1. (a) It is easily seen that
\[ X_1(t + s) - X_1(s) = k \left[ Z \left( \frac{t + s}{k^2} \right) - Z \left( \frac{s}{k^2} \right) \right] \]
is normally distributed with mean zero and variance \( k^2 \left( \frac{t + s}{k^2} - \frac{s}{k^2} \right) = t \). Also the increments \( X_1(t_{i+1}) - X_1(t_i) = k[Z(t_{i+1}/k^2) - Z(t_i/k^2)] \) over disjoint time intervals \([t_i, t_{i+1}], \, i = 1, 2, \cdots, n - 1\) are independent and \( X_1(t) \) is continuous at \( X_1(0) = 0 \). Hence \( X_1(t) \) is a standard Brownian motion.

(b) Since \( Z \left( \frac{1}{t} \right) - Z \left( \frac{1}{t + s} \right) \) and \( Z \left( \frac{1}{s} \right) \) are independent, it follows that
\[ X_2(t + s) - X_2(s) = tZ \left( \frac{1}{t + s} \right) - s \left[ Z \left( \frac{1}{s} \right) - Z \left( \frac{1}{t + s} \right) \right] \]
is normally distributed with mean zero and variance \( \frac{t^2}{t + s} + s^2 \left( \frac{1}{s} - \frac{1}{t + s} \right) = t \). To prove that the increments \( X_2(t_{i+1}) - X_2(t_i) \) over disjoint time intervals \([t_i, t_{i+1}], \, i = 1, 2, \cdots, n - 1\) are independent, since they are normal, it suffices to show that \( \text{cov}(X_2(t_{i+1}) - X_2(t_i), X_2(t_{j+1}) - X_2(t_j)) = 0 \), for \( i < j \). This follows from
\[
E[(X_2(t_{i+1}) - X_2(t_i))[X_2(t_{j+1}) - X_2(t_j)]] = t_{i+1}t_{j+1}E\left[Z \left( \frac{1}{t_{i+1}} \right) Z \left( \frac{1}{t_{j+1}} \right) \right] - t_{i+1}t_jE\left[Z \left( \frac{1}{t_{i+1}} \right) Z \left( \frac{1}{t_j} \right) \right] \\
- t_{i}t_{j+1}E\left[Z \left( \frac{1}{t_i} \right) Z \left( \frac{1}{t_{j+1}} \right) \right] + t_{i}t_jE\left[Z \left( \frac{1}{t_i} \right) Z \left( \frac{1}{t_j} \right) \right] \\
= t_{i+1}t_{j+1} - t_{i+1}t_j - t_{i}t_{j+1} + t_{i}t_j \\
= 0.
\]
To show that \( X_2(t) \) starts at \( t = 0 \) almost surely, we establish
\[
E \left( \lim_{t \to 0^+} |X_2(t)| \right) = \lim_{t \to 0^+} E |X_2(t)| \\
= \lim_{t \to 0^+} \sqrt{\frac{2t}{\pi}} \int_0^\infty xe^{-x^2/2} dx \\
= \lim_{t \to 0^+} \sqrt{\frac{2t}{\pi}} \\
= 0,
\]
implying \( P[\lim_{t \to 0^+} X_2(t) = 0] = 1 \). Hence \( X_2(t) \) is a standard Brownian motion.
2. We have

\[ E \left[ \int_t^T [Z(u) - Z(t)] \, du \right] = \int_t^T E[Z(u) - Z(t)] \, du = 0 \]

and

\[ \var \left( \sigma \int_t^T [Z(u) - Z(t)] \, du \right) = \sigma^2 E \left[ \int_t^T [Z(u) - Z(t)] \, du \right]^2 = \sigma^2 \int_t^T \int_t^T [Z(u) - Z(t)][Z(v) - Z(t)] \, dudv = \sigma^2 \int_t^T \int_t^T E[\{Z(u) - Z(t)\}\{Z(v) - Z(t)\}] \, dudv = \sigma^2 \int_t^T \int_t^T \min(u, v) - t] \, dudv = \sigma^2 \left[ \int_t^T (u - t) \, du \right] \int_t^T \int_t^T (v - t) \, dv \int_t^T du ) = \sigma^2 (T - t)^3/3. \]

3. By virtue of the properties of normal distributions and the definition of a Brownian motion, we observe that \( \sigma_1 dZ_1(t) + \sigma_2 dZ_2(t) \) is a Brownian motion with mean 0 and variance rate \( \sigma^2 \), where \( \sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2 \). Define \( Z(t) = \frac{\sigma_1 Z_1(t) + \sigma_2 Z_2(t)}{\sigma} \), which is seen to be a Brownian motion with zero mean and unit variance rate. Note that \( dZ_1 dZ_2 = \rho_{12} dt \) in the mean square sense. For \( f = S_1 S_2 \), it follows that

\[
\begin{align*}
    df &= S_1 dS_2 + S_2 dS_1 + dS_1 dS_2 \\
    &= S_1 S_2 (\mu_2 dt + \sigma_2 dZ_2) + S_2 S_1 (\mu_1 dt + \sigma_1 dZ_1) + S_1 S_2 \sigma_1 \sigma_2 dZ_1 dZ_2 \\
    &= f(\mu_1 + \mu_2 + \rho_{12}\sigma_1\sigma_2) dt + f(\sigma_1 dZ_1 + \sigma_2 dZ_2) \\
    &= f \mu dt + f \sigma dZ.
\end{align*}
\]

