1. \( \Leftarrow \) part: The trading strategy \( \mathcal{H} \) with \( V_0 < 0 \) and \( V_1(\omega) \geq 0, \forall \omega \in \Omega \), dominates the zero-holding trading strategy \( \widehat{\mathcal{H}} = (0 \ 0 \ \cdots \ 0)^T \). The zero-holding strategy gives \( \widehat{V}_1(\omega) = V_0 < 0 \), so \( V_1(\omega) > \widehat{V}_1(\omega) \) for all \( \omega \in \Omega \). Thus, \( \mathcal{H} \) dominates \( \widehat{\mathcal{H}} \).

\( \Rightarrow \) part: Existence of a dominant trading strategy implies that there exists a trading strategy \( \mathcal{H} = (h_1 \ \cdots \ h_M)^T \) such that \( V_0 = 0 \) and \( V_1(\omega) > 0, \forall \omega \in \Omega \). Let \( G_{min}^* = \min_{\omega} G^*(\omega) = \min \sum_{m=1}^{M} h_m \Delta S^*_m. \) Since \( G^*(\omega) = V_1^* - V_0^* > 0 \), we have \( G_{min}^* > 0 \). Consider the new trading strategy with
\[
\widehat{h}_m = h_m \text{ for } m = 1, \cdots, M, \\
\widehat{h}_0 = -G_{min}^* - \sum_{m=1}^{M} h_m S_m^*(0).
\]
Now, \( \widehat{V}_0^* = \widehat{h}_0 + \sum_{m=1}^{M} \widehat{h}_m S_m^*(0) = -G_{min}^* < 0 \); while
\[
\widehat{V}_1^*(\omega) = \widehat{h}_0 + \sum_{m=1}^{M} \widehat{h}_m S_m^*(1; \omega) \\
= -G_{min}^* + \sum_{m=1}^{M} h_m \Delta S_m^*(\omega) \geq 0,
\]
by virtue of the definition of \( G_{min}^* \). Thus, \( \widehat{\mathcal{H}} = (\widehat{h}_1 \ \cdots \ \widehat{h}_M)^T \) is a trading strategy that gives \( \widehat{V}_0 < 0, \widehat{V}_1(\omega) \geq 0, \forall \omega \in \Omega \).

2. For the given securities model, we have the discounted terminal payoff matrix:
\[
S(1; \Omega) = \begin{pmatrix} 1.1 & 1.1 \\ 1.1 & 2.2 \\ 1.1 & 3.3 \end{pmatrix}
\]
and initial price vector \( S(0) = (1 \ 4). \)

(a) With \( h_0 = 4, h_1 = -1 \), we obtain
\[
V_0 = \begin{pmatrix} 1 \ 4 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} = 0 \\
V_1(\omega) = S(1; \Omega) \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3.3 \\ 2.2 \\ 1.1 \end{pmatrix} > 0, \quad V_1^*(\omega) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.
\]
Thus \( \begin{pmatrix} 4 \\ -1 \end{pmatrix} \) is a dominant trading strategy.

(b) \( G^* = V_1^* - V_0^* = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \).

(c) We shall use the result in Question 1. Now, \( G_{\text{min}}^* = \min_\omega G^*(\omega) = 1 \) so that 
\[
\hat{h}_0 = -1 - (-1)(4) = 3.
\]
Take \( \hat{h} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \), then
\[
\hat{V}_0 = (1 - 4) \begin{pmatrix} 3 \\ -1 \end{pmatrix} = -1 < 0
\]
\[
\hat{V}_1 = S(1; \Omega) \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2.2 \\ 1.1 \\ 0 \end{pmatrix} \geq 0.
\]

Thus \( \hat{H} \) is a trading strategy that starts with negative wealth \( \hat{V}_0 \) and ends with non-negative wealth \( \hat{V}_1 \) for sure.

