1. (a) Explain how the forward price on a tradeable commodity can be enforced using an appropriate replication procedure. Give the financial interpretation of the cost-of-carry formula for the forward price.

(b) Given two tradeable unit par discount bonds with maturity dates $T_1$ and $T_2$, where $t < T_1 < T_2$. Here, $t$ is the current time. How do you determine the “no-arbitrage” fixed rate in a forward rate agreement that is applied over the future period $[T_1, T_2]$ in terms of the time-$t$ bond prices $B_t(T_1)$ and $B_t(T_2)$?

(c) Consider the flexible notional currency forward where any fractional amount of the notional can be used to buy the underlying foreign currency at predetermined dates but the full notional amount must be exercised by the maturity date of the currency forward. Explain why the optimal strategy is either no action or exercise the full notional amount on any predetermined exercisable date.

2. (a) Explain why an interest rate swap can be visualized as an exchange of a floating rate bond with a fixed rate bond, where the common bond par of the two bonds is the notional of the swap contract. Write down the formula for the fixed rate in an interest rate swap in terms of the current discount bond prices $B(t, T_j)$, $j = 1, 2, \ldots, N$, at preset swap dates: $T_1, T_2, \ldots, T_N$.

(b) A swaption can be used to monetize the callable right embedded in a callable bond. Suppose the callable bond pays fixed rate coupons, show how a swaption (holder becomes the fixed rate payer upon exercise) can be used to monetize the callable right as the premium of the swaption?

(c) The asset swap spread $s_A(0)$ at initial time 0 is given by

$$s_A(0) = \frac{1}{A(0)}[C(0) - \overline{C}(0)],$$

where $A(0)$ is the value of the unit-dollar annuity stream with the same tenor as that of the fixed rate payment stream of the interest rate swap in the asset swap. Here, we take the notional of the asset swap to be unity. Also, $\overline{C}(0)$ denotes the time-0 price of the underlying defaultable bond and $C(0)$ denotes the time-0 price of the non-defaultable counterpart. Give an elegant financial argument to show the validity of the above asset swap spread formula.

3. (a) “It is never optimal to exercise an American call option on a non-dividend paying asset.” Give financial argument to justify the above statement.
“It is never optimal to exercise an American put option if the riskfree interest rate is zero.” Give financial argument to justify the above statement.

A necessary condition for optimal early exercise of an American call option on a dividend paying asset is given by

\[ D > X [1 - B(\tau)], \]

where \( X \) is the strike price of the call, \( B(\tau) \) is the price of \( \tau \)-maturity unit par discount bond and \( D \) is the present value of all discrete dividends paid during the remaining life of the call. Explain why.

For an American put option on a discrete dividend paying asset, explain why it is non-optimal to exercise within certain time interval prior to an ex-dividend date \( t_d \). More precisely, show that when the current time \( t \) falls within \( \left[ t_d - \frac{\ln(1+r)}{r}, t_d \right] \), where \( D \) is the discrete dividend amount, \( r \) is the riskless interest rate and \( X \) is the strike price, it is never optimal to exercise prematurely the American put option.

4. For each of the following statements, determine whether it is true or false. Provide a proof if it is true and present a counter example if it is false.

(a) Suppose the law of one price holds, then redundant securities do not exist.
(b) Suppose the market is complete, then all state prices exist and they are all positive.

5. Consider a one-period securities model with the following initial price vector: \( S(0) = (1 \hspace{1em} 4 \hspace{1em} 5) \) and discounted terminal payoff matrix: \( S^*(1) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 5 & 6 \end{pmatrix} \).

(a) Find an arbitrage opportunity associated with the above securities model. Give details on demonstrating the arbitrage.
(b) How to modify \( S(0) \) such that the securities model does not admit arbitrage? Find the corresponding set of risk neutral measures.