\hat{d}_1 = \frac{\ln \frac{S}{X} + (r - q + \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}, \quad \hat{d}_2 = \hat{d}_1 - \sigma \sqrt{\tau}.$$ 

Alternatively, knowing that the expected rate of return of $S_t$ is $r - q$, we can deduce that

$$c(S, \tau) = e^{-r\tau}[Se^{(r-q)\tau} N(\hat{d}_1) - XN(\hat{d}_2)].$$
Similarly, the European put formula with continuous dividend yield $q$ can be deduced from the Black-Scholes put price formula to be

$$p = X e^{-r\tau} N(-\hat{d}_2) - S e^{-q\tau} N(-\hat{d}_1).$$

The new put and call prices satisfy the put-call parity relation

$$p = c - S e^{-q\tau} + X e^{-r\tau}.$$

Furthermore, the following put-call symmetry relation can also be deduced from the above call and put price formulas

$$c(S, \tau; X, r, q) = p(X, \tau; S, q, r).$$

That is, the put price formula can be obtained from the corresponding call price formula by interchanging $S$ with $X$ and $r$ with $q$ in the formula.
Recall that a call option entitles its holder the right to exchange the riskless asset for the risky asset, and vice versa for a put option. The dividend yield earned from the risky asset is $q$ while that from the riskless asset is $r$.

If we interchange the roles of the riskless asset and risky asset in a call option, the call becomes a put option, thus giving the justification for the put-call symmetry relation.

As a verification, consider

$$
p(X, \tau; S, q, r) = Se^{-q\tau} N \left( -\frac{\ln \frac{X}{S} + \left( q - r - \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}} \right) - X e^{-r\tau} N \left( -\frac{\ln \frac{X}{S} + \left( q - r + \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}} \right) = Se^{-q\tau} N(d_1) - X e^{-r\tau} N(d_2) = c(S, \tau; X, r, q).$$
Time dependent parameters

Suppose the model parameters become time dependent functions, the Black-Scholes equation has to be modified as follows

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2(\tau)}{2} S^2 \frac{\partial^2 V}{\partial S^2} + [r(\tau) - q(\tau)] S \frac{\partial V}{\partial S} - r(\tau)V, \quad 0 < S < \infty, \quad \tau > 0,$$

where $V$ is the price of the derivative security.

When we apply the following transformations: $y = \ln S$ and $w = e^{\int_0^\tau r(u) \, du} V$, then

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2(\tau)}{2} \frac{\partial^2 w}{\partial y^2} + \left[ r(\tau) - q(\tau) - \frac{\sigma^2(\tau)}{2} \right] \frac{\partial w}{\partial y}.$$

Consider the following form of the fundamental solution

$$f(y, \tau) = \frac{1}{\sqrt{2\pi s(\tau)}} \exp \left( -\frac{(y + e(\tau))^2}{2s(\tau)} \right),$$

which satisfies the initial condition: $f(y, 0^+) = \delta(y)$. 
By direct differentiation, it can be shown that $f(y, \tau)$ satisfies the parabolic equation

$$\frac{\partial f}{\partial \tau} = \frac{1}{2} s'(\tau) \frac{\partial^2 f}{\partial y^2} + e'(\tau) \frac{\partial f}{\partial y}.$$ 

Suppose we let

$$s(\tau) = \int_0^\tau \sigma^2(u) \, du$$

$$e(\tau) = \int_0^\tau [r(u) - q(u)] \, du - \frac{s(\tau)}{2},$$

then $f$ satisfies the same differential equation as that for $w(y, \tau)$. One can deduce that the fundamental solution is given by

$$\phi(y, \tau) = \frac{1}{\sqrt{2\pi \int_0^\tau \sigma^2(u) \, du}} \exp\left(-\frac{\{y + \int_0^\tau [r(u) - q(u) - \frac{\sigma^2(u)}{2}] \, du\}^2}{2 \int_0^\tau \sigma^2(u) \, du}\right).$$

Given the initial condition $w(y, 0)$, the solution can be expressed as

$$w(y, \tau) = \int_{-\infty}^\infty w(\xi, 0) \phi(y - \xi, \tau) \, d\xi.$$
Note that the time dependency of the coefficients $r(\tau), q(\tau)$ and $\sigma^2(\tau)$ will not affect the spatial integration with respect to $\xi$. We may simply make the following substitutions in the option price formulas

- $r$ is replaced by $\frac{1}{\tau} \int_0^\tau r(u) \, du$
- $q$ is replaced by $\frac{1}{\tau} \int_0^\tau q(u) \, du$
- $\sigma^2$ is replaced by $\frac{1}{\tau} \int_0^\tau \sigma^2(u) \, du$.

For example, the European call price formula is modified as follows:

$$c = S e^{-\int_0^\tau q(u) \, du} \, N(\widetilde{d}_1) - X e^{-\int_0^\tau r(u) \, du} \, N(\widetilde{d}_2)$$

where

$$\widetilde{d}_1 = \ln \frac{S}{X} + \int_0^\tau [r(u) - q(u) + \frac{\sigma^2(u)}{2}] \, du \quad \widetilde{d}_2 = \widetilde{d}_1 - \sqrt{\int_0^\tau \sigma^2(u) \, du}.$$
4.2 Exchange options

- An exchange option is an option that gives the holder the right but not the obligation to exchange one risky asset for another.

- Let $X_t$ and $Y_t$ be the price processes of the two risky assets.

- The terminal payoff of a European exchange option at maturity $T$ of exchanging $Y_T$ for $X_T$ is given by $\max(X_T - Y_T, 0)$.

Under the risk neutral measure $Q$, let $X_t$ and $Y_t$ be governed by

$$
\frac{dX_t}{X_t} = (r - q_X)\, dt + \sigma_X\, dZ^Q_{X,t} \quad \text{and} \quad \frac{dY_t}{Y_t} = (r - q_Y)\, dt + \sigma_Y\, dZ^Q_{Y,t},
$$

where $r$ is the constant riskless interest rate, $\sigma_X$ and $\sigma_Y$ are the constant volatility of $X_t$ and $Y_t$, respectively, $q_X$ and $q_Y$ are the dividend yield of $X_t$ and $Y_t$, respectively. Also, the two standard Brownian motions are correlated with $dZ^Q_{X,t} dZ^Q_{Y,t} = \rho\, dt$, where $\rho$ is correlation coefficient.
Numeraire Invariance Theorem

The choice of the money market account $M(t)$ as the numeraire is not unique in order that the risk neutral valuation formula holds. We may choose the price of a tradable asset as the numeraire. Let $N(t)$ be a numeraire whereby we have the existence of an equivalent probability measure $Q_N$ such that all security prices normalized with respect to $N(t)$ are $Q_N$-martingale. The self-financing trading strategy to achieve replication of an attainable contingent claim remains invariance under the change of numeraire.

For convenience, we take the current time to be time zero. The time-0 arbitrage price $V(0)$ of a contingent claim $Y$ as given by the martingale pricing approach under both measures should agree. Since the deflated prices $\frac{V(t)}{M(t)}$ and $\frac{V(t)}{N(t)}$ are $Q$-martingale and $Q_N$-martingale, respectively, $0 \leq t \leq T$, we should expect

$$V(0) = M(0)E_Q \left[ \frac{Y}{M(T)} | \mathcal{F}_0 \right] = N(0)E_{Q_N} \left[ \frac{Y}{N(T)} | \mathcal{F}_0 \right].$$
**Determination of the Radon-Nikodym derivative**

How to find the appropriate $\frac{dQ_N}{dQ}|_{\mathcal{F}_0}$ such that the above formula holds? Suppose we adopt the change of measure from $Q_N$ to $Q$ by the choice of the following Radon-Nikodym derivative, where

$$\frac{dQ_N}{dQ}|_{\mathcal{F}_0} = \frac{N(T)}{N(0)} / \frac{M(T)}{M(0)}.$$

Observing that both $N(0)$ and $M(0)$ are measurable with respect to $\mathcal{F}_0$, we then have

$$N(0)E_{Q_N} \left[ \frac{Y}{N(T)} \mid \mathcal{F}_0 \right] = N(0)E_Q \left[ \frac{Y}{N(T)} \frac{N(T)}{N(0)} / \frac{M(T)}{M(0)} \mid \mathcal{F}_0 \right]$$

$$= M(0)E_Q \left[ \frac{Y}{M(T)} \mid \mathcal{F}_0 \right],$$

which agrees with the earlier martingale pricing formula under the change of numeraire.
Use of the underlying asset as the numeraire (share measure) and the associated change of measure

Recall that the risk neutral measure uses the money market account as the numeraire. Let the starting time be time zero for notational convenience. Consider the Radon-Nikodym derivative $L_t$ as a stochastic process that is defined by taking the ratio of the asset numeraire and the money market account

$$L_t = \left. \frac{dQ^S}{dQ} \right|_{\mathcal{F}_0} = e^{qt} \frac{S_t}{S_0} / \frac{M_t}{M_0}, \quad t \in (0, T],$$

where $M_t = e^{rt}$ is the money market account and $q$ is the dividend yield of the underlying asset. The inclusion of the factor $e^{qt}$ means one unit of asset initially grows to $e^{qt}$ units after time $t$ if all dividends are invested into the purchase of new units of the risky asset. We examine the change of measure from $Q$ to $Q^S$ as effected by $L_t$. 
Symbolically, we write

\[ L_t = \left. \frac{dQ^S}{dQ} \right|_{\mathcal{F}_t}, \quad t \in (0, T]. \]

Under the risk neutral measure \( Q \), the dynamics of \( S_t \) is governed by

\[ \frac{dS_t}{S_t} = (r - q) dt + \sigma dZ_t^Q, \quad Z_t^Q \text{ is } Q\text{-Brownian}. \]

The solution to \( S_t \) is given by

\[ S_t = S_0 e^{(r-q-\frac{\sigma^2}{2})t+\sigma Z_t^Q} \]

so that

\[ L_t = e^{qt} \left. \frac{S_t}{S_0} \right/ e^{rt} = e^{-\frac{\sigma^2}{2}t+\sigma Z_t^Q}, \quad t \in (0, T]. \]

This corresponds to the choice of \( \gamma = -\sigma \) in the Girsanov theorem. We then deduce that

\[ Z_t^{Q^S} = Z_t^Q - \sigma t \text{ is a } Q^S\text{-Brownian}. \]

We commonly call \( Q^S \) to be the share measure with respect to \( S_t \).
As a check, we consider

\[ V_0 = e^{-rT} E_Q[V_T(S_T)] = e^{-rT} E_Q S \left[ \frac{V_T(S_T)}{L_T} \right] = S_0 e^{-qT} E_Q S \left[ \frac{V_T(S_T)}{S_T} \right], \]

so that

\[ \frac{V_0}{\widehat{S}_0} = E_Q S \left[ \frac{V_T(S_T)}{\widehat{S}_T} \right], \quad \text{where } \widehat{S}_t = e^{qt} S_t. \]

Hence, \( V_t/\widehat{S}_t \) is \( Q^S \)-martingale.

Write \( Q^X \) as the share measure with respect to the asset price process \( X_t \). We deduce that

\[ Z_{X,t}^{Q^X} = Z_{X,t}^Q - \sigma_X t \]

is \( Q^X \)-Brownian.

Since \( X_t \) and \( Y_t \) have correlation coefficient \( \rho \), we expect that subtracting \( \rho \sigma_X t \) from \( Z_{Y,t}^Q \) would make \( Z_{Y,t}^Q - \rho \sigma_X t \) to be \( Q^X \)-Brownian.
How to verify that $Z_{Y,t}^{QX} = Z_{Y,t}^Q - \rho \sigma_X t$ is $Q^X$-Brownian?