3. (a) If the law of one price does not hold, then there exist two trading strategies \( \hat{h} \) and \( \hat{h}' \) such that
\[
S^*(1) \hat{h} = S^*(1) \hat{h}' \text{ but } S^*(0) \hat{h} > S^*(0) \hat{h}'.
\]

For any payoff \( x \) in the asset span, it can be expressed as \( x = S^*(1) \hat{h} \) for some \( \hat{h} \). Using the relation: \( S^*(1) \hat{h} = S^*(1) \hat{h}' \), we have
\[
x = S^*(1) \hat{h} + kS^*(1) \hat{h} - kS^*(1) \hat{h}'
= S^*(1) [\hat{h} + k(\hat{h} - \hat{h}')], \text{ for any value of } k.
\]

The initial price of the portfolio that generates \( x \) is given by
\[
S^*(0) \hat{h} + k[S^*(0) \hat{h} - S^*(0) \hat{h}'], \text{ for any value of } k.
\]

As \( S^*(0) \hat{h} - S^*(0) \hat{h}' \neq 0 \), the initial price of the portfolio with payoff \( x \) can assume any value.

(b) Uniqueness of the price of any security in the asset span is equivalent to satisfaction of law of one price. Consider the securities model
\[
S^*(1) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } S^*(0) = \begin{pmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ \frac{3}{3} & \frac{2}{3} \end{pmatrix}.
\]

The state prices \( (\pi_1 \  \pi_2 \  \pi_3) \) can be found by solving
\[
(\pi_1 \  \pi_2 \  \pi_3) \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ \frac{3}{3} & \frac{2}{3} \end{pmatrix}.
\]
giving \((\pi_1 \ \pi_2 \ \pi_3) = \left( \frac{1}{3} \ - \frac{1}{3} \ 1 \right)\). It can be shown that by taking the portfolio \(h = \left( \begin{array}{c} -6 \\ 2 \\ 5 \end{array} \right)\), we have
\[
V_0^* = \left( \begin{array}{ccc} 1 & 4 & 2 \\ 3 & 3 & 3 \end{array} \right) \left( \begin{array}{c} -6 \\ 2 \\ 5 \end{array} \right) = 0
\]
while
\[
V_1^* = \left( \begin{array}{ccc} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{c} -6 \\ 2 \\ 5 \end{array} \right) = \left( \begin{array}{c} 3 \\ 6 \\ 1 \end{array} \right) > 0.
\]
This indicates that \(h = \left( \begin{array}{c} -6 \\ 2 \\ 5 \end{array} \right)\) represents an arbitrage opportunity. Indeed, \(V_0^*\) and \(V_1^*\) are related by
\[
0 = V_0^* = (\pi_1 \ \pi_2 \ \pi_3) V_1^* = \left( \begin{array}{ccc} 1 & 6 & 3 \\ 1 & 2 & 2 \\ 1 & 12 & 6 \end{array} \right) = 0.
\]

4. The state prices \((\pi_1 \ \pi_2 \ \pi_3)\) are found by solving
\[
\left( \begin{array}{ccc} 1 & 3 & 2 \\ 2 & 1 & 1 \end{array} \right) = (\pi_1 \ \pi_2 \ \pi_3) \left( \begin{array}{ccc} 1 & 6 & 3 \\ 1 & 2 & 2 \\ 1 & 12 & 6 \end{array} \right).
\]
The solution is found to be: \((\pi_1 \ \pi_2 \ \pi_3) = \left( \begin{array}{c} 2 \\ 1 \\ -1 \end{array} \right)\). The state prices are \(\pi_i, i = 1, 2, 3\). Positivity of the state prices is not observed so the securities model admits arbitrage opportunity. To find an arbitrage opportunity (for simplicity, we take \(h_1 = 0\)), we seek for \((h_0 \ h_2)^T\) such that
\[
V_0^* = (1 \ 2) \left( \begin{array}{c} h_0 \\ h_2 \end{array} \right) = h_0 + 2h_2 = 0
\]
while
\[
V_1^*(\omega) = \left( \begin{array}{ccc} 1 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 6 & 6 \end{array} \right) \left( \begin{array}{c} h_0 \\ h_2 \end{array} \right) = \left( \begin{array}{c} h_0 + 3h_2 \\ h_0 + 2h_2 \\ h_0 + 6h_2 \end{array} \right) \geq \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right),
\]
with at least one strict inequality. A possible arbitrage portfolio is \((h_0 \ h_2)^T = (-2 \ 1)^T\). We short sell 2 units of the risk free asset, long hold one unit of the second risky asset and zero unit of the first risky asset (since \(h_1 = 0\)). The resulting discounted payoff of the portfolio is given by
\[
V_1^*(\omega) = \left( \begin{array}{c} 1 \\ 0 \\ 4 \end{array} \right).
\]
5. Let $x_1$ and $x_2$ be two discounted terminal payoff vectors in the asset span $S$. This would imply that there exist $h_1, h_2$ such that $x_i = S^*(1)h_i$ for $i = 1, 2$. By the law of one price, the pricing functional is given by $F(x_i) = S^*(0)h_i$ for $i = 1, 2$. For any scalars $\alpha_1$ and $\alpha_2$, we consider

