1. There will be two coupons before delivery: one in 6 months and one just prior to delivery. Using the present value formula:

$$94.6 = \frac{F + 4}{e^{0.1}} + \frac{4}{e^{0.05}},$$

where $F$ is the forward price, $4$ is the coupon amount and interest is compounded continuously at the rate of 10%. We obtain

$$F = 94.6e^{0.1} - 4e^{0.05} - 4 = 96.34.$$

Alternatively, we can interpret coupons as negative cost of carry and apply the formula:

$$F = \frac{S - D}{B(\tau)},$$

where $D$ is the sum of present value of the coupons. We then have

$$F = \frac{94.6 - 4e^{-0.05} - 4e^{-0.1}}{e^{-0.1}} = 96.34.$$ 

2. Consider the fixed-leg and floating-leg of a swap of unit notional. Suppose the current time is indexed by 0 (i.e. $t = 0$), and $t = 1$ means one year away from now.

**Fixed-leg**

At $t = \frac{1}{4}$, in-flow of $\frac{10\%}{2} = 0.05$ of interest.

At $t = \frac{3}{4}$, in-flow of $\frac{10\%}{2} = 0.05$ of interest.

**Floating-leg**

At $t = \frac{1}{4}$, in-flow of $L_{\frac{1}{2}} \left( \frac{-1}{4} \right) \left( \frac{3}{4} - \frac{1}{4} \right) = \frac{1}{2} L_{\frac{1}{2}} \left( \frac{-1}{4} \right)$.

Here, $L_{\frac{1}{2}} \left( \frac{-1}{4} \right)$ denotes half-year LIBOR reset at an earlier time $t = -\frac{1}{4}$.

At $t = \frac{3}{4}$, receives $L_{\frac{1}{2}} \left( \frac{1}{4} \right) \left( \frac{3}{4} - \frac{1}{4} \right) = \frac{1}{2} L_{\frac{1}{2}} \left( \frac{1}{4} \right)$.

Discount bond prices observed at $t = 0$: $B \left( 0, \frac{1}{4} \right) = 0.972, B \left( 0, \frac{3}{4} \right) = 0.918$. The floating rate bond that receives $1 + \frac{1}{2} L_{\frac{1}{2}} \left( \frac{-1}{4} \right)$ at time $\frac{1}{4}$ is now priced at 0.992.
This gives the present value of
\[
\frac{1}{2} L_{\frac{1}{4}} \left(-\frac{1}{4}\right) = 0.992 - B\left(0, \frac{1}{4}\right) = 0.992 - 0.972 = 0.02.
\]

Also, the implied present value of
\[
\frac{1}{2} L_{\frac{1}{4}} \left(\frac{1}{4}\right) = B\left(0, \frac{1}{4}\right) - B\left(0, \frac{3}{4}\right) = 0.972 - 0.918 = 0.054.
\]

Therefore, the present value of the floating-leg payments is $0.02 + $0.054 = $0.074
per unit notional. The present value of the fixed-leg payments is
\[
0.05 \left[B\left(0, \frac{1}{4}\right) + B\left(0, \frac{3}{4}\right)\right] = (0.05)(0.972 + 0.918) = 0.0945
\]
per unit notional.

The value of the swap to the fixed-rate payer with notional one million
\[
= \quad 1,000,000 \times (0.074 - 0.0945)
= \quad -20,500.
\]

3. We can rewrite the caplet payoff as
\[
(1 + K_R) \max \left(\frac{R_T(T, T + s) - K_R}{\left[1 + R_T(T, T + s)\right](1 + K_R)}, 0\right)
= \quad (1 + K_R) \max \left(\frac{1}{1 + K_R} - \frac{1}{1 + R_T(T, T + s)}, 0\right).
\]
It is the same as the payoff of $1 + K_R$ put options on an $s$-period bond with strike
price $\frac{1}{1 + K_R}$.

4. Consider a portfolio consisting of a call of strike $X_1$ and a discount bond with par
$X_1$, both have the same date of maturity. The terminal payoff of the portfolio is
max($S_T, X_1$). We compare this portfolio with another similar portfolio, except that
$X_1$ is replaced by $X_2$, where $X_2 > X_1$. Since
\[
\max(S_T, X_1) \leq \max(S_T, X_2),
\]
so the second portfolio dominates over the first portfolio. By no arbitrage principle, the present value of the second portfolio is greater than or equal to that of the first portfolio. We then have
\[
c(S, \tau; X_1) - c(S, \tau; X_2) \leq B(\tau)(X_2 - X_1)
\]
\[
B(\tau) \geq -\frac{c(S, \tau; X_2) - c(S, \tau; X_1)}{X_2 - X_1}.
\]
By taking the limit $X_1 \to X_2$, we then obtain

$$B(\tau) \geq -\frac{\partial c}{\partial X}(S, \tau; X_2) \quad \text{or} \quad \frac{\partial c}{\partial X}(S, \tau; X) \geq -B(\tau).$$

Also, the call price function is a decreasing function of the strike price, so it is obvious that

$$\frac{\partial c}{\partial X}(S, \tau; X) \leq 0.$$ 

(a) The above result holds for European options on a dividend paying asset since the holder of a European option is not entitled to receive the dividends. The terminal payoffs of the two portfolios remain the same even the underlying asset is dividend paying.