By considering the moment generating function, it suffices to show

$$E_{Q^X}[\exp(\alpha Z_{Y}^{QX}(T))] = E_{Q^X}[\exp(\alpha Z_{Y}^Q(T) - \alpha \rho \sigma_X T)] = \exp\left(\frac{\alpha^2}{2} T\right).$$

Recall

$$L_T = \left. \frac{dQ^X}{dQ} \right|_{\mathcal{F}_0} = \exp\left(-\frac{\sigma_X^2}{2} T + \sigma_X Z_{X}^Q(T)\right),$$

so

$$E_{Q^X}[\exp(\alpha Z_{Y}^{QX}(T))]$$

$$= E_Q \left[ \exp(\alpha Z_{Y}^Q(T) - \alpha \rho \sigma_X T) \exp\left(-\frac{\sigma_X^2}{2} T + \sigma_X Z_{X}^Q(T)\right) \right]$$

$$= \exp\left(-\frac{\sigma_X^2}{2} T - \alpha \rho \sigma_X T\right) E_Q[\exp \left( \alpha Z_{Y}^Q(T) + \sigma_X Z_{X}^Q(T) \right)]$$

$$= \exp\left(-\frac{\sigma_X^2}{2} T - \alpha \rho \sigma_X T\right) \exp\left(\frac{\alpha^2 + 2\rho \alpha \sigma_X + \sigma_X^2}{2} T\right)$$

$$= \exp\left(\frac{\alpha^2}{2} T\right).$$
Derivation of the price formula of an exchange option

Suppose we choose $e^{qX_t}X_t$ as the numeraire, the corresponding Radon-Nikodym derivative that effects the change from $Q$ to $Q^X$ is given by

$$L_T = e^{(qX-r)T} \frac{X_T}{X_0}.$$ 

The price function of the exchange option with maturity $T$ and initial asset values $X_0$ and $Y_0$ is given by

$$V(X_0, Y_0; T) = e^{-rT}E_Q[\max(X_T - Y_T, 0)]$$

$$= e^{-rT}E_{Q^X} \left[ \frac{X_0e^{(r-qX)T}}{X_T}X_T \left(1 - \frac{Y_T}{X_T}\right) 1_{\{Y_T/X_T < 1\}} \right].$$

Setting $W_T = Y_T/X_T$, then

$$\frac{V(X_0, Y_0; T)}{X_0} = e^{-qX T}E_{Q^X}[(1 - W_T) 1_{\{W_T < 1\}}].$$

This nice feature of dimension reduction of the option model does not work if the terminal payoff becomes $\max(X_T - Y_T - K, 0)$. 
From Ito’s lemma, the dynamics of $W_t$ under $Q$ is given by

$$\frac{dW_t}{W_t} = [(r - q_Y) - (r - q_X) - \rho \sigma_X \sigma_Y + \sigma_X^2] dt + \sigma_Y dZ_{Y,t}^Q - \sigma_X dZ_{X,t}^Q.$$ 

We observe that $Z_{X,t}^Q$ and $Z_{Y,t}^Q$ as defined by

$$dZ_{X,t}^Q = dZ_{X,t}^Q - \sigma_X dt$$

and

$$dZ_{Y,t}^Q = dZ_{Y,t}^Q - \rho \sigma_X dt$$

are $Q^X$-Brownian motions. The dynamics of $W_t$ under $Q^X$ becomes

$$\frac{dW_t}{W_t} = (q_X - q_Y) dt + \sigma_Y dZ_{Y,t}^Q - \sigma_X dZ_{X,t}^Q.$$ 

We deduce that $W_t$ remains to be a Geometric Brownian motion, and

$$\sigma_W^2 = \sigma_Y^2 - 2\rho \sigma_X \sigma_Y + \sigma_X^2$$

and

$$\mu_W = q_X - q_Y = (r - q_Y) - (r - q_X)$$

under $Q^X$. We may write

$$\frac{dW_t}{W_t} = (q_X - q_Y) dt + \sigma_W dZ_{W,t}^Q,$$

where $Z_{W,t}^Q$ is $Q_X$-Brownian.
The payoff \((1 - W_T)1_{\{W_T < 1\}}\) resembles a put payoff with unit strike and underlying \(W_t\). Using the put price formula, we deduce

\[
E_{Q_X}[(1 - W_T)1_{\{W_T < 1\}}] = N(d_X) - W_0 e^{(q_X - q_Y)T} N(d_Y), \quad W_0 = \frac{Y_0}{X_0},
\]

where

\[
d_X = \frac{\ln \frac{X_0}{Y_0} + (q_Y - q_X)T + \frac{\sigma_Y^2 T}{2}}{\sigma_W \sqrt{T}},
\]

\[
d_Y = \frac{\ln \frac{X_0}{Y_0} + (q_Y - q_X)T - \frac{\sigma_Y^2 T}{2}}{\sigma_W \sqrt{T}}.
\]

Finally, the price function of the exchange option is given by

\[
V(X_0, Y_0; T) = e^{-q_X T} X_0 N(d_X) - e^{-q_Y T} Y_0 N(d_Y).
\]

\[
= e^{-r T} \left[ e^{(r - q_X)T} X_0 N \left( \frac{\ln \frac{X_0}{Y_0} + \left[ (r - q_X) - (r - q_Y) + \frac{\sigma_Y^2}{2} \right] T}{\sigma_W \sqrt{T}} \right) \\
- e^{(r - q_Y)T} Y_0 N \left( -\frac{\ln \frac{Y_0}{X_0} + \left[ (r - q_Y) - (r - q_X) + \frac{\sigma_Y^2}{2} \right] T}{\sigma_W \sqrt{T}} \right) \right].
\]

Suppose we take \(Y_t\) to be the fixed strike price \(K\). By setting \(Y_0 = K\), \(q_Y = r\) and \(\sigma_Y^2 = \sigma_X^2\) (since \(\sigma_Y = 0\)), we recover the usual call price formula.
4.3 Quanto option – equity options with exchange rate risk exposure

- A quanto option is an option on a foreign currency denominated asset but the payoff is in domestic currency.
- The holder of a quanto option is exposed to both exchange rate risk and equity risk.

Some examples of quanto call options are listed below:

1. Foreign equity call struck in foreign currency

\[ c_1(S_T, F_T, T) = F_T \max(S_T - X_f, 0) . \]

Here, \( F_T \) is the terminal exchange rate, \( S_T \) is the terminal price of the underlying foreign currency denominated asset and \( X_f \) is the strike price in foreign currency.
2. Foreign equity call struck in domestic currency

\[ c_2(S_T, T) = \max(F_T S_T - X_d, 0) \]

Here, \( X_d \) is the strike price in domestic currency.

3. Fixed exchange rate foreign equity call

\[ c_3(S_T, T) = F_0 \max(S_T - X_f, 0) \]

Here, \( F_0 \) is some predetermined fixed exchange rate.

4. Equity-linked foreign exchange call

\[ c_4(S_T, T) = S_T \max(F_T - X_F, 0) \]

Here, \( X_F \) is the strike price on the exchange rate. The holder plans to purchase the foreign asset any way but wishes to place a floor value \( X_F \) on the exchange rate. If it happens that the terminal exchange rate \( F_T \) shoots beyond \( X_F \), she receives compensation from the positive payoff received through holding the foreign exchange call.
Quanto prewashing techniques

• Let $S_t$ and $F_t$ denote the stochastic process of the foreign asset price and exchange rate, respectively.

• Define $S^*_t = F_t S_t$, which is the foreign asset price in domestic currency.

• Let $r_d$ and $r_f$ denote the constant domestic and foreign interest rate, respectively, and let $q$ denote the dividend yield of the foreign asset.

• We assume that both $S_t$ and $F_t$ follow the Geometric Brownian motion.
• Under the domestic risk neutral measure $Q_d$, the drift rate of $S^*$ and $F$ are

$$\delta^d_{S^*} = r_d - q \quad \text{and} \quad \delta^d_F = r_d - r_f.$$  

• The reciprocal of $F$ can be considered as the foreign currency price of one unit of domestic currency.

• The drift rate of $S$ and $1/F$ under the foreign risk neutral measure $Q_f$ are given by

$$\delta^f_S = r_f - q \quad \text{and} \quad \delta^f_{1/F} = r_f - r_d,$$

respectively. Note that the dividend yield is the same for the foreign asset in the two-currency world.

• “Quanto prewashing” means finding $\delta^d_S$, that is, the drift rate in the stochastic price process of the foreign currency denominated asset $S$ under the domestic risk neutral measure $Q_d$. 
Let the dynamics of $S_t$ and $F_t$ under $Q_d$ be governed by

$$\frac{dS_t}{S_t} = \delta^d_S dt + \sigma_S dZ^d_S$$
$$\frac{dF_t}{F_t} = \delta^d_F dt + \sigma_F dZ^d_F,$$

where $dZ^d_S dZ^d_F = \rho dt$, $\sigma_S$ and $\sigma_F$ are the volatility of $S_t$ and $F_t$, respectively. Since $S^*_t = F_t S_t$, we obtain from Ito’s lemma (see Problem 3 in HW3):

$$\delta^d_{S^*} = \delta^d_{FS} = \delta^d_F + \delta^d_S + \rho \sigma_F \sigma_S.$$

The extra drift rate $\rho \sigma_F \sigma_S$ arises from the correlated diffusion movements of $F_t$ and $S_t$, where $dZ^d_S dZ^d_F = \rho dt$. We then obtain

$$\delta^d_S = \delta^d_{S^*} - \delta^d_F - \rho \sigma_F \sigma_S = (r_d - q) - (r_d - r_f) - \rho \sigma_F \sigma_S = r_f - q - \rho \sigma_F \sigma_S.$$

Comparing with $\delta^f_S = r_f - q$, we need to add the quanto prewashing term $-\rho \sigma_F \sigma_S$ when we specify the dynamics of $S_t$ changing from $Q_f$ to $Q_d$. 

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**Siegel’s paradox** \( \delta_{1/F}^d = r_f - r_d + \sigma_F^2 = \delta_{1/F}^f + \sigma_F^2 \)

Given that the dynamics of \( F_t \) under \( Q_d \) is

\[
\frac{dF_t}{F_t} = (r_d - r_f) \, dt + \sigma_F \, dZ_d,
\]

then the process for \( 1/F_t \) under \( Q_d \) is (see Problem 3 in HW3)

\[
\frac{d(1/F_t)}{1/F_t} = (r_f - r_d + \sigma_F^2) \, dt - \sigma_F \, dZ_d.
\]

This is seen as a puzzle to many people since the risk neutral drift rate for \( 1/F \) is expected to be \( r_f - r_d \) instead of \( r_f - r_d + \sigma_F^2 \).