$$\alpha_1 F(x_1) + \alpha_2 F(x_2) = \alpha_1 S^*(0)h_1 + \alpha_2 S^*(0)h_2 = S^*(0)(\alpha_1 h_1 + \alpha_2 h_2)$$

while

$$S^*(1)(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 S^*(1)h_1 + \alpha_2 S^*(1)h_2 = \alpha_1 x_1 + \alpha_2 x_2 \in S.$$

Knowing that $\alpha_1 x_1 + \alpha_2 x_2 \in S$, $F(\alpha_1 x_1 + \alpha_2 x_2)$ is given by $S^*(0)(\alpha_1 h_1 + \alpha_2 h_2)$ as deduced from the relation: $\alpha_1 x_1 + \alpha_2 x_2 = S^*(1)(\alpha_1 h_1 + \alpha_2 h_2)$. We then have

$$F(\alpha_1 x_1 + \alpha_2 x_2) = S^*(0)(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 F(x_1) + \alpha_2 F(x_2).$$

This proves the linearity of the pricing functional.

6. Consider $S^*(1) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and $S^*(0) = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$.

Since the three rows of $S^*(1)$ are independent, so that the row space of $S^*(1)$ spans the whole $\mathbb{R}^3$. Hence, $S^*(0)$ is sure to lie in the row space of $S^*(1)$. Therefore, we can conclude that the law of one price holds for the given securities model. However, we observe that $(-1 1 1)^T$ dominates the trading strategy $(0 0 0)^T$ as $V^*_0 = S^*(0)(-1 1 1)^T = 0$ and

$$V^*_1 = S^*(1) \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} > 0.$$  

7. Let $q = (q(\omega_1) \quad q(\omega_2) \quad q(\omega_3))$. Since the initial bet is one dollar, we have to solve

$$q S(1; \Omega) = (1 \quad 1 \quad 1),$$

giving

$$q(\omega_i) = \frac{1}{d_i + 1} > 0 \quad \text{for} \quad i = 1, 2, 3.$$  

(1)

We also have to observe $\sum_{i=1}^{3} q(\omega_i) = 1$, that is,

$$\sum_{i=1}^{3} \frac{1}{d_i + 1} = 1.$$  

(2)

Eqs. (1) and (2) state the required conditions for the existence of a risk neutral probability measure for the betting game. An example would be $d_1 = 1, d_2 = 3$ and $d_3 = 3$. The betting game pays out $2$ if $\omega_1$ occurs and $4$ if either $\omega_2$ or $\omega_3$ occurs.
8. Note that the last two columns are seen to be
\[
\begin{pmatrix}
\frac{3}{4} \\
\frac{6}{1}
\end{pmatrix} = \begin{pmatrix}
1 \\
1
\end{pmatrix} + \begin{pmatrix}
2 \\
3
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
\frac{4}{5} \\
\frac{7}{1}
\end{pmatrix} = 2 \begin{pmatrix}
1 \\
1
\end{pmatrix} + \begin{pmatrix}
2 \\
3
\end{pmatrix}.
\]

The rank of $\tilde{S}^*(1; \Omega)$ is 2. We also observe that
\[
\begin{align*}
S_2^*(1; \Omega) &= S_0^*(1; \Omega) + S_1^*(1; \Omega) \quad \text{while} \quad S_2(0) \neq S_0(0) + S_1(0); \\
S_3^*(1; \Omega) &= S_0^*(1; \Omega) + S_2^*(1; \Omega) \quad \text{while} \quad S_3(0) \neq S_0(0) + S_2(0).
\end{align*}
\]

Hence, the law of one price does not hold. In fact, $\tilde{S}^*(0) = (1 \ 3 \ 5 \ 9)$ does not lie in the row space of $\tilde{S}^*(1; \Omega)$. This is equivalent to saying that solution to the linear system
\[
\tilde{S}^*(0) = q\tilde{S}^*(1; \Omega)
\]
does not exist.