From \( S_2(t) = S_2(0) \exp \left[ \left( \mu_2 - \frac{\sigma_2^2}{2} \right) t + \sigma_2 Z_2(t) \right] \), we deduce

\[
S_2^{-1}(t) = \frac{1}{S_2(0)} \exp \left[ \left( - \mu_2 + \frac{\sigma_2^2}{2} \right) t - \sigma_2 Z_2(t) \right].
\]

The corresponding dynamic equation is seen to be

\[
\frac{dS_2^{-1}}{S_2^{-1}} = (-\mu_2 + \sigma_2^2) dt - \sigma_2 dZ_2.
\]

From the first result, for \( g = S_1/S_2 \), it follows that

\[
dg = g\mu dt + g\sigma d\tilde{Z},
\]

where \( \mu = \mu_1 - \mu_2 - \rho_{12}\sigma_1\sigma_2 + \sigma_2^2 \), \( \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2 \) and

\[
\tilde{Z}(t) = \frac{\sigma_1 Z_1(t) - \sigma_2 Z_2(t)}{\sigma}.
\]
4. First, we have \( \text{var}_P(X) = 2/3 \). By solving the following equations:

\[
E_P[X] = 2\tilde{P}[\omega] + 3\tilde{P}[\omega] + 4\tilde{P}[\omega] = 3.5,
\]

\[
\text{var}_P(X) = (2 - 3.5)^2\tilde{P}[\omega] + (3 - 3.5)^2\tilde{P}[\omega] + (4 - 3.5)^2\tilde{P}[\omega] = 2/3,
\]

and

\[
\tilde{P}[\omega] + \tilde{P}[\omega] + \tilde{P}[\omega] = 1,
\]

there is a unique solution: \( \tilde{P}[\omega] = 5/24, \tilde{P}[\omega] = 1/12, \tilde{P}[\omega] = 17/24 \). Since \( \tilde{P}[\omega], i = 1, 2, 3 \) are all nonnegative, we do obtain the required probability measure \( \tilde{P} \) and it is unique.

5. Let \( \gamma = \mu - \mu' \) and consider the Radon-Nikodym derivative:

\[
\frac{d\tilde{P}}{dP} = \rho(t)
\]

where

\[
\rho(t) = \exp\left(\int_0^t -\gamma(s) \, dZ(s) - \frac{1}{2} \int_0^t \gamma(s)^2 \, ds\right).
\]

Under the measure \( \tilde{P} \), the stochastic process

\[
\tilde{Z}_t = Z_t + \int_0^t \gamma(s) \, ds
\]

is \( \tilde{P} \)-Brownian by the Girsanov Theorem. It is seen that when we set \( \gamma = \mu - \mu' \), then

\[
\mu' dt + \sigma d\tilde{Z}_t = \mu' dt + \sigma dZ_t + \gamma dt = \mu dt + \sigma dZ_t.
\]

Therefore, \( S_t \) is governed by

\[
\frac{dS_t}{S_t} = \mu' dt + \sigma d\tilde{Z}_t
\]

under the measure \( \tilde{P} \).

6. Let \( K \) be the delivery price of the commodity forward, then the value of the forward contract is given by

\[
f = S - Ke^{-r\tau} = S - V(\tau),
\]

where \( V(\tau) \) denotes the price of a bond with par value \( K \) at time \( t \), hence the hedge ratio \( \Delta \) is always one. The forward contract is an agreement where the holder agrees to buy the commodity at the delivery time \( T \) for the delivery price \( K \). It can be replicated by holding one unit of the commodity and shorting one unit of a bond with par value \( K \), implying the hedge ratio is one. Setting \( f = 0 \) we get \( K = Se^{r\tau} \).