We observe directly from the above SDE’s for \( 1/F_t \) that

\[\sigma_F = \sigma_{1/F} \quad \text{and} \quad \rho_{F,1/F} = -1.\]

Note that \( \delta_{1/F}^f = r_f - r_d \). This is also consistent with the quanto prewashing technique when it is applied to \( 1/F \), where the added prewashing term \( -\rho \sigma_F \sigma_{1/F} \) becomes \(-(-1)\sigma_F^2 = \sigma_F^2 \).
An interesting application of Siegel’s paradox

Suppose the terminal payoff of an exchange rate option is $F_T \mathbf{1}_{\{F_T > K\}}$. Let $V^d(F, t)$ denote the value of the option in the domestic currency world. Define

$$V^f(F_t, t) = V^d(F_t, t)/F_t,$$

so that the terminal payoff of the exchange rate option in foreign currency world is $\mathbf{1}_{\{F_T > K\}}$. Now

$$V^f(F, t) = e^{-r_f(T-t)} E_t^Q [\mathbf{1}_{\{F_T > K\}} | F_t = F].$$
From $\delta^d_{1/F} = \delta^f_{1/F} + \sigma^2_F$ and observing $\sigma_F = \sigma_{1/F}$, we deduce that

$$\delta^f_F = \delta^d_F + \sigma^2_F.$$ 

This result is consistent with the Siegel formula if we interchange the foreign and domestic currency worlds. We obtain

$$V^d(F,t) = FV^f(F,t) = e^{-rf(T-t)}FN(d) = e^{-rd}\tau e^{\delta^d_F \tau}FN(d)$$

where $\tau = T - t$ and

$$d = \frac{\ln \frac{F}{K} + \left(\delta^f_F - \frac{\sigma^2_F}{2}\right)\tau}{\sigma \sqrt{\tau}}$$

$$= \frac{\ln \frac{F}{K} + \left(r_d - rf + \frac{\sigma^2_F}{2}\right)\tau}{\sigma \sqrt{\tau}}.$$
Price formulas of various quanto options

1. Foreign equity call struck in foreign currency

Let $c^f_1(S, \tau)$ denote the usual vanilla call option on the foreign currency asset in the foreign currency world. The terminal payoff is

$$c^f_1(S, 0) = \max(S - X_f, 0).$$

We treat this call as if it is structured in the foreign currency world. Its value can always be converted into domestic currency using the prevailing exchange rate.
\[ c_1(S, F, \tau) = Fc_1^f(S, \tau) = F \left[ Se^{-q\tau} N(d_1^{(1)}) - X_f e^{-r_f \tau} N(d_2^{(1)}) \right], \]

where

\[ d_1^{(1)} = \frac{\ln \frac{S}{X_f} + \left( \delta_S^f + \frac{\sigma_S^2}{2} \right) \tau}{\sigma_S \sqrt{\tau}}, \quad d_2^{(1)} = d_1^{(1)} - \sigma_S \sqrt{\tau}. \]

Note that both the correlation risk \( \rho \) and exchange rate risk \( \sigma_F \) do not appear in the price formula! This is reasonable since we do not set the exchange rate to some fixed value \( F_0 \).
2. Foreign equity call struck in domestic currency

The terminal payoff in domestic currency is

\[ c_2(S, F, 0) = \max(S^* - X_d, 0), \]

where \( S^* = FS \) is the price of a domestic currency denominated asset. Note that

\[ \delta_{S^*}^d = r_d - q \quad \text{and} \quad \sigma_{S^*}^2 = \sigma_S^2 + 2\rho\sigma_S\sigma_F + \sigma_F^2. \]

The price formula of the foreign equity call is then given by

\[ c_2(S, F, \tau) = S^* e^{-q\tau} N(d_1^{(2)}) - X_d e^{-r_d\tau} N(d_2^{(2)}), \]

where

\[ d_1^{(2)} = \frac{\ln \frac{S^*}{X_d} + \left( \delta_{S^*}^d + \frac{\sigma_{S^*}^2}{2} \right) \tau}{\sigma_{S^*}\sqrt{\tau}}, \quad d_2^{(2)} = d_1^{(2)} - \sigma_{S^*}\sqrt{\tau}. \]
3. Fixed exchange rate foreign equity call

The terminal payoff is denominated in the domestic currency world, so the drift rate $\delta^d_S$ of the foreign asset in $Q_d$ should be used. The price function of the fixed exchange rate foreign equity call is given by

$$c_3(S, \tau) = F_0 e^{-r_d \tau} \left[ S e^{\delta^d_S \tau} N(d_1^{(3)}) - X_f N(d_2^{(3)}) \right],$$

where

$$d_1^{(3)} = \frac{\ln \frac{S}{X_f} + \left( \delta^d_S + \frac{\sigma^2_S}{2} \right) \tau}{\sigma_S \sqrt{\tau}}, \quad d_2^{(3)} = d_1^{(3)} - \sigma_S \sqrt{\tau}.$$

- The price formula does not depend on the exchange rate $F$ since the exchange rate has been chosen to be the fixed value $F_0$.
- The currency exposure of the call is embedded in the quanto-prewashing term $-\rho \sigma_S \sigma_F$ in $\delta^d_S$. This call option has exposure to both correlation risk and exchange rate risk.
4. Equity-linked foreign exchange call

Write the terminal payoff in the form of an exchange option

\[ c_4(S, F, 0) = \max(S^* - XS, 0). \]

Taking the two assets to be an exchange \( XS \) for \( S^* \), the ratio of the two assets is \( \frac{S^*}{XS} = \frac{F}{X} \) and the difference of the drift rates under \( Q_d \) is \( \delta_{S^*}^d - \delta_S^d = r_d - q - (r_f - q - \rho \sigma_S \sigma_F) = r_d - r_f + \rho \sigma_F \sigma_S. \)

\[
c_4(S, \tau) = e^{-r_d \tau} \left[ S^* e^{\delta_{S^*}^d \tau} N(d_1^{(4)}) - XS e^{\delta_S^d \tau} N(d_2^{(4)}) \right] = S e^{-q \tau} \left[ F N(d_1^{(4)}) - X e^{(r_f - r_d - \rho \sigma_F \sigma_S) \tau} N(d_2^{(4)}) \right],
\]

where

\[
d_1^{(4)} = \ln \frac{F}{X} + \left( r_d - r_f + \rho \sigma_F \sigma_S + \frac{\sigma_F^2}{2} \right) \tau \div \sigma_F \sqrt{\tau}, \quad d_2^{(4)} = d_1^{(4)} - \sigma_F \sqrt{\tau}.
\]
Digital quanto option relating 3 currency worlds

$F_{S\backslash U} = \text{SGD currency price of one unit of USD currency}$

$F_{H\backslash S} = \text{HKD currency price of one unit of SGD currency}$

**Example 1**

Digital quanto option payoff: pay one HKD if $F_{S\backslash U}$ is above some strike level $K$.

The dynamics of $F_{S\backslash U}$ under $Q^S$ is governed by

$$
\frac{dF_{S\backslash U}}{F_{S\backslash U}} = (r_{SGD} - r_{USD}) \, dt + \sigma_{F_{S\backslash U}} \, dZ^S_{F_{S\backslash U}}.
$$
Given \( \delta^S_{FS\setminus U} = r_{SGD} - r_{USD} \), how to find \( \delta^H_{FS\setminus U} \), which is the risk neutral drift rate of the SGD asset denominated in Hong Kong dollar?

Treating \( FS\setminus U \) as the foreign asset and \( FH\setminus S \) as the exchange rate, by the quanto-prewashing technique

\[
\delta^H_{FS\setminus U} = \delta^S_{FS\setminus U} - \rho \sigma_{FS\setminus U} \sigma_{FH\setminus S}.
\]

Digital option value \( = e^{-r_{HKD}\tau}E^t_{QH}\left[1_{\{FS\setminus U > K\}}\right] = e^{-r_{HKD}\tau}N(d) \)

where

\[
d = \ln \frac{FS\setminus U}{K} + \left( \delta^H_{FS\setminus U} - \frac{\sigma^2_{FS\setminus U}}{2} \right) \tau \\
\sigma_{FS\setminus U} \sqrt{\tau}.
\]
Example 2

The quanto option pays $F_{H\backslash S}$ Hong Kong dollars when $F_{S\backslash U} > K$. This is equivalent to pay one Singaporean dollar. Value of the quanto option in Singaporean dollar is

$$e^{-r_{SGD} \tau} E_{Q_S}^t \left[ 1 \{F_{S\backslash U} > K\} \right] = e^{-r_{SGD} \tau} N(\hat{d})$$

where

$$\hat{d} = \ln \frac{F_{S\backslash U}}{K} + \left( \delta_{FS\backslash U}^S - \frac{\sigma_{FS\backslash U}^2}{2} \right) \tau$$

$$\delta_{FS\backslash U}^S = r_{SGD} - r_{USD}.$$

This option model is similar to $c_1(S, F, \tau)$, where the option payoff in foreign currency is converted into domestic currency using the prevailing exchange rate at maturity. The most efficient approach is to perform valuation of the option under the foreign currency world. The value of the quanto option in Hong Kong dollar is

$$F_{H\backslash S} e^{-r_{SGD} \tau} N(\hat{d}).$$
Example 3

The quanto option pays $F_{H\backslash U}$ Hong Kong dollars when $F_{S\backslash U} > K$. This is equivalent to pay one US dollars.

**Method One**

Observe that $F_{H\backslash U} = F_{H\backslash S}F_{S\backslash U}$ so that it is like paying $F_{S\backslash U}$ Singaporean dollars when $F_{S\backslash U} > K$.

Value of the quanto option in Hong Kong dollars is

$$F_{H\backslash U}e^{-r_{SGD}\tau}E_{Q_S}[F_{S\backslash U}\mathbf{1}_{\{F_{S\backslash U} > K\}}] = F_{H\backslash U}e^{-r_{SGD}\tau}e^{(r_{SGD}\tau - r_{USD})\tau}F_{S\backslash U}N(d_1)$$

$$= F_{H\backslash U}e^{-r_{USD}\tau}N(d_1)$$

where

$$d_1 = \frac{\ln \frac{F_{S\backslash U}}{K} + \left(r_{SGD} - r_{USD} + \frac{\sigma_{FS\backslash U}^2}{2}\right)\tau}{\sigma_{FS\backslash U}\sqrt{\tau}}.$$
Method Two

The quanto option pays one US dollars when $F_{S\setminus U} > K \iff \frac{1}{K} > \frac{1}{F_{S\setminus U}} = F_{U\setminus S}$. Later, we multiply the option value in US currency by the exchange rate $F_{H\setminus U}$ to convert into Hong Kong dollars.

Value of the quanto option in US dollars is

$$e^{-r_{USD}\tau}E_{QU}^{t} \left[ 1 \{ F_{U\setminus S} < \frac{1}{K} \} \right] = e^{-r_{USD}\tau}N(-d_2),$$

where

$$d_2 = \frac{\ln \frac{F_{U\setminus S}}{1/K} + \left( r_{USD} - r_{SGD} - \frac{\sigma_{F_{U\setminus S}}^2}{2} \right) \tau}{\sigma_{F_{U\setminus S}}\sqrt{\tau}} = -d_1.$$

Remark  The quanto option value in Hong Kong dollars using the two approaches agree with each other.
4.4 Implied volatilities and volatility smiles

The difficulties of setting volatility value in the option price formulas lie in the fact that the input value should be the forecast volatility value over the remaining life of the option rather than an estimated volatility value from the past market data of the asset price (historical volatility).

The Black-Scholes model assumes a lognormal probability distribution of the asset price at all future times. Since volatility is the only unobservable parameter in the Black-Scholes model, the model gives the option price as a function of volatility. The Black-Scholes implied volatility $\sigma_{imp}(X,T)$ is the unique solution to

$$V_{market}(X,T) = V^{BS}(S,t; X, T, \sigma_{imp}(X, T)).$$

The above equation is an answer to: what volatility is implied in observed option prices, if the Black-Scholes model is a valid description of the market conditions?
Volatility smiles and volatility term structures

- In financial markets, it becomes a common practice for traders to quote an option’s market price in terms of implied volatility $\sigma_{imp}$.

- In particular, several implied volatility values obtained simultaneously from different options with varying maturities and strike prices on the same underlying asset provide an extensive market view about the volatility at varying strikes and maturities.

- The Black-Scholes (BS) implied volatility computed from the market option price by inverting the BS price formula varies with strike price and time to expiration – volatility smile (skew) and volatility term structure, respectively. The plot of implied volatilities against moneyness ($X/S$) and time to expiration $T-t$ generates the implied volatility surface.
Implied volatility surface

DAX option implied volatilities (as black dots) on 2000/05/02. The lower left axis is moneyness and right axis is time to expiration measured in years.

- $\sigma_{imp}(X,T)$ is non-linear in strikes and time to expiration; and if observed over in calendar time, it is also time-dependent.
Patterns of volatility smiles before and after 1987 market crash

If we plot the implied volatility of the exchange-traded options, like index options, against their strike price for a fixed maturity, the curve is typically convex in shape, rather than a straight horizontal line as suggested by the simple Black-Scholes model. This phenomenon is commonly called the *volatility smile* by market practitioners.

These smiles exhibit widely differing properties, depending on whether the market data were taken before or after the October, 1987 market crash.
The implied volatility values are obtained by averaging over exchange-traded European index options of different maturities.

