1. Consider the discrete multi-period version of the *Fundamental Theorem of Asset Pricing*. Recall that the probability measure $Q$ is said to be a martingale measure if

(i) $Q(\omega) > 0$ for all $\omega \in \Omega$.

(ii) Every discounted price process $S^*_m$ in the securities model is a martingale under $Q$.

(a) Give the definition of an arbitrage opportunity. 

(b) Show that the existence of $Q$ implies non-existence of arbitrage opportunities.

(c) Explain why if there are no arbitrage opportunities in the multi-period model, then there will be no arbitrage opportunities in any underlying single period. Use this result to show that non-existence of arbitrage opportunities implies existence of a martingale measure.

2. Let the dynamics of the stock price process $S_t$ under a martingale pricing measure $Q$ be governed by

$$\frac{dS_t}{S_t} = r \, dt + \sigma \, dZ_t^Q,$$

where $Z_t^Q$ is $Q$-Brownian, $r$ is the riskless interest rate and $\sigma$ is the volatility. Let $V(S_t, t)$ be the price function of a financial derivative with the underlying asset price $S_t$ and $M_t$ be the money market account.

(a) Define $V^*(S_t, t) = V(S_t, t)/M_t$, state the principle used to obtain

$$V^*(S_t, t) = E_t[V^*(S_T, T)].$$

(b) Deduce the governing equation for the price function $V(S, t)$ based on the above expectation formula using the *Feynman-Kac representation theorem*.

3. In the Black-Scholes formulation of riskless hedging, we consider the formation of a riskless hedged portfolio whose value $\Pi_t$ is given by

$$\Pi_t = -V_t + \Delta_t S_t,$$

where $\Delta_t$ is the hedge ratio, $V_t$ is the time-$t$ value of the derivative, and $S_t$ is the asset price at time $t$.

(a) Black and Scholes compute the differential of $\Pi_t$ as

$$d\Pi_t = -dV_t + \Delta_t dS_t$$

without the inclusion of $S_t \, d\Delta_t$. Explain why they manage to arrive at the correct option pricing equation even the product rule of calculus:

$$d(\Delta_t S_t) = \Delta_t dS_t + S_t d\Delta_t$$

is not observed.
(b) Suppose the actual dynamics of $S_t$ is governed by
\[
\frac{dS_t}{S_t} = \rho \, dt + \sigma \, dZ_t,
\]
where $\rho$ is the expected rate of return and $\sigma$ is the volatility, find the appropriate hedge ratio $\Delta_t$ so that the portfolio is instantaneously riskless at all times. \[1\]

(c) How do you modify the above riskless hedging approach if the price of the derivative is dependent on some stochastic index that is not tradeable? Demonstrate how to form a riskless hedged portfolio under such scenario. No detailed calculations to derive the governing equation are required. \[3\]

(d) An investor is said to be risk neutral if he demands zero market price of risk on his risky investments. In the Black-Scholes option pricing equation, it appears apparently that we price a derivative by assuming risk neutrality of the investor. Actually, we simply use the mathematical convenience of risk neutrality. The absolute derivative price does depend on the risk preference of the investor (the actual market price of risk demanded by the investor). Give your comment on the above argument. \[3\]

(e) Recall that two hedgeable securities should have the same market price of risk. How would you use this principle to derive the Black-Scholes equation?\[5\]

Hint: Recall that the expected rate of return of a financial derivative is
\[
\rho_V = \left( \frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) / V,
\]
where the price function $V(S, t)$ satisfies
\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho S \frac{\partial V}{\partial S} - \rho_V V = 0.
\]

With the use of the Ito lemma, we obtain
\[
S \frac{\partial V}{\partial S} = \frac{\sigma V}{\sigma}.
\]

4. Let $q$ be the constant dividend yield of a stock whose price dynamics follows the usual Geometric Brownian motion assumption as in the Black-Scholes framework.

(a) Explain why the risk neutral drift rate is $r - q$, where $r$ is the riskless interest rate. \[2\]

Hint: Consider the wealth process of holding one unit of stock initially and additional units are acquired from the dividends received.

(b) Let $c(S, \tau; X, r, q)$ and $p(S, \tau; X, r, q)$ denote the call price function and put price function on the underlying asset $S$ that pays dividend yield $q$. Here, $X$ is the strike price. Give the financial interpretation of the following put-call symmetry relation:
\[
c(S, \tau; X, r, q) = p(X, \tau; S, q, r). \]

[2]

(c) Suppose the call price formula is
\[
c(S, \tau; X, r, q) = S e^{-q \tau} N(d_1) - X e^{-r \tau} N(d_2),
\]
where
\[
d_1 = \frac{\ln \frac{S}{X} + (r - q + \frac{\sigma^2}{2}) \tau}{\sigma \sqrt{\tau}}, \quad d_2 = d_1 - \sigma \sqrt{\tau}, \quad \tau = T - t.
\]

Derive the put price formula using the above put-call symmetry relation. \[3\]
5. Let $F_t$ denote the exchange rate process that gives the time-$t$ domestic currency price of one unit of foreign currency. Let $Q_d$ and $Q_f$ denote the risk neutral measure in the domestic currency world and foreign currency world, respectively. Let $r_d$ and $r_f$ denote the domestic and foreign riskless interest rate, respectively.

(a) Let $F_{S/U}$ denote the Singaporean currency price of one unit of US currency and $F_{H/S}$ denote the Hong Kong currency price of one unit of Singaporean currency. Assume $F_{S/U}$ to be governed by the following dynamics under the risk neutral measure $Q_S$ in the Singaporean currency world:

$$\frac{dF_{S/U}}{F_{S/U}} = (r_{SGD} - r_{USD}) dt + \sigma_{F_{S/U}} dZ^S_{F_{S/U}},$$

where $r_{SGD}$ and $r_{USD}$ are the Singaporean and US riskless interest rates, respectively.