Next, we check whether \[
\begin{pmatrix}
6 \\
8 \\
12
\end{pmatrix}
\]
is attainable by asking whether solution to the following linear system
\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 3 & 4 & 5 \\
1 & 5 & 6 & 7
\end{pmatrix}
\begin{pmatrix}
h_0 \\
h_1 \\
h_2 \\
h_3
\end{pmatrix}
= \begin{pmatrix}
6 \\
8 \\
12
\end{pmatrix}
\]
exists. The Gaussian elimination procedure gives
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 6 \\
1 & 3 & 4 & 5 & 8 \\
1 & 5 & 6 & 7 & 12
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 2 & 3 & 4 & 6 \\
0 & 1 & 1 & 1 & 2 \\
0 & 3 & 3 & 3 & 6
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 2 & 3 & 4 & 6 \\
0 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 1 & 2 & 2 \\
0 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The set of all possible trading strategies that generate the payoff is seen to be
\[
\begin{pmatrix}
h_0 \\
h_1 \\
h_2 \\
h_3
\end{pmatrix} = \begin{pmatrix}
2 - h_2 - 2h_3 \\
2 - h_2 - h_3 \\
h_2 \\
h_3
\end{pmatrix} \quad \text{for any values of} \quad h_2, h_3 \in \mathbb{R}.
\]

Thus, \[
\begin{pmatrix}
6 \\
8 \\
12
\end{pmatrix}
\]
lies in the asset span. For example, we take $h_2 = h_3 = 1$ so that $h_1 = 0$ and $h_0 = -1$, giving the following replicating strategy:
\[
\begin{pmatrix}
6 \\
8 \\
12
\end{pmatrix} = - \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} + \begin{pmatrix}
3 \\
4 \\
6
\end{pmatrix} + \begin{pmatrix}
4 \\
5 \\
7
\end{pmatrix}.
\]
Note that $\widetilde{S}(0) = 8 + h_2 + 4h_3$. The cost of the replicating portfolio is dependent on $h_2$ and $h_3$. This verifies that the law of one price does not hold in this securities model. There are infinitely many possible prices for this contingent claim.

9. From $\begin{cases} 1 = \Pi_u R + \Pi_d R \\ S = \Pi_u uS + \Pi_d dS \end{cases}$, the state prices $\Pi_u$ and $\Pi_d$ can be expressed in terms of $u, d$ and $R$: $\Pi_u = \frac{R - d}{u - d} \frac{1}{R}$ and $\Pi_d = \frac{u - R}{u - d} \frac{1}{R}$.

The call value under the binomial model is given by $c = \Pi_u c_u + \Pi_d c_d = \frac{R - d}{u - d} \frac{c_u}{R} + \frac{u - R}{u - d} \frac{c_d}{R} = \frac{pc_u + (1 - p)c_d}{R}$, where $p = \frac{R - d}{u - d}$.

10. We test whether a risk neutral measure $Q = (Q_1 \ Q_2 \ Q_3)$ exists for the given securities model. This is done by solving $(Q_1 \ Q_2 \ Q_3) \begin{pmatrix} 1 & 4 & 5 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} = (1 \ 2 \ 3)$. We obtain the set of risk neutral measures $R$ as characterized by $(Q_1 \ Q_2 \ Q_3) = (\lambda \ 1 - 3\lambda \ 2\lambda)$, $0 < \lambda < \frac{1}{3}$. For $Y^* = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$, we have $E_Q[Y^*] = (\lambda \ 1 - 3\lambda \ 2\lambda) \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 4 + \lambda$. We deduce that $V_+ = \sup \{E_Q[Y^*] : Q \in R\} = 4 + \frac{1}{3} = \frac{13}{3}$, $V_- = \inf \{E_Q[Y^*] : Q \in R\} = 4$.

Hence, in order to avoid arbitrage, the range of reasonable initial price is $[4, \frac{13}{3}]$.

11. For the securities model, it is easy to check that the set of risk neutral measures is characterized by $(Q_1 \ Q_2 \ Q_3) = (\alpha \ 1 - 2\alpha \ \alpha)$, $0 < \alpha < \frac{1}{2}$.
Consider $E_Q[Y^*] = (Q_1 \ Q_2 \ Q_3) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} / S_0(1) = \frac{\alpha (y_1 - 2y_2 + y_3) + y_2}{S_0(1)}$, which is independent of $\alpha$ if and only if $y_1 - 2y_2 + y_3 = 0$. Since attainability of a contingent claim is equivalent to uniqueness of risk neutral price, so the necessary and sufficient condition for $Y$ to be attainable is $y_1 - 2y_2 + y_3 = 0$.