A typical pattern of pre-crash smile. The implied volatility curve is convex with a dip.
A typical pattern of post-crash smile. The implied volatility drops against $X/S$, indicating that out-of-the-money puts ($X/S < 1$) are traded at higher implied volatility than out-of-the-money calls ($X/S > 1$). The market price of the out-of-the-money puts became more expensive than the Black-Scholes price after the 1987 crash (investors are generally worried about market clashes and buy puts for protection).
Comparison of the risk neutral probability density of asset price (solid curve) implied from market data and the theoretical lognormal distribution (dotted curve). The risk neutral probability density is thicker at the left tail and thinner at the right tail, indicating that there is a higher change of more acute drop when $S$ is low and a lower chance of further increase when $S$ is high.
Negative correlation between stock price process and volatility process

In real market situation, it is a common occurrence that when the asset price is high, volatility tends to decrease, making it less probable for a higher asset price to be realized. When the asset price is low, volatility tends to increase, that is, it is more probable that the asset price plummets further down. In other words, stock price process and volatility process are in general negatively correlated.
Extreme events in stock price movements

Probability distributions of stock market returns have typically been estimated from historical time series. Unfortunately, common hypotheses may not capture the probability of extreme events. The clash events are rare and may not be present in the historical record.

Examples

1. On October 19, 1987, the two-month S&P 500 futures price fell 29%. Under the lognormal hypothesis of annualized volatility of 20%, this is a $-27$ standard deviation event with probability $10^{-160}$ (virtually impossible).

2. On October 13, 1989, the S&P 500 index fell about 6%, a $-5$ standard deviation event. Under the maintained hypothesis, this should occur only once in 14,756 years.
Different volatilities across time

Supply and demand
When markets are very quiet, the implied volatilities of the near month options are generally lower than those of the far month. When markets are very volatile, the reverse is generally true.

- In very volatile markets, everyone wants or needs to load with gamma. Near-dated options provide the most gamma and the resultant buying pressure will have the effect of pushing prices up.

- In quiet markets, no one wants a portfolio long of near dated options.
Different volatilities for different strike prices

1. *Stock options* – higher volatilities at lower strike and lower volatilities at higher strikes

- In a falling market, everyone needs out-of-the-money puts for insurance and will pay a higher price for the lower strike options.

- Equity fund managers are long billions of dollars worth of stock and writing out-of-the-money call options against their holdings as a way of generating extra income. This pushes the value of out-of-the-money call options down.
2. *Commodity options* – higher volatilities at higher strike and lower volatilities at lower strikes

- Government intervention – no worry about a large price fall. Speculators are tempted to sell puts aggressively.

- Risk of shortages – no upper limit on the price. Demand for higher strike price options.
Numerical calculations of implied volatilities

Since $\sigma$ cannot be solved explicitly in terms of $S, X, r, \tau$ and option price $V$ from the pricing formulas, the determination of the implied volatility must be accomplished by an iterative algorithm as commonly performed for the root-finding procedure for a non-linear equation. The sequence of iterates $\{x_n\}$ generated by the Newton method for solving $f(x) = 0$ is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$  

Recall $V(\sigma_{imp}) = V_{market}$. When applied to the implied volatility calculations, the Newton-Raphson iterative scheme is given by

$$\sigma_{n+1} = \sigma_n - \frac{V(\sigma_n) - V_{market}}{V'(\sigma_n)},$$

where $\sigma_n$ denotes the $n^{th}$ iterate of $\sigma_{imp}$. Provided that the first iterate $\sigma_1$ is properly chosen, the limit of the sequence $\{\sigma_n\}$ converges to the unique solution $\sigma_{imp}$. 

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The above iterative scheme may be rewritten in the following form

\[
\frac{\sigma_{n+1} - \sigma_{imp}}{\sigma_n - \sigma_{imp}} = 1 - \frac{V(\sigma_n) - V(\sigma_{imp})}{\sigma_n - \sigma_{imp}} \frac{1}{V'(\sigma_n)} = 1 - \frac{V'(\sigma^*_n)}{V'(\sigma_n)}.
\]

By virtue of the Mean Value Theorem in calculus, \(\sigma^*_n\) lies between \(\sigma_n\) and \(\sigma_{imp}\). The first iterate \(\sigma_1\) is chosen such that \(V'(\sigma)\) is maximized by \(\sigma = \sigma_1\) to guarantee convergence. We obtain

\[
V'(\sigma) = Sn(d_1) \frac{\partial d_1}{\partial \sigma} - X e^{-r\tau n} d_2(\sigma) \frac{\partial d_2}{\partial \sigma} = \frac{S \sqrt{\tau} e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} > 0 \quad \text{for all } \sigma,
\]

and so

\[
V''(\sigma) = \frac{S \sqrt{\tau} d_1 d_2 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi} \sigma} = \frac{V'(\sigma) d_1 d_2}{\sigma}.
\]

The critical points of the function \(V'(\sigma)\) are given by \(d_1 = 0\) and \(d_2 = 0\), which lead respectively to

\[
\sigma^2 = -2 \frac{\ln \frac{S}{X} + r\tau}{\tau} \quad \text{and} \quad \sigma^2 = 2 \frac{\ln \frac{S}{X} + r\tau}{\tau}.
\]
The above two values of $\sigma^2$ both give $V'''(\sigma) < 0$. We can choose the first iterate $\sigma_1$ to be

$$\sigma_1 = \sqrt{\frac{2}{\tau} \left( \ln \frac{S}{X} + r\tau \right)}.$$

With this choice of $\sigma_1$, $V'(\sigma)$ is maximized at $\sigma = \sigma_1$. Setting $n = 1$ and observing $V'(\sigma_1^*) < V'(\sigma_1)$ [note that $V'(\sigma)$ is maximized at $\sigma = \sigma_1$], we obtain

$$0 < \frac{\sigma_2 - \sigma_{imp}}{\sigma_1 - \sigma_{imp}} < 1.$$

In general, suppose we can establish

$$0 < \frac{\sigma_{n+1} - \sigma_{imp}}{\sigma_n - \sigma_{imp}} < 1, \quad n \geq 1,$$

then the sequence $\{\sigma_n\}$ is monotonic and bounded, so $\{\sigma_n\}$ converges to the unique solution $\sigma_{imp}$. 
Term structure of volatility

The Black-Scholes formulas remain valid under time dependent volatility except that \( \sqrt{\frac{1}{T-t} \int_t^T \sigma(\tau)^2 d\tau} \) is used to replace \( \sigma \).

How to obtain the term structure of volatility \( \sigma(t) \) given the implied volatility measured at time \( t^* \) of a European option expiring at time \( t \)? For an option with time to expiry \( t - t^* \), the substitution of the implied volatility \( \sigma_{imp}(t^*, t) \) into the standard Black-Scholes formula under constant volatility gives the option price. The equivalence of giving the same observed option price by adopting the two different forms of volatility in the two separate option price formulas leads to

\[
\int_{t^*}^{t} \sigma(u)^2 du = \sigma_{imp}^2(t^*, t)(t - t^*). \]

Differentiating with respect to \( t \), we obtain the term structure of volatility in terms of the term structure of implied volatility

\[
\sigma(t) = \sqrt{\sigma_{imp}(t^*, t)^2 + 2(t - t^*)\sigma_{imp}(t^*, t) \frac{\partial \sigma_{imp}(t^*, t)}{\partial t}}. 
\]
Approximation of \( \sigma(t) \) as a piecewise constant function

Practically, we do not have a continuous differentiable implied volatility function \( \sigma_{imp}(t^*, t) \), but rather implied volatilities are available at discrete instants \( t_i, \ i = 1, 2, \ldots, n \). Suppose we assume \( \sigma(t) \) to be piecewise constant over \((t_{i-1}, t_i)\), where \( \sigma(t) = \sigma_i, \ t_{i-1} < t < t_i, \ i = 1, 2, \ldots, n \). We then have

\[
\begin{align*}
\int_{t^*}^{t_i} \sigma^2(\tau) \, d\tau - \int_{t^*}^{t_{i-1}} \sigma^2(\tau) \, d\tau &= (t_i - t^*) \sigma_{imp}^2(t^*, t_i) - (t_{i-1} - t^*) \sigma_{imp}^2(t^*, t_{i-1}) \\
&= \int_{t_{i-1}}^{t_i} \sigma^2(\tau) \, d\tau = \sigma_i^2(t_i - t_{i-1}), \quad t_{i-1} < t < t_i,
\end{align*}
\]

giving

\[
\sigma_i = \sqrt{\frac{(t_i - t^*) \sigma_{imp}^2(t^*, t_i) - (t_{i-1} - t^*) \sigma_{imp}^2(t^*, t_{i-1})}{t_i - t_{i-1}}}, \quad t_{i-1} < t < t_i.
\]
Risk neutral density function

- Let $\psi(S_T, T; S_t, t)$ denote the transition density function of the asset price. The time-$t$ price of a European call with maturity date $T$ and strike price $X$ is given by

$$c(S_t, t; X, T) = e^{-r(T-t)} \int_{X}^{\infty} (S_T - X)\psi(S_T, T; S_t, t) \, dS_T.$$ 

- If we differentiate $c$ with respect to $X$, we obtain

$$\frac{\partial c}{\partial X} = -e^{-r(T-t)} \int_{X}^{\infty} \psi(S_T, T; S_t, t) \, dS_T;$$

and differentiate once more, we have

$$\psi(X, T; S_t, t) = e^{r(T-t)} \frac{\partial^2 c}{\partial X^2}.$$ 

- Suppose that market European option prices at all strikes are available, the risk neutral density function can be inferred completely from the market prices of options with the same maturity and different strikes, without knowing the volatility function.
**Dupire equation and local volatility function**

Assuming that the asset price dynamics under the risk neutral measure is governed by

\[
\frac{dS_t}{S_t} = (r - q)dt + \sigma(S_t, t)dz_t,
\]

where the local volatility function is assumed to have both state and time dependence. Write \( c = c(X, T) \), the Dupire equation takes the form

\[
\frac{\partial c}{\partial T} = -qc - (r - q)X \frac{\partial c}{\partial X} + \frac{\sigma^2(X, T)}{2} X^2 \frac{\partial^2 c}{\partial X^2}.
\]
Proof

We differentiate $\psi(X,T;S_t,t)$ with respect to $T$ to obtain

$$\frac{\partial \psi}{\partial T} = e^{r(T-t)} \left( r \frac{\partial^2 c}{\partial X^2} + \frac{\partial^2 c}{\partial X^2 \partial T} \right),$$

and $\psi(X,T;S,t)$ satisfies the forward Fokker-Planck equation, where

$$e^{-r(T-t)} \frac{\partial \psi}{\partial T} = \frac{\partial^2}{\partial X^2} \left[ \frac{\sigma^2(X,T)}{2} X^2 \psi \right] - \frac{\partial}{\partial X} [(r - q)X \psi].$$

See Problem 3.8 on P.166 in Kwok’s text for a proof of the forward Fokker-Planck equation.
Combining the above equations and eliminating the common factor $e^{r(T-t)}$, we have

$$r \frac{\partial^2 c}{\partial X^2} + \frac{\partial^2}{\partial X^2} \frac{\partial c}{\partial T} = \frac{\sigma^2(X,T)}{2} X^2 \frac{\partial^2 c}{\partial X^2} - \frac{\partial}{\partial X} \left[ (r - q) X \frac{\partial^2 c}{\partial X^2} \right].$$

Integrating the above equation with respect to $X$ twice, we obtain

$$\frac{\partial c}{\partial T} + r c + \left( r - q \right) \left( X \frac{\partial c}{\partial X} - c \right) = \frac{\sigma^2(X,T)}{2} X^2 \frac{\partial^2 c}{\partial X^2} + \alpha(T) X + \beta(T),$$

where $\alpha(T)$ and $\beta(T)$ are arbitrary functions of $T$.