The quanto option pays $F_{H/S}$ Hong Kong dollars if $F_{S/U}$ is above the strike level $X$.

Find the value of the quanto option in Hong Kong currency in terms of the riskless interest rates of different currency worlds and volatility values $\sigma_{F_{S/U}}$ and $\sigma_{F_{H/S}}$.

(b) For the pricing of the following quanto option with terminal payoff $c(S, F, T) = S \max(F - X, 0)$.

Let $r_d$ and $r_f$ be the riskless interest rate in domestic currency and foreign currency. Explain how to use the pricing formula of an exchange option to derive the corresponding pricing formula.

Hint: (i) Compute $\delta^*_S - \delta^d_S$, where $S^* = FS$, in terms of $\sigma_S$, $\sigma_F$ and $\rho$.

(ii) For the exchange option with terminal payoff: $\max(X_T - Y_T, 0)$, where

$$\frac{dX_t}{X_t} = (r - q_X)dt + \sigma_X dZ^Q_{X,t} \quad \text{and} \quad \frac{dY_t}{Y_t} = (r - q_Y)dt + \sigma_Y dZ^Q_{Y,t},$$

with $\rho dt = dZ^Q_{X,t}dZ^Q_{Y,t}$, the pricing formula is

$$e^{-q_X T} X_0 N \left( \frac{\ln \frac{X_0}{Y_0} + (q_Y - q_X + \frac{\sigma_X^2}{2})T}{\sigma_W \sqrt{T}} \right) - e^{-q_Y T} Y_0 N \left( \frac{\ln \frac{X_0}{Y_0} + (q_Y - q_X - \frac{\sigma_Y^2}{2})T}{\sigma_W \sqrt{T}} \right),$$

where $\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho \sigma_X \sigma_Y$.

6. The debt issuer’s firm value process $A_t$ evolves according to the stochastic process of the form

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma dZ_t,$$

where $\mu_A$ is the instantaneous expected rate of return, $\sigma$ is the volatility of the firm asset value process. The liabilities of the firm consist only of a single debt with face value $F$. The debt has zero coupon and no embedded option features.

(a) Explain why the holders of the risky debt have essentially sold a put option to the issuer. That is, the value of the risky debt $V(A, \tau)$ is $Fe^{-r\tau}$ less the put option. Here, $r$ is the riskless interest rate and $\tau$ is the time to expiry.
(b) By invoking the standard assumption of continuous time no-arbitrage pricing framework (continuous trading and short selling of the firm assets, perfectly divisible assets, no borrowing-lending spread, etc.), we obtain
\[
\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} A^2 \frac{\partial^2 V}{\partial A^2} + r A \frac{\partial V}{\partial A} - r A.
\]
Find the price formula for the risky debt \( V(A, \tau) \).

(c) Explain how the put option sold to the issuer can be interpreted as the default probability times expected discounted loss given default. Find the analytic expression of the expected discounted loss.

(d) The yield to maturity \( Y(\tau) \) of the risky debt is defined as the rate of return of the debt, where
\[
V(A, \tau) = Fe^{-Y(\tau)\tau}, \quad \tau = T - t.
\]
Show that the credit spread is
\[
Y(\tau) - r = -\frac{1}{\tau} \ln \left( N(d_2) + \frac{1}{d} N(d_1) \right),
\]
where
\[
d = \frac{Ee^{-rt}}{A}, \quad d_1 = \frac{\ln d - \sigma \sqrt{\tau}}{\sigma \sqrt{\tau}} \quad \text{and} \quad d_2 = - \frac{\ln d - \sigma \sqrt{\tau}}{\sigma \sqrt{\tau}}.
\]

7. Consider the Leland transaction cost model, where \( k \) denotes the round trip transaction cost per unit dollar of transaction.

(a) Let \( \Delta \) denote the hedge ratio. The change in the number of units of asset held for hedging in time \( \delta t \) is given by
\[
\delta \Delta = \frac{\partial V}{\partial S} (S + \delta S, t + \delta t) - \frac{\partial V}{\partial S} (S, t),
\]
where \( \Delta = \frac{\partial V}{\partial S} \). Assuming that the dynamics of \( S \) under \( Q \) is governed by
\[
\frac{dS}{S} = r dt + \sigma dZ.
\]
Explain why the leading order of \( |\delta \Delta| \) is found to be
\[
|\delta \Delta| \approx \sigma S \left| \frac{\partial^2 V}{\partial S^2} \right| |\delta Z|.
\]
*Hint:* Recall that \( \delta Z \) is of order \( O(\sqrt{\delta t}) \).

(b) If we define the Leland number to be \( Le = \sqrt{2 \pi} \left( \frac{k}{\sigma \sqrt{\delta t}} \right) \), we obtain the governing equation for the option value \( V(S, t) \) as follows
\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\sigma^2}{2} LeS^2 \left| \frac{\partial^2 V}{\partial S^2} \right| + r S \frac{\partial V}{\partial S} - r V = 0.
\]
One may rewrite the equation into the form that resembles the Black-Scholes equation
\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0.
\]
where the modified volatility is given by
\[
\tilde{\sigma}^2 = \sigma^2[1 + Le \text{ sign}(\Gamma)].
\]

When \( Le \) is small and \( \Gamma \) is always positive, show that the total transaction costs can be approximated by
\[
\mathcal{T} \approx \frac{\partial V}{\partial \tilde{\sigma}} (\tilde{\sigma} - \sigma),
\]

Explain why \( \tilde{\sigma} - \sigma \approx \frac{k}{\sqrt{2\pi \delta t}} \) when \( Le \ll 1 \). [3]

— End —