It follows that the forward price \( F(S, \tau) \) is given by

\[
F(S, \tau) = Se^{r\tau} = S/B(t, T), \quad \tau = T - t.
\]
7. When the self-financing trading strategy is adopted, the purchase of additional units of asset is financed by the sale of the riskless asset, hence \( Sd\Delta + dM = 0 \) and it follows that

\[
d\Pi = \Delta dS + Sd\Delta + rMdt + dM = \Delta dS + rMdt.
\]

By substituting \( dS = \rho S dt + \sigma S dZ \) into the above equation, we obtain

\[
d\Pi = (\rho S\Delta + rM) dt + \sigma S\Delta dZ.
\]

On the other hand, Ito’s Lemma implies

\[
dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} dt
\]

\[
= \left( \frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dZ.
\]

Since \( \Pi \) is a replicating portfolio, in order to match \( d\Pi = dV \), it is necessary to choose

\[ \Delta = \frac{\partial V}{\partial S}. \]

This leads to

\[ \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - rM = 0. \]

Note that \( M = V - \Delta S = V - S \frac{\partial V}{\partial S} \), the Black-Scholes equation for \( V \) is then given by

\[ \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \]

8. Consider \( \Delta_c = N(d_1) \) and \( \Delta_p = -N(-d_1) \), where

\[ d_1 = \frac{\ln \frac{S}{X} + (r + \frac{\sigma^2}{2}) \tau}{\sigma\sqrt{\tau}}. \]

We have

\[ \frac{\partial d_1}{\partial \sigma} = \frac{\sqrt{\tau}}{2} - \frac{\ln \frac{S}{X} + r\tau}{\sigma^2 \sqrt{\tau}}, \quad \frac{\partial d_1}{\partial \tau} = \frac{-\ln \frac{S}{X} + (r + \frac{\sigma^2}{2}) \tau}{2\sigma^2 \tau^{\frac{3}{2}}}. \]

The financial interpretation of the above results is presented below. When the option is sufficiently out-of-the-money currently, a higher volatility of the asset price or a longer time to expiry implies a greater value of delta. Therefore, it is more likely for the option to expire in-the-money. We have the opposite effect when the option is currently in-the-money.

9. The convexity of the European call price with respect to the asset price implies

\[ \frac{c(S, \tau; X) - c(S', \tau; X)}{S - S'} \geq \frac{c(S, \tau; X)}{S}. \]
Let \( S' \to S \), we get \( \frac{dV}{dS} \geq \frac{e}{S} \), hence \( e_c \geq 1 \).

For European options, the elasticity gives the measure of the percentage change in the option price for a unit percentage change in the asset price, so

\[
e_V \sim \frac{S \frac{\partial V}{\partial S}}{S - X e^{-r \tau}}.
\]

For a greater value of \( X/S \) and a smaller value of \( \tau \), the option becomes more out-of-the-money and closer to expiry. These would give greater elasticity in absolute value. Since

\[
e_p = \left( \frac{\partial p}{\partial S} \right) \left( \frac{S}{p} \right) = \frac{-SN(-d_1)}{X e^{-r \tau} N(-d_2) - SN(-d_1)},
\]

the European put’s elasticity has absolute value less than one when the put is sufficiently out-of-the-money.

10. (a) By Ito’s Lemma and observing \( \Delta = 0 \), it follows that

\[
df \; = \; \left( \frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} \right) dt + \frac{\partial f}{\partial S} dS
\]

\[
\; = \left( \Theta + \frac{\sigma^2 S^2}{2} \Gamma \right) dt
\]

By virtue of no arbitrage, we set \( df = rfdt \) and obtain

\[
\Theta + \frac{\sigma^2 S^2}{2} \Gamma = rf.
\]

(b) When the asset value is sufficiently high, the call will always be exercised with terminal payoff \( S_T - X \). Therefore, the call price tends asymptotically to \( S - X e^{-r \tau} \). It is seen that the theta tends asymptotically to \(-rX e^{-r \tau} \) from below.

11. Using the risk neutral valuation principle, the value of the European call option is given by

\[
c_M(S, \tau; X, M) = e^{-r \tau} EQ[c_M(S_T, 0; X, M)]
\]

\[
= e^{-r \tau} EQ[\max(S_T - X, 0) + M - \max(\max(S_T - X, 0), M)]
\]

\[
= e^{-r \tau} EQ[\max(S_T - X, 0)] + e^{-r \tau} EQ[\max(S_T - X - M, 0)]
\]

\[
= c(S, \tau; X) - c(S, \tau; X + M).
\]

12. The terminal payoff function of this call is given by

\[
c_L(S, \tau; X, \alpha) = \min(\max(S_T - X, 0), \alpha S_T).
\]

Using the risk neutral valuation principle, it follows that

\[
c_L(S, \tau; X, \alpha) = e^{-r \tau} EQ[c_L(S_T, 0; X, \alpha S_T)]
\]

\[
= e^{-r \tau} EQ[\max(S_T - X, 0) + \alpha S_T - \max(\max(S_T - X, 0), \alpha S_T)]
\]

\[
= e^{-r \tau} EQ[\max(S_T - X, 0)] - (1 - \alpha)e^{-r \tau} EQ \left[ \max \left( \frac{S_T - X}{1 - \alpha}, 0 \right) \right]
\]

\[
= c(S, \tau; X) - (1 - \alpha)c \left( S, \tau; \frac{X}{1 - \alpha} \right).
\]