Since all functions involving $c$ in the above equation vanish as $X$ tends to infinity, hence $\alpha(T)$ and $\beta(T)$ must be zero (one needs to check $\lim_{X \to \infty} X \frac{\partial c}{\partial X} = 0$). Grouping the remaining terms in the equation, we obtain the Dupire equation.
From the Dupire equation, we may express the local volatility \( \sigma(X, T) \) explicitly in terms of the call price function and its derivatives, where

\[
\sigma^2(X, T) = \frac{2 \left[ \frac{\partial c}{\partial T} + qc + (r - q)X \frac{\partial c}{\partial X} \right]}{X^2 \frac{\partial^2 c}{\partial X^2}}.
\]

• Suppose a sufficiently large number of market option prices are available at many maturities and strikes, we can estimate the local volatility from the above equation by approximating the derivatives of \( c \) with respect to \( X \) and \( T \) using the market data.

• In real market conditions, market prices of options are available only at limited number of maturities and strikes.
Relationship between local volatility and implied volatility

Dupire’s equation shows how to compute $\sigma_{loc}(X,T)$ from market prices of European options. On the other hand, the market quote prices for European options are in terms of their implied volatilities. One may want to relate $\sigma_{loc}(X,T)$ with $\sigma_{imp}(X,T)$. We have (see Problem 11 in HW4)

$$\sigma_{loc}^2(X,T) = \frac{\sigma_{imp}^2 + 2T \sigma_{imp} \frac{\partial \sigma_{imp}}{\partial T} + 2(r - q)XT \sigma_{imp} \frac{\partial \sigma_{imp}}{\partial X}}{(1 + Xd_1T \frac{\partial \sigma_{imp}}{\partial X})^2 + X^2T \sigma_{imp} \left[ \frac{\partial^2 \sigma_{imp}}{\partial X^2} - d_1T \left( \frac{\partial \sigma_{imp}}{\partial X} \right)^2 \right]},$$

where

$$d_1 = \frac{\ln \frac{S}{X} + \left[ r - q + \frac{\sigma_{imp}^2(X,T)}{2} \right] T}{\sigma_{imp}(X,T) \sqrt{T}}.$$
4.5 Volatility trading: variance and volatility swaps, VIX

*Characteristics of volatility (hidden stochastic process)*

- Likely to grow when uncertainty and risk increase. May serve as a proxy for market confidence – fear gauge.

- Volatilities appear to revert to the mean (non-linear drift).
  - After a large volatility spike, the volatility can potentially decrease rapidly.
  - After a low volatility period, it may start to increase slowly.

- Volatility is often negatively correlated with stock or index level, and tends to stay high after large downward moves.

- Stock options are impure: they provide exposure to both direction of the stock price and its volatility. If one hedges an option according to the Black-Scholes prescription, then she can remove the exposure to the stock price.
Delta-hedging is at best inaccurate since volatility cannot be accurately estimated, stocks cannot be traded continuously, together with transaction costs and jumps in stock prices.

Volatility swaps are forward contracts on future realized stock volatility; and similarly, variance swaps on future variance (square of future volatility). They provide pure exposure to volatility and variance, respectively.

*Businesses that are implicitly short volatility* (lose when volatility increases)

- Investors following active benchmarking strategies may require more frequent rebalancing and incur higher transaction expenses during volatile periods.

- Equity funds are probably short volatility due to the negative correlation between index level and volatility.
Replication of variance swaps - continuous model

The fair strike of a variance swap (continuously monitored) is given by

\[ K_{\text{Var}} = E_0[V_R] = E_0 \left( \frac{1}{T} \int_0^T \sigma_t^2 \, dt \right) \]

Suppose the asset price process \( S_t \) follows the following Brownian motion under a pricing measure \( Q \):

\[ \frac{dS_t}{S_t} = r \, dt + \sigma_t \, dW_t, \]

where \( W_t \) is the standard Brownian motion and \( \sigma_t \) may be state dependent. We may rewrite the dynamics equation as follows:

\[ d \ln S_t = \left( r - \frac{\sigma_t^2}{2} \right) dt + \sigma_t \, dW_t. \]

The additional drift term \( -\frac{\sigma_t^2}{2} \, dt \) arises from the Ito lemma, where

\[ \frac{1}{2} \sigma_t^2 S_t^2 (dZ_t)^2 \frac{\partial^2}{\partial S_t^2} \ln S_t = -\frac{\sigma_t^2}{2} \, dt. \]
Subtracting the two dynamic equations, we obtain

\[
\frac{dS_t}{S_t} - d\ln S_t = \frac{\sigma_t^2}{2} \, dt
\]

The continuous realized variance \( V_R \) is then given by

\[
V_R = \frac{1}{T} \int_0^T \sigma_t^2 \, dt = \frac{2}{T} \left( \int_0^T \frac{dS_t}{S_t} - \ln \frac{S_T}{S_0} \right).
\]

The formula dictates the strategy that can be adopted to replicate the continuous realized variance.

We take \( \frac{1}{S_t} \) units of stock at time \( t \) paying $1, and enter into a “static” short position at time 0 in a forward contract which at maturity has a payoff equals to the logarithm of the total return of the stock \( \ln \frac{S_T}{S_0} \), where \( \frac{S_T}{S_0} \) is the total return over \([0, T]\).
This is a self-financing strategy

Suppose the stock price goes up, the investor sells \( \frac{1}{S_t} \) units of stock and buys \( \frac{1}{S_{t+dt}} \) units paying $1. Over the same period, the forward log contract changes in value of amount \( \ln \frac{S_{t+dt}}{S_t} \). The short position in the log contract offsets the gain on the long stock position. By observing \( \ln(1 + x) \approx x \) when \( x \) is small, and taking \( x = \frac{S_{t+dt}}{S_t} - 1 \), we have

\[
\ln \frac{S_{t+dt}}{S_t} \approx \frac{S_{t+dt}}{S_t} - 1.
\]

To find the fair strike \( K_{\text{var}} \) of the variance swap, we consider

\[
K_{\text{var}} = E_0[V_R] = \frac{2}{T} E_0 \left[ \int_0^T \frac{dS_t}{S_t} - \ln \frac{S_T}{S_0} \right]
\]

\[
= \frac{2}{T} \left\{ E_0 \left[ \int_0^T r \ dt \right] + E_0 \left[ \int_0^T \sigma_t \ dW_t \right] - E_0 \left[ \ln \frac{S_T}{S_0} \right] \right\}.
\]

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The expectation of the long stock position gives $rT$ since the dollar value of the stock position is always $1$. How to replicate the log contract using basic instruments of forward contracts, calls and puts?

**Technical result** For any twice-differentiable function $f: \mathbb{R} \to \mathbb{R}$, and any non-negative $S_*$, we have

$$f(S_T) = f(S_*) + f'(S_*)(S_T - S_*) + \int_0^{S_*} f''(K)(K - S_T)^+ dK$$

$$+ \int_{S_*}^{\infty} f''(K)(S_T - K)^+ dK.$$

$$\int_0^\xi \delta(x - x_0) \, dx = \begin{cases} 0 & \text{if } \xi < x_0 \\ 1 & \text{if } \xi > x_0 \end{cases} = 1_{\{x_0 < \xi\}}.$$
To establish the technical results, we perform repeated integration by parts in order to generate the option payoff terms: \((S_T - K)^+\) and \((K - S_T)^+\).

\[
f(S_T) = \int_0^{S_*} f(K) \delta(S_T - K) \, dK + \int_{S_*}^\infty f(K) \delta(S_T - K) \, dK
\]

\[
= f(K) \mathbb{1}_{\{S_T < K\}} \bigg|_0^{S_*} - \int_0^{S_*} f'(K) \mathbb{1}_{\{S_T < K\}} \, dK
\]

\[
+ f(K) \mathbb{1}_{\{S_T \geq K\}} \bigg|_{S_*}^{\infty} - \int_{S_*}^{\infty} f'(K) \mathbb{1}_{\{S_T \geq K\}} \, dK
\]

\[
= f(S_*) \mathbb{1}_{\{S_T < S_*\}} - \left[f'(K)(K - S_T)^+\right]_0^{S_*} + \int_0^{S_*} f''(K)(K - S_T)^+ \, dK
\]

\[
+ f(S_*) \mathbb{1}_{\{S_T \geq S_*\}} - \left[f'(K)(S_T - K)^+\right]_{S_*}^{\infty} + \int_{S_*}^{\infty} f''(K)(S_T - K)^+ \, dK
\]

\[
= f(S_*) + f'(S_*)[(S_T - S_*)^+ - (S_* - S_T)^+]
\]

\[
+ \int_0^{S_*} f''(K)(K - S_T)^+ \, dK + \int_{S_*}^{\infty} f''(K)(S_T - K)^+ \, dK
\]

Rearranging the terms, we obtain

\[
f(S_T) - f(S_*) = f'(S_*) (S_T - S_*) + \int_0^{S_*} f''(K)(K - S_T)^+ \, dK
\]

\[
+ \int_{S_*}^{\infty} f''(K)(S_T - K)^+ \, dK.
\]
Applying the technical formula for \( f(S_T) = \ln S_T \), we have

\[
\ln S_T - \ln S_* = \frac{S_T - S_*}{S_*} - \int_0^{S_*} \frac{1}{K^2} (K - S_T)^+ \ dK - \int_{S_*}^{\infty} \frac{1}{K^2} (S_T - K)^+ \ dK.
\]

- Hold a long position in \( \frac{1}{S_*} \) forward contracts with forward price \( S_* \);
- Short positions in \( \frac{1}{K^2} \) put options with strike \( K \), \( K \) from 0 to \( S_* \); short positions in \( \frac{1}{K^2} \) call options with strike \( K \), \( K \) from \( S_* \) to \( \infty \).

All contracts have the same maturity \( T \).
Valuation of the fair strike of the continuous variance swap

\[ K_{\text{var}} = \frac{2}{T} \left\{ rT - E_0 \left[ \ln \frac{S_0}{S_0} + \frac{S_T - S_*}{S_*} - \int_0^{S_*} \frac{1}{K^2} (K - S_T)^+ \, dK - \int_{S_*}^\infty \frac{1}{K^2} (S_T - K)^+ \, dK \right] \right\}. \]

Note that

\[ S_0 = e^{-rT} E_0[S_T], \quad C_0(K) = e^{-rT} E_0[(S_T - K)^+], \]
\[ P_0(K) = e^{-rT} E_0[(K - S_T)^+]. \]

We then have

\[ K_{\text{var}} = \frac{2}{T} \left[ rT - \left( \frac{S_0}{S_*} e^{rT} - 1 \right) - \ln \frac{S_*}{S_0} \right.
+ e^{rT} \int_0^{S_*} \frac{1}{K^2} P_0(K) \, dK
+ e^{rT} \int_{S_*}^\infty \frac{1}{K^2} C_0(K) \, dK \left. \right]. \]

The strike price formula requires an infinite number of strikes in order to be exact, while the market provides only options with finite number of strikes.
**Volatility swaps**

Even though the variance swaps can be priced and replicated easily, they are still less actively traded compared to the volatility swaps since investors are more accustomed to volatility than variance of the underlying asset price process.

Consider the second order Taylor expansion of $g(V_R) = \sqrt{V_R}$ around $K_{\text{var}} = E_0[V_R]$, we have

$$\sqrt{V_R} \approx \sqrt{K_{\text{var}}} + \frac{1}{2\sqrt{K_{\text{var}}}}(V_R - K_{\text{var}}) - \frac{1}{8(K_{\text{var}})^{3/2}}(V_R - K_{\text{var}})^2.$$  

Taking the expected values on both sides, we obtain

$$K_{s/d} = E_0[\sqrt{V_R}] \approx \sqrt{K_{\text{var}}} - \frac{1}{8(K_{\text{var}})^{3/2}} \underbrace{E_0[(V_R - K_{\text{var}})^2]}_{\text{var}_0(V_R)} \underbrace{- \frac{1}{8(K_{\text{var}})^{3/2}} \underbrace{E_0[(V_R - K_{\text{var}})^2]}_{\text{convexity correction}}}_{\text{convexity correction}}.$$
The convexity correction represents the mismatch between $K_{s/d}$ and $\sqrt{K_{\text{var}}}$. Under this approximation, we observe $K_{s/d} < \sqrt{K_{\text{var}}}$.