12. Note that $\mathcal{F}$ is generated by the partition $\mathcal{P} = \{-3, -2\}, \{-1, 1\}, \{2, 3\}$.

(i) Since $\{2, 3\} \in \mathcal{P}$ and $X(2) = 4 \neq X(3) = 9$, $X$ is not $\mathcal{F}$-measurable.

(ii) Since $\{2, 3\} \in \mathcal{P}$ and $X(2) = 2 \neq X(3) = 3$, $X$ is not $\mathcal{F}$-measurable.

Define the random variable $X(\omega) = \max(\omega, 3)$. Now, $X(\omega) = 3$ for all $\omega \in \Omega$, hence $X$ is $\mathcal{F}$-measurable.

13. (a) Suppose $\mathcal{F}$ is generated by a partition $\mathcal{P}$. It suffices to show that this property is valid for every $B \in \mathcal{P}$. Consider

\[
E[I_B E[X|\mathcal{F}]] = \sum_{\omega \in B} E[X|B]P(\omega) = E[X|B]P(B) = \sum_{\omega \in B} X(\omega)(P(\omega)/P(B))P(B) = \sum_{\omega \in B} X(\omega)P(\omega) = E[XI_B].
\]

(b) Recall $E[X|\mathcal{F}] = \sum_{j=1}^{J} E[X|B_j]1_{B_j}$, and consider

\[
E[\max(X_1, \ldots, X_n)|\mathcal{F}]
= \sum_{j=1}^{J} E[\max(X_1, \ldots, X_n)|B_j]1_{B_j}
= \sum_{j=1}^{J} K_j \sum_{k=1}^{K_j} \max(X_1(\omega_{k,j}), \ldots, X_n(\omega_{k,j}))P(\omega_{k,j})1_{B_j},
\]

while

\[
\max(E[X_1|\mathcal{F}], \ldots, E[X_n|\mathcal{F}])
= \max \left( \sum_{j=1}^{J} K_j \sum_{k=1}^{K_j} X_1(\omega_{k,j})P(\omega_{k,j})1_{B_j}, \ldots, \sum_{j=1}^{J} K_j \sum_{k=1}^{K_j} X_n(\omega_{k,j})P(\omega_{k,j})1_{B_j} \right).
\]

It is obvious that the maximum value among the various sums of $X_\ell(\omega_{k,j})$, $\ell = 1, \ldots, n$, cannot be greater than the value obtained by taking the maximum value among $X_1(\omega_{k,j}), \ldots, X_n(\omega_{k,j})$ and performing the summation afterward. Hence, we obtain the desired result.
14. The property: $E[X_{t+1} - X_t | \mathcal{F}_t] = 0, t = 0, 1, \ldots, T - 1$, does imply that $X$ is a martingale because

$$E[X_T | \mathcal{F}_t] = \sum_{s=t}^{T-1} E[X_{s+1} - X_s | \mathcal{F}_t] + E[X_t | \mathcal{F}_t]$$

$$= \sum_{s=t}^{T-1} E[E[X_{s+1} - X_s | \mathcal{F}_s] | \mathcal{F}_t] + X_t$$

$$= X_t, \quad t = 0, 1, \ldots, T - 1.$$

15. Note that $N_{k+1} - N_k$ is independent of $\mathcal{F}_k$ since the successive binomial trials are independent and $p = \text{probability of success in the } (k + 1)^{th} \text{ trial}$, we have

$$E[N_{k+1} - (k + 1)p - (N_k - kp) | \mathcal{F}_k] = E[N_{k+1} - N_k - p | \mathcal{F}_k]$$

$$= E[N_{k+1} - N_k - p]$$

$$= 0.$$

Then from Problem 14, we deduce that $Y_k$ is a martingale.

16. Consider a portfolio consisting of 4 units of money market account with interest rate $r = 0.3$ and shorting one unit of asset, then we have

$$V(0) = 4 - S(0) = 0, V(2; \omega_i) = 4(1 + r)^2 - S(2; \omega_i) \geq 0.76, \quad i = 1, 2, 3, 4.$$

Hence, this is an arbitrage opportunity.