(b) Consider the two American calls on a dividend paying asset with different strike prices, $X_2 > X_1$. Recall

$$C(S, \tau; X_1) = c(S, \tau; X_1) + \text{early exercise premium of } X_1\text{-strike call}$$
$$C(S, \tau; X_2) = c(S, \tau; X_2) + \text{early exercise premium of } X_2\text{-strike call}.$$ 

Unfortunately, we have the following price behavior:

$$\text{early exercise premium of } X_2\text{-strike call} \leq \text{early exercise premium of } X_1\text{-strike call}$$

based on the following two observations.

(i) The chance of early exercise of the lower strike American call is higher than that of its higher strike counterpart. This is because the lower strike American call faces a smaller loss in the time value of strike and a lower chance of regret of early exercise.

(ii) Upon early exercise, the gain $(qS - rX)\delta t$ over the infinitesimal time interval $\delta t$ is higher for the lower strike American call. Note that $qS\delta t$ represents the dividend amount received while $rX\delta t$ represents the interest payment paid.

Hence, we cannot establish

$$C(S, \tau; X_1) - C(S, \tau; X_2) \leq B(\tau)(X_2 - X_1).$$

Indeed, when $S$ is sufficiently high, $C(S, \tau; X)$ tends to $S - X$ so that

$$C(S, \tau; X_1) - C(S, \tau; X_2) \to (S - X_1) - (S - X_2) = X_2 - X_1;$$

and hence

$$\frac{\partial C}{\partial X}(S, \tau; X) \to -1.$$
5. The put price function is homogeneous of degree one, that is,

\[ p(\lambda S; \lambda X) = \lambda p(S; X). \]

Let \( h_1 > h_2 \) so that \( h_1 \geq \lambda h_1 + (1 - \lambda)h_2 \) and \( \lambda h_1 + (1 - \lambda)h_2 \geq h_2, \forall \lambda \in [0, 1] \).

Let \( \mu = \frac{\lambda h_1}{\lambda h_1 + (1 - \lambda)h_2} \), and observe \( \frac{X}{h_1} \leq \frac{X}{\lambda h_1 + (1 - \lambda)h_2} \leq \frac{X}{h_2} \), we obtain from the convexity properties of the option price functions with respect to the strike price:

\[
p \left( X; \frac{X}{\lambda h_1 + (1 - \lambda)h_2} \right) \leq \mu p \left( X; \frac{X}{h_1} \right) + (1 - \mu) p \left( X; \frac{X}{h_2} \right)
\]

\[ \Leftrightarrow [\lambda h_1 + (1 - \lambda)h_2] \mu \left( X; \frac{X}{\lambda h_1 + (1 - \lambda)h_2} \right) \leq \lambda h_1 p \left( X; \frac{X}{h_1} \right) + (1 - \lambda) h_2 p \left( X; \frac{X}{h_2} \right) \]

Using the homogeneous property, we obtain

\[ p(\lambda h_1 X + (1 - \lambda)h_2 X; X) \leq \lambda p(h_1 X; X) + (1 - \lambda) p(h_2 X; X). \]

Lastly, by setting \( S_1 = h_1 X \) and \( S_2 = h_2 X \), we deduce that

\[ p(\lambda S_1 + (1 - \lambda)S_2; X) \leq \lambda p(S_1; X) + (1 - \lambda) p(S_2; X), \] where \( \lambda \in [0, 1] \).

6. Consider the following two portfolios:

A : \( c + X \)

B : \( P + S - D \)

Let \( r \) denote the risk-free interest rate. First, suppose there is no early exercise, then the terminal payoff values are

<table>
<thead>
<tr>
<th>( S_T \leq X )</th>
<th>( S_T &gt; X )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio A</td>
<td>( X e^{rT} )</td>
</tr>
<tr>
<td>Portfolio B</td>
<td>( X - S_T + S_T = X )</td>
</tr>
</tbody>
</table>

| | Portfolio A | Portfolio B |
|-----------------|-----------------|
| \( V(A) > V(B) \) | \( V(A) = V(B) \) |

Portfolio B is worth \( \max(X, S_T) \) at expiry while portfolio A is worth at least the value of portfolio B. By no arbitrage argument, the current value of portfolio B is less than portfolio A.