- The above formula does not give a straightforward formula for $K_{s/d}$ since the conditional variance of the realized variance has to be estimated. The computation of $\text{var}_0(V_R)$ can be quite cumbersome.

- Broadie and Jain (2008) show that this convexity correction formula for approximating the fair volatility strike may not provide good estimates in jump-diffusion models.
The tale of VIX

In response to the emergence of the over-the-counter volatility derivatives market, the Chicago Board of Options Exchange (CBOE) introduced the CBOE volatility index (VXO) that was designed to reveal the market forecast of the future realized volatility of S&P 100 index based on the traded prices of options.

- Compute an average of the Black-Scholes option implied volatility with strike prices close to the current spot index level and maturities interpolated at about one month.

- Implied volatility measure (so does VXO) is used as an indicator of market stress. However, this is based on the Black-Scholes model (not model free).
Steps

1. Consider 2 maturities $T_1$ and $T_2$ that bracket maturity date on one month later.

At maturity $T_i, i = 1, 2$, the near-the-money Black-Scholes implied volatility for each $K_{j}^{(i)}$ is obtained by averaging the BS implied volatilities of one call and one put at $T_i$ and $K_{j}^{(i)}$. Taking two strikes $K_1^{(i)}$ and $K_2^{(i)}$ that bracket the current spot index level, the CBOE interpolates linearly the implied volatilities to obtain the at-the-money BS implied volatility [as denoted by $\text{ATMV}(t,T_i)$].
2. Since the Black-Scholes implied volatility is based on “actual calendar days/365”, the CBOE converts \( \text{ATMV}(t, T_i) \) into trading day volatility \( \text{TV}(t, T_i) \) following the “trading days/252” convention:

\[
\text{TV}(t, T_i) = \text{ATMV}(t, T_i) \sqrt{\frac{\text{NC}(t, T_i)}{\sqrt{\text{NT}(t, T_i)}}}.
\]

\( \text{NC}(t, T_i) \) = number of actual calendar days between \( t \) and \( T_i \)

\( \text{NT}(t, T_i) \) = number of trading days between \( t \) and \( T_i \)

\[
= \text{NC}(t, T_i) - 2 \times \text{int} \left( \frac{\text{NC}(t, T_i)}{7} \right)
\]

3. The CBOE interpolates linearly \( \text{TV}(t, T_1) \) and \( \text{TV}(t, T_2) \) to estimate the 22 trading days (one month) at-the-money implied volatility

\[
\text{VXO}_t = \frac{\text{TV}(t, T_1)(NT_2 - 22) + \text{TV}(t, T_2)(22 - NT_1)}{NT_2 - NT_1}.
\]
Mathematical derivation of VIX

VIX expresses volatility in percentage points. It is calculated as 100 times the square root of the expected 30-day variance (var) of the S&P 500 rate of return.

\[
VIX = 100\sqrt{\text{forward price of expected realized cumulative variance}}
\]

Suppose the forward price \( F_t \) of the index under \( Q \) follows

\[
\frac{dF_t}{F_t} = \sigma_t \, dW_t \quad \text{so that} \quad d\ln F_t = -\frac{\sigma_t^2}{2} \, dt + \sigma_t \, dW_t.
\]

Subtracting the two equations, we obtain the cumulative variance over \([0, T]\)

\[
\frac{dF_t}{F_t} - d\ln F_t = \frac{\sigma_t^2}{2} \, dt, \quad \text{so} \quad \int_0^T \sigma_t^2 \, dt = 2 \left[ \int_0^T \frac{dF_t}{F_t} - \ln \frac{F_T}{F_0} \right].
\]

From a well known result in Taylor expansion, we have

\[
f(F_T) - f(F_0) = f'(F_0)(F_T - F_0) + \int_0^{F_0} f''(K)(K - F_T)^+ dK
\]

\[
+ \int_{F_0}^\infty f''(K)(F_T - K)^+ dK; \quad F_0 = \text{time-0 forward price.}
\]
Taking $f(F_T) = \ln F_T$, we have
\[
\ln \frac{F_T}{F_0} = \frac{F_T - F_0}{F_0} - \int_0^{F_0} \frac{(K - F_T)^+}{K^2} dK - \int_{F_0}^{\infty} \frac{(F_T - K)^+}{K^2} dK.
\]
Combining the results, the total variance is
\[
\text{var}_T = 2 \left[ \int_0^T \frac{dF_t}{F_t} - \frac{F_T - F_0}{F_0} + \int_0^{F_0} \frac{(K - F_T)^+}{K^2} dK + \int_{F_0}^{\infty} \frac{(F_T - K)^+}{K^2} dK \right].
\]
How does CBOE construct the VIX based on traded option prices?

Note that
\[
E_Q \left[ \int_0^T dF_t \right] = E_Q \left[ \int_0^T \sigma_t \, dW_t \right] = 0
\]
and
\[
E_Q[F_T] = F_0
\]
so that the expectation under $Q$ of the first two terms is zero.
• The last two terms represent continuum of puts whose strikes are below $F_0$ and calls whose strikes are above $F_0$, respectively. They represent out-of-the-money options with respect to $F_0$. The CBOE’s choice is more natural since out-of-the-money options tend to be more liquid contracts.

• In reality, we can only approximate the continuum of options by strips of out-of-the-money puts and calls, whose strikes have finite distance $\Delta K$ apart. Let $K_0$ be the closest listed strike below $F_0$. An additional term \( \left( \frac{F_0}{K_0} - 1 \right)^2 \) is subtracted due to the adjustment compensating for the strips of options that are not centered around a strike exactly at $F_0$. This adjustment term becomes zero when $F_0 = K_0$.

The adjustment can be visualized as an approximation to the correction required when the limits of integration in the two integrals of the put and call options are changed from $F_0$ to $K_0$. 
The error is seen to be

$$\int_{K_0}^{F_0} \frac{(K - F_T)^+}{K^2} dK - \int_{K_0}^{F_0} \frac{(F_T - K)^+}{K^2} dK$$

$$\approx \int_{K_0}^{F_0} \frac{K - K_0}{K_0^2} dK = \left( \frac{F_0}{K_0} - 1 \right)^2.$$

The forward price of the expected realized cumulative variance is approximated by

$$2e^{rT} \left[ \sum_{0}^{K_0} \frac{\Delta K}{K^2}\text{put}_K + \sum_{K_0}^{\infty} \frac{\Delta K}{K^2}\text{call}_K \right] - \left( \frac{F_0}{K_0} - 1 \right)^2,$$

where put$_K$ and call$_K$ are prices of out-of-the-money put and call, respectively.
Finally, we multiply the above expected realized cumulative variance by the product of the annualization conversion factor $\frac{365}{30}$ and percentage point factor 100 to obtain

$$VIX_t^2 = 100^2 \left\{ \frac{2}{30/365} \sum_i \frac{\Delta K_i}{K_i^2} e^{r(30/365)} Q(K_i) - \frac{1}{30/365} \left( \frac{F}{F_0} - 1 \right)^2 \right\}$$

Here, $K_0$ is the first strike below the forward index level $F_0$, $Q(K_i)$ is the time-$t$ out-of-the-money option price with strike $K_i$. 
Plot of the VIX index (02/01/1990–02/01/2009).
Timer options

A standard timer call option can be viewed as a call option with random maturity which depends on the time needed for a pre-specified variance budget to be fully consumed. A variance budget is calculated as the target volatility squared, multiplied by the target maturity.

Suppose the asset price is monitored at \( t_j = j \Delta t, \ j = 0, 1, \ldots, n \).

The annualized realized variance for the period \([0, T]\) is defined as

\[
\hat{\sigma}^2_T = \frac{1}{(n - 1) \Delta t} \sum_{i=0}^{n-1} \left( \ln \frac{S_{i+1}}{S_i} \right)^2, \ T = n \Delta t.
\]

Here, \( \Delta t \) is in units of year. The realized log returns over successive time intervals \( \ln \frac{S_{t_{i+1}}}{S_{t_i}}, \ i = 0, 1, \ldots, n - 1 \), are considered as sampling of the independent Brownian increments with variance rate \( \sigma^2_T \) and time interval \( \Delta t \). The factor \( n - 1 \) is related to the Bessel correction due to the loss of one degree of freedom.
The realized variance over time period \([0, T]\) is
\[
V_T = n \Delta t \hat{\sigma}_T^2 \approx \sum_{i=0}^{n-1} \left( \ln \frac{S_{i+1}}{S_i} \right)^2.
\]

The investor specifies a variance budget
\[
B = \sigma_0^2 T_0,
\]
where \(T_0\) is the estimated investment horizon and \(\sigma_0\) is the forecast volatility during the investing period.

The timer call with random maturity pays \(\max(S_{t_j} - X, 0)\) at the first time \(t_j\) when the realized variance exceeds \(B\). That is,
\[
t_j = \min \left\{ k > 0, \sum_{i=0}^{k-1} \left( \ln \frac{S_{i+1}}{S_i} \right)^2 \geq B \right\}.
\]

Suppose the variance budget is never met within \([0, T]\), the timer call option expires at the mandated maximum expiration date \(T\) with terminal payoff \(\max(S_T - X, 0)\).
Option strategy without paying the volatility risk premium

The price of a vanilla call option is determined by the level of implied volatility quoted in the market (as well as maturity and strike price). The level of implied volatility is often higher than the realized volatility (risk premium due to the uncertainty of future market direction).

“High implied volatility means call options are often overpriced. In the timer option, the investor only pays the real cost of the call and does not suffer from high implied volatility.”
The first trade was in April 2007 on HSBC with a June expiry.

- The implied volatility on the plain vanilla call was slightly above 15%, but the client sets a target volatility level of 12%, a little higher than the prevailing realized volatility level of around 10%.

- The premium of the timer call has 20% discount compared to the vanilla call counterpart. Here, the vanilla call price is calculated based on $T_0$ and implied volatility of 15%.

- The realized volatility has been around 9.5% since the inception of the trade. The maturity of the timer call is 60% longer than the original vanilla call.
Bullish view of the market

Long a timer call, short a vanilla call. Usually, stock price and volatility are negatively correlated.

- The implied volatility in the market is too high currently, and subsequent realized volatility will be less than that implied in the market.

By setting the volatility target to be below the current implied volatility level (price of timer call would be less than that of the comparative vanilla call).

If the stock shifts higher over the period, the tenor of the timer option would be longer; and a net credit would be received that captures the value due to the difference in the time value of the two options (since longer-lived options are more expensive).
4.6 Merton’s model for risky debts

Default is assumed to occur when the market value of the issuer’s assets has fallen to a low level such that the issuer cannot meet the par payment at maturity.

The issuer is essentially granted an option to default on its debt. When the value of firm’s assets is less than the total debt, the debt holders can only receive the value of the firm. In the literature, the approach that uses the firm value as the fundamental state variable determining default is termed the *structural approach* or *firm value approach*.

To analyze the credit risk structure of a risky debt using the structural approach, it is necessary to characterize the issuer’s firm value process together with the information on the capital structure of the firm.
Firm value

The value of a firm is the value of its business as a going concern. The firm’s business constitutes its assets, and the present assessment of the future returns from the firm’s business constitutes the current value of the firm’s assets.

The value of the firm’s assets is different from the bottom line on the firm’s balance sheet. When the firm is bought or sold, the value traded is the ongoing business. The difference between the amount paid for that value and the amount of book assets is usually accounted for as the “good will”. 
The value of the firm’s assets can be measured by the market prices of the various *claims on its assets*. The claimants may include the debt holders, equity holder, etc.

\[
\text{market value of firm asset} = \text{market value of equity} + \text{market value of bonds} \\
= \text{share price} \times \text{total number of shares} \\
+ \text{sum of market bond prices}
\]

For more details, read Chapter 14 in “Corporate Finance” by Berk and DeMarzo.
The debt issuer’s firm $A_t$ evolves according to the geometric Brownian motion of the form

$$\frac{dA_t}{A_t} = \mu_A \ dt + \sigma \ dZ_t,$$

where $\mu_A$ is the instantaneous expected rate of return, $\sigma$ is the volatility of the firm asset value process. The liabilities of the firm consist only of a single debt with face value $F$. The debt has zero coupon and no embedded option features.

- At debt’s maturity, the payment to the debt holders is the minimum of the face value $F$ and the firm value at maturity $A_T$.

- Default can be triggered only at maturity and this occurs when $A_T < F$, that is, the firm asset value cannot meet its debt claim.

- Upon default, the firm is liquidated at zero cost and all the proceeds from liquidation are transferred to the debt holder.
• The terminal payoff to the debt holders can be expressed as

\[ \min(A_T, F) = F - \max(F - A_T, 0), \]

where the last term can be visualized as a put payoff. The debt holders have essentially sold a put option to the issuer since the issuer has the right to put the firm assets at the price of the par value \( F \).

• Let \( A \) denote the firm asset value at current time, \( \tau = T - t \) is the time to expiry and we view the value of the risky debt \( V(A, \tau) \) as a contingent claim on the firm asset value.

• By invoking the standard assumption of continuous time no-arbitrage pricing framework (continuous trading and short selling of the firm assets, perfectly divisible assets, no borrowing-lending spread, etc.), we obtain the usual Black-Scholes pricing equation:
\[
\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} A^2 \frac{\partial^2 V}{\partial A^2} + r A \frac{\partial V}{\partial A} - r A.
\]

The terminal payoff becomes the “initial” condition at \( \tau = 0 \):

\[
V(A, 0) = F - \max(F - A, 0).
\]

By linearity of the Black-Scholes equation, \( V(A, \tau) \) can be decomposed into

\[
V(A, \tau) = Fe^{-r\tau} - p(A, \tau), \quad \tau = T - t,
\]

where \( p(A, \tau) \) is the price function of a European put option as given by

\[
p(A, \tau) = Fe^{-r\tau} N(-d_2) - AN(-d_1),
\]

\[
d_1 = \ln\left(\frac{A}{F}\right) + \left( r + \frac{\sigma^2}{2} \right) \tau \quad \sigma \sqrt{\tau}, \quad d_2 = d_1 - \sigma \sqrt{\tau}.
\]
The value of the risky debt $V(A, \tau)$ is seen to be the value of the default free debt $Fe^{-r\tau}$ less the present value of the expected loss to the debt holders. The expected loss is simply the value of the put option granted to the issuer.

The equity value $E(A, \tau)$ (or shareholders’ stake) is the firm value less the debt liability.

$$E(A, \tau) = A - V(A, \tau)$$
$$= A - [Fe^{-r\tau} - p(A, \tau)] = c(A, \tau),$$

where $c(A, \tau)$ is the price function of the European call. This is not surprising since the shareholders have the call payoff at maturity equals $\max(A_T - F, 0)$. 
Term structure of credit spreads

The yield to maturity $Y(\tau)$ of the risky debt is defined as the rate of return of the debt, where

$$V(A, \tau) = Fe^{-Y(\tau)\tau}.$$

Rearranging the terms, we have

$$Y(\tau) = -\frac{1}{\tau} \ln \frac{V(A, \tau)}{F} = \frac{1}{\tau} \ln \frac{Fe^{-r\tau}[1 - N(-d_2)] + AN(-d_1)}{F} = r - \ln \left( N(d_2) + \frac{A}{Fe^{-r\tau}} N(-d_1) \right).$$

The credit spread is the difference between the yields of risky and default free zero-coupon debts. This represents the risk premium demanded by the debt holders to compensate for the potential risk of default.
Under the assumption of constant riskfree interest rate, the credit spread is found to be

\[ Y(\tau) - r = \frac{1}{\tau} \ln \left( N(\hat{d}_2) + \frac{1}{d} N(\hat{d}_1) \right), \]

where

\[ d = \frac{F e^{-r\tau}}{A}, \quad \hat{d}_1 = \frac{\ln d}{\sigma \sqrt{\tau}} - \frac{\sigma \sqrt{\tau}}{2} \quad \text{and} \quad \hat{d}_2 = \frac{-\ln d}{\sigma \sqrt{\tau}} - \frac{\sigma \sqrt{\tau}}{2}. \]

- The quantity \( d \) is the ratio of the default free debt \( F e^{-r\tau} \) to the firm value \( A \), thus it is coined the term “quasi” debt-to-firm ratio. The adjective “quasi” is added since all valuations are performed under the risk neutral measure instead of the “physical” measure.
As time approaches maturity, the credit spread always tends to zero when $d \leq 1$ but tends toward infinity when $d > 1$.

At times far from maturity, the credit spread has low value for all values of $d$ since sufficient time has been allowed for the firm value to have a higher potential to grow beyond $F$. 
**Time dependent behaviors of credit spreads**

- Downward-sloping for highly leveraged firms.
- Hump shaped for medium leveraged firms.
- Upward-sloping for low leveraged firms.

**Possible explanation**

For high-quality bonds, credit spreads widen as maturity increases since the upside potential is limited and the downside risk is substantial.

**Remark**

Most banking regulations do not recognize the term structure of credit spreads. When allocating economic capital to cover potential defaults and credit downgrades, a one-year risky bond is treated the same as a ten-year counterpart.
Shortcomings

1. Default can never occur by surprise since the firm value is assumed to follow a diffusion process – may be partially remedied by introducing jump effect into the firm value process.

2. Actual spreads are larger than those predicted by Merton’s model. Other types of risks, like liquidity risk, have not been incorporated into Merton’s model.

3. Default premiums are shown to be inversely related to firm size as revealed from empirical studies. In Merton’s model, $Y(\tau) - r$ is a function of $d$ and $\sigma^2\tau$ only, with no explicit dependence on $A$. 
Interpretation of the put option value sold to the issuer

Write the expected loss (put option value) as

\[ N(-d_2) \left[ F e^{-r(T-t)} - \frac{N(-d_1)}{N(-d_2)} A \right], \]

where \( \frac{N(-d_1)}{N(-d_2)} A \) is visualized as the expected discounted recovery value.

Risky bond value

\[ = \text{present value of par} - \text{default probability} \times \text{expected discounted loss given default} \]

and

\[ \text{default probability} = N(-d_2) = P[A_T \leq F]. \]
Numerical example

Data

At the current time $t$, we observe: $A_t = 100$, $\sigma_V = 40\%$, $d_t = \text{quasi-debt-leverage ratio} = 60\%$; $F_t = d_t A_t = 60$; $T - t = 1$ year and $r = \ln(1 + 5\%)$.

Calculations

1. Given $d_t = \frac{F_t}{A_t} = \frac{F e^{-r(T-t)}}{A_t} = 0.6$,

   then we can deduce the par value $F = 100 \times 0.6 \times (1 + 5\%) = 63$. 

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2. Expected discounted recovery value given default

\[ \frac{N(-d_1)}{N(-d_2)} A_t = \frac{0.069829}{0.140726} \times 100 = 49.62. \]

3. Expected discounted loss given default = 60 – 49.62 = 10.38.

4. Cost of default = value of the put sold to the issuer

\[ = N(-d_2) \times \text{expected discounted loss given default} \]

\[ = 14.07\% \times 10.38 = 1.46. \]

Finally, the value of credit risky bond is given by

\[ F_t – \text{put value} = 60 – 1.46 = 58.54. \]
Example – Risky commodity-linked bond

- A silver mining company offered bond issues backed by silver. Each $1,000 bond is linked to 50 ounces of silver, pays a coupon rate of 8.5% and has a maturity of 15 years.
- At maturity, the company guarantees to pay the holders either $1,000 or the market value of 50 ounces of silver.

Rationale  The issuer is willing to share the potential price appreciation in exchange for a lower coupon rate or other favorable bond indentures.

Terminal payoff of the risky commodity-linked bond

\[ \overline{B}(V, S, T) = \min(V, F + \max(S - F, 0)), \]

where \( V \) is the firm value, \( r \) is the interest rate, \( S \) is the value of 50 ounces of silver, \( F \) is the face value.
Potential extensions in risky debt models

1. *Interest rate uncertainty*
   Debts are relatively long-term interest rate sensitive instruments. The assumption of constant rates is not quite realistic.

2. Jump-diffusion process of the firm value.
   We allow for a jump process to shock the firm value process. This would make remedy to the unrealistic small short-maturity spreads as predicted by the pure diffusion model. As a result, default may occur by surprise.
3. **Bankruptcy–triggering mechanism**

Black and Cox (1976) assume a cut-off level whereby intertemporal default can occur. The cut-off may be considered as a safety covenant which protects bondholder or liability level for the firm below which the firm bankrupts.

4. **Deviation from the strict priority rule**

Empirical studies show that the absolute priority rule is enforced in only 25% of corporate bankruptcy cases. The write-down of creditor claims is usually the outcome of a bargaining process which results in shifts of gains and losses among corporate claimants relative to their contractual rights.
Quality spread differentials between fixed rate debt and floating rate debt

- In fixed rate debts, the par paid at maturity is fixed. A floating rate debt is similar to a money market account, where the par at maturity is the sum of principal and accrued interests. The amount of accrued interests depends on the realization of the stochastic interest rate over the life of bond.

Question: Would the default premiums demanded by investors be equal for both types of debts? Also, does the swap rate in an interest rate swap depend on which party is serving as the fixed rate payer since the two counterparties have different creditworthiness levels?

Empirical studies reveal that the yield premiums for fixed rate debts are in general higher than those for floating rate debts. On the other hand, when the yield curve is upward sloping, the floating rate debt holders should demand a higher floating spread.
Credit valuation model

1. Credit risk should be measured in terms of *probabilities and mathematical expectations*, rather than assessed by qualitative ratings.

2. Credit risk model should be based on current, rather than historical observed data. The relevant variables are the *actual market values rather than accounting values*. It should reflect the development in the borrower’s credit standing through time.
3. An assessment of the future earning power of the firm, company's operations, projection of cash flows, etc., has already been made by the aggregate of the market participants in the stock market. The stock price will be the first to reflect the changing prospects. The challenge is how to interpret the changing share prices properly. Accurate and timely information from the equity market provides a continuous credit monitoring process that is difficult and expensive to duplicate using traditional credit analysis.

4. The various liabilities of a firm are claims on the firm's value, which often take the form of options, so the credit model should be consistent with the theory of option pricing.
Industrial implementation: KMV model

*Expected default frequency* (EDF) is a forward-looking measure of actual probability of default. EDF is firm specific.

The KMV model is based on the structural approach to calculate EDF (credit risk is driven by the firm value process).

- It is best when applied to publicly traded companies, where the value of equity is determined by the stock market.
- The market information contained in the firm’s stock price and balance sheet are translated into an implied risk of default.

Annual reviews and other traditional credit processes cannot maintain the same degree of “on guard” that EDFs calculated on a monthly or a daily basis can provide.
Key features in KMV model

1. Distance to default ratio determines the level of default risk.
   - This key ratio compares the firm’s net worth $E(V_T) - d^*$ to its volatility over the given time horizon.
   - The net worth is based on values from the equity market, so it is both timely and superior estimate of the firm value.

2. Ability to adjust to the credit cycle and quickly reflect any deterioration in credit quality.

3. Work best in highly efficient liquid market conditions.
Three steps used to derive the actual probability of default

1. Estimation of the market value and volatility of the firm asset value.

2. Calculation of the distance to default, an index measure of default risk.

3. Scaling of the distance to default to actual probability of default using a default database.
• Changes in EDF tend to anticipate at least one year earlier than the downgrading of the bond issuer by rating agencies like Moodys and S&P.
• According to KMV’s empirical studies, the log-asset returns confirm quite well to a normal distribution and $\sigma_V$ stays relatively constant.