Suppose the American put option is exercised at time \( s \), the values of the portfolios are

\[ A : c_s + X e^{rs}, \quad B : X - D_s. \]

Here, \( D_s \) is the present value of future dividends at time \( s \),

\( c_s \) is the value of the European call option,

\( S_s \) is the spot price of the underlying at time \( s \).

\[
V(A) - V(B) = c_s + X e^{rs} - (X - D_s)
= c_s + D_s + X[e^{rs} - 1] > 0,
\]
since \( c_s > 0 \) and \( D_s > 0 \). We then have
\[
c + X > P + S - D \text{ and together with } c < C,
\]
we finally obtain
\[
C - P > S - X - D.
\]

7. To show the left inequality, we consider the following two portfolios:
\[
A : c + X
\]
[European call on foreign currency plus domestic discount bond of par \( \frac{X}{B(\tau)} \) ]
\[
B : P + SB_f(\tau)
\]
(American put on foreign currency plus a unit par foreign discount bond).
If there were no early exercise, then the portfolio values at maturity are
\[
\begin{array}{c|c|c}
  & S_T < X & S_T \geq X \\
  A & \frac{X}{B(\tau)} & \frac{X}{B(\tau)} + S_T - X \\
  B & X - S_T + S_T = X & S_T \\
\end{array}
\]
If we exercise the American put early at time \( s \in [0, T) \), and this would occur only when \( S_s \) is smaller than the strike \( X \), then the values of the portfolios are
\[
A : c_s + \frac{X}{B(T - s)}
\]
\[
B : X - S_s + S_sB_f(T - s) = X + S_s[B_f(T - s) - 1]
\]
Since \( c_s > 0 \) and \( B_f(T - s) < 1 \), so \( V(A) > V(B) \). So, we have \( c + X > P + SB_f(\tau) \) at initiation. Therefore,
\[
c - P > SB_f(\tau) - X \text{ and together with } c < C,
\]
we obtain
\[
C - P > c - P > SB_f(\tau) - X. \quad (i)
\]
To show the right inequality, we consider
\[
C : C + XB(\tau)
\]
(American call on foreign currency plus domestic discount bond of par \( X \))
\[
D : p + S
\]
(European put on foreign currency plus a unit par foreign discount bond).
If there were no early exercise, then the portfolio values at maturity are
Again, suppose we exercise the American call prematurely at any time $s \in [0, \tau)$, which means $S_s > X$. The portfolio values are

\[
C : S_s - X + X B(T - s) = S_s + X [B(T - s) - 1] \\
D : p_s + \frac{S_s}{B(T - s)} .
\]

Since $p_s > 0$ and $B(T - s) < 1$, so we obtain $V(D) > V(C)$. Therefore,

\[
C - p < S - X B(\tau)
\]

we obtain

\[
C - P < S - X B(\tau) . \tag{ii}
\]

Combining the above pair of inequalities, we have

\[
SB_f(\tau) - X < C - P < S - X B(\tau).
\]

8. When the strike price is growing at the riskless interest rate, there will be no gain on the time value of the strike price upon early exercise of the American put. In this case, the American early exercise right is rendered worthless, so the price of the American put is the same as that of its European counterpart.

(i) When $X = 0$, the American put becomes worthless since the exercise of the American put always gives zero value.

(ii) Once $S = 0$, the asset price stays at zero value forever. There will be no loss in dividends from the asset and insurance value associated with the holding of the American put. The American put should be exercised immediately to receive the strike price $X$. Hence, the value of the American put is equal to $X$.

9. Consider the following payoff table:

<table>
<thead>
<tr>
<th>Transaction</th>
<th>time $t$</th>
<th>time $T$</th>
<th>$S_T \leq Q_T$</th>
<th>$S_T &gt; Q_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy call</td>
<td>$-c(S_t, Q_t, T - t)$</td>
<td>0</td>
<td>$S_T - Q_T$</td>
<td></td>
</tr>
<tr>
<td>sell put</td>
<td>$p(S_t, Q_t, T - t)$</td>
<td>$S_T - Q_T$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>sell forward on A</td>
<td>$F_{t,T}^P(S)$</td>
<td>$-S_T$</td>
<td>$-S_T$</td>
<td></td>
</tr>
<tr>
<td>buy forward on B</td>
<td>$-F_{t,T}^P(Q)$</td>
<td>$Q_T$</td>
<td>$Q_T$</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>$-c(S_t, Q_t, T - t)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ + p(S_t, Q_t, T - t) \]
\[ + F_{t,T}^P(S) - F_{t,T}^P(Q) \]
Since the time-$T$ portfolio value is always zero under all possible cases, the time-$t$ portfolio value must also be zero. If otherwise, then arbitrage opportunity arises. We then have the put-call parity relation.