• From the sample of several hundred companies, firms default when the asset value reaches a level somewhere between the value of total liabilities and the value of the short-term debts.
**Distance to default**

Default point, $d^* = \text{short-term debts} + \frac{1}{2} \times \text{long-term debts}$. Why 1/2? Why not!

From $V_T = V_0 \exp \left( \left( \mu - \frac{\sigma_V^2}{2} \right) T + \sigma_V Z_T \right)$, where $V_0$ is the current market value of firm, $\mu$ is the expected rate of return on firm value and $\sigma_V$ is the annualized firm value volatility, the probability of finishing below $d^*$ at time $T$ is

\[ P[V_T \leq d^*] = N \left( -\frac{\ln \frac{V_0}{d^*} + \left( \mu - \frac{\sigma_V^2}{2} \right) T}{\sigma_V \sqrt{T}} \right). \]
Motivated by the above result, the KMV model defines the distance to default by

\[ d_f = \frac{E[V_T] - d^*}{\sigma_V \sqrt{T}}. \]

It appears to be more intuitive to use the difference \( E[V_T] - d^* \) normalized by volatility over the time horizon \( T \) as the distance to default, though the proper definition should be \( \ln E[V_T] - \ln d^* \) under the lognormal diffusion dynamics of \( V_T \).

The probability of default is a function of the firm’s capital structure, the volatility of the asset returns and the current asset value.
Estimation of firm value $V$ and volatility of firm value $\sigma_V$

- Usually, only the price of equity for most public firms is directly observable. In some cases, part of the debt is directly traded.

- Using option pricing approach:

  - equity value, $E = f(V, \sigma_V, K, c, r)$  \hspace{1cm} (Merton’s risky debt model)
  - volatility of equity, $\sigma_E = g(V, \sigma_V, K, c, r)$

where $K$ denotes the leverage ratio in the capital structure, $c$ is the average coupon paid on the long-term debt, $r$ is the riskfree rate. Actually, the relation between $\sigma_E$ and $\sigma_V$ is obtained via the Ito lemma: $E \sigma_E = \frac{\partial f}{\partial V} V \sigma_V$, where the ratio of the volatility of derivative value process to the volatility of underlying price process is the hedge ratio. That is, $g(V, \sigma_V) = \frac{V \partial f}{E \partial V} \sigma_V$.

- Note that $E$ and $\sigma_E$ can be inferred from market data of equity prices. Solve for $V$ and $\sigma_V$ from the above 2 equations.
Based on historical information on a large sample of firms, for each time horizon, one can estimate the proportion of firms of a given default distance (say, $d_f = 4.0$) which actually defaulted after one year.
**Example Federal Express** (dollars in billion of US$)

<table>
<thead>
<tr>
<th></th>
<th>November 1997</th>
<th>February 1998</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market capitalization</td>
<td>$7.9</td>
<td>$7.3</td>
</tr>
<tr>
<td>(price × shares outstanding)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Book liabilities</td>
<td>$4.7</td>
<td>$4.9</td>
</tr>
<tr>
<td>Market value of assets</td>
<td>$12.6</td>
<td>$12.2</td>
</tr>
<tr>
<td>Expected one-year firm value</td>
<td>$12.9</td>
<td>$12.5</td>
</tr>
<tr>
<td>Asset volatility</td>
<td>15%</td>
<td>17%</td>
</tr>
<tr>
<td>Default point</td>
<td>$3.4</td>
<td>$3.5</td>
</tr>
<tr>
<td>Default distance</td>
<td>$12.9 − 3.4</td>
<td>$12.5 − 3.5</td>
</tr>
<tr>
<td>EDF</td>
<td>0.06% (6 bps) = AA−</td>
<td>0.11% (11 bps) = A−</td>
</tr>
</tbody>
</table>

The causes of change for the EDF are due to variations in the *stock price, debt level* (leverage ratio) and *asset volatility*.

**Remark**

Book liabilities is the sum of all debts while default point is short-term debts $+ \frac{1}{2} \times$ long-term debts.
Weaknesses of the KMV approach

- It requires some *subjective estimation* of the input parameters.
- It is difficult to construct theoretical EDFs without the *assumption of log-normality* of asset returns.
- *Private firms EDFs* can be calculated only by using some compatibility analysis based on accounting data.
- It does not *distinguish* among different types of long-term bonds according to their seniority, collateral, covenants or convertibility.
4.7 Transaction costs models

How to construct the hedging strategy that best replicates the payoff of a derivative security in the presence of transaction costs?

Recall that one can create a portfolio containing $\Delta$ units of the underlying asset and money market account which replicates the payoff of the option. By the portfolio replication argument, the value of an option is equal to the initial cost of setting up the replicating portfolio which mimics the payoff of the option.

Leland proposes a modification to the Black-Scholes model where the portfolio is adjusted at regular time intervals. His model assumes proportional transaction costs where the costs in buying and selling the asset are proportional to the monetary value of the transaction.
Let $k$ denote the round trip transaction cost per unit dollar of transaction. Suppose $\alpha$ units of assets are bought ($\alpha > 0$) or sold ($\alpha < 0$) at the price $S$, then the transaction cost is given by $\frac{k}{2} |\alpha| S$ in either buying or selling.

We consider a hedged portfolio of the writer of the option, where he is shorting one unit of option and long holding $\Delta$ units of the underlying asset. The value of this hedged portfolio at time $t$ is given by

$$\Pi(t) = -V(S,t) + \Delta S,$$

where $V(S,t)$ is the value of the option and $S$ is the asset price at time $t$. Let $\delta t$ denote the fixed and small finite time interval between successive rebalancing of the portfolio.
After the small time interval $\delta t$, the change in value of the portfolio is

$$\delta \Pi = -\delta V + \Delta \delta S - \frac{k}{2}|\delta \Delta|S,$$

where $\delta S$ is the change in asset price and $\delta \Delta$ is the change in the number of units of asset held in the portfolio.

A cautious reader may doubt why the proportional transaction cost term $-\frac{k}{2}|\delta \Delta|S$ appears in $\delta \Pi$ while the term $S \delta \Delta$ is missing.

- The transaction cost term represents the single trip transaction cost paid due to rebalancing of the position in the underlying asset.

- By following the “pragmatic” approach used by Black and Scholes (1973), the number of units $\Delta$ is taken to be instantaneously constant.
By Ito’s lemma, the change in option value in time $\delta t$ to leading orders is given by

$$
\delta V \approx \frac{\partial V}{\partial S} \delta S + \left( \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) \delta t.
$$

In order to cancel the stochastic terms, one chooses $\Delta = \frac{\partial V}{\partial S}$. The change in the number of units of asset in time $\delta t$ is given by

$$
\delta \Delta = \frac{\partial V}{\partial S} (S + \delta S, t + \delta t) - \frac{\partial V}{\partial S} (S, t).
$$

From the dynamics of $S_t$, we observe $\delta S \approx \rho S \delta t + \sigma S \delta Z$. Note that $\delta Z \approx 0(\sqrt{\delta t})$, so the leading order of $|\delta \Delta|$ is found to be

$$
|\delta \Delta| \approx \left| \frac{\partial^2 V}{\partial S^2} \right| |\delta S| \approx \sigma S \left| \frac{\partial^2 V}{\partial S^2} \right| |\delta Z|.
$$
Formally, we may treat $\delta Z$ as $\tilde{x}\sqrt{\delta t}$, where $\tilde{x}$ is the standard normal variable. The expectation of the reflected Brownian motion $|\delta Z|$ is given by

$$E(|\delta Z|) = 2 \left( \int_0^\infty t \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt \right) \sqrt{\delta t}$$

$$= 2 \left( \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-u} \, du \right) \sqrt{\delta t}, \quad u = t^2/2,$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{\delta t}.$$

The risk associated with transaction costs is investor-specific, so it should not be compensated. By the Capital Asset Pricing Model, the hedged portfolio should earn an expected rate of return same as that of a riskless asset. This gives

$$E[\delta \Pi] = \left( -\frac{\partial V}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) \delta t - \frac{k}{2} \sigma S^2 \sqrt{\frac{2}{\pi}} \left| \frac{\partial^2 V}{\partial S^2} \right| \sqrt{\delta t}$$

$$= r \left( -V + \frac{\partial V}{\partial S} S \right) \delta t.$$
By putting all the above results together, the above equation can be rewritten as

\[
\left( -\frac{\partial V}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \frac{\sigma^2}{2} S^2 \sqrt{\frac{2}{\pi}} \frac{k}{\sigma \sqrt{\delta t}} \left| \frac{\partial^2 V}{\partial S^2} \right| \right) \delta t = r \left( -V + \frac{\partial V}{\partial S} S \right) \delta t.
\]

If we define the Leland number to be \( Le = \sqrt{\frac{2}{\pi}} \left( \frac{k}{\sigma \sqrt{\delta t}} \right) \), we obtain

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\sigma^2}{2} Le S^2 \left| \frac{\partial^2 V}{\partial S^2} \right| + rS \frac{\partial V}{\partial S} - rV = 0.
\]

The Leland number is related to the ratio of \( k \) and standard deviation of the asset price process over the rebalancing time interval \( \delta t \).
In the *proportional transaction costs model*, the term \( \frac{\sigma^2}{2} Le S^2 \left| \frac{\partial^2 V}{\partial S^2} \right| \) is in general non-linear, except when the comparative static \( \Gamma = \frac{\partial^2 V}{\partial S^2} \) does not change sign for all \( S \). The transaction cost term is dependent on \( \Gamma \), where \( \Gamma \) measures the sensitivity of the hedge ratio \( \Delta \) to the underlying asset price \( S \).

One may rewrite the equation into the form that resembles the Black-Scholes equation

\[
\frac{\partial V}{\partial t} + \frac{\tilde{\sigma}^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0,
\]

where the modified volatility under transaction costs is given by

\[
\tilde{\sigma}^2 = \sigma^2 [1 + Le \text{ sign}(\Gamma)].
\]

The governing equation becomes mathematically ill-posed when \( \tilde{\sigma}^2 \) becomes negative. This occurs when \( \Gamma < 0 \) and \( Le > 1 \).
It is known that $\Gamma$ is always positive for the vanilla European call and put options in the absence of transaction costs. If we postulate the same sign behavior for $\Gamma$ in the presence of transaction costs, then $\tilde{\sigma}^2 = \sigma^2(1 + Le) > \sigma^2$.

The governing equation then becomes linear under the above assumption so that the Black-Scholes formulas become applicable except that the modified volatility $\tilde{\sigma}$ is now used as the volatility parameter.

We can deduce $V(S,t)$ to be an increasing function of $Le$ since we expect a higher option value for a high value of modified volatility. Financially speaking, the more frequent the rebalancing (smaller $\delta t$) the higher the transaction costs and so the writer of an option should charge higher for the price of the option.
Let $V(S,t;\tilde{\sigma})$ and $V(S,t;\sigma)$ denote the option values obtained from the Black-Scholes formula with volatility values $\tilde{\sigma}$ and $\sigma$, respectively. The total transaction costs associated with the replicating strategy is then given by

$$\mathcal{T} = V(S,t;\tilde{\sigma}) - V(S,t;\sigma).$$

When $Le$ is small, $\mathcal{T}$ can be approximated by

$$\mathcal{T} \approx \frac{\partial V}{\partial \sigma} (\tilde{\sigma} - \sigma).$$

Since $\tilde{\sigma} = \sigma[1 + Le \text{ sign}(\Gamma)]^{1/2} \approx \sigma \left[ 1 + \frac{Le}{2} \text{sign}(\Gamma) \right]$, so $\tilde{\sigma} - \sigma \approx \frac{k}{\sqrt{2\pi\delta t}}$.

Note that $\frac{\partial V}{\partial \sigma} = \frac{S\sqrt{T-te^{-\frac{d_2}{2}}}}{\sqrt{2\pi}}$ is the same for both call and put options. For $Le \ll 1$, the total transaction costs for either a call or a put is approximately given by

$$\mathcal{T} \approx \frac{kSe^{-\frac{d_1}{2}}}{2\pi} \sqrt{\frac{T-t}{\delta t}}.$$