1. Consider a one-year forward contract whose underlying asset is a coupon paying bond with maturity date beyond the expiration date of the forward contract. Assume that the bond pays coupon semi-annually at the coupon rate of 8%, and the face value of the bond is $100 (that is, each coupon payment is $4). The current market price of the bond is $94.6, and the previous coupon has just been paid. Taking the riskless interest rate to be at the constant value of 10%, find the forward price of this bond forward.

Hint: The coupon payments may be considered as negative cost of carry.

2. Consider an interest rate swap of notional principal $1 million and remaining life of 9 months, the terms of the swap specify that six-month LIBOR is exchanged for the fixed rate of 10% per annum (quoted with semi-annual compounding). The market prices of unit par zero coupon bonds with maturity dates 3 months and 9 months from now are $0.972 and $0.918, respectively, while the market price of unit par floating rate bond with maturity date 3 months from now is $0.992. Find the value of the interest rate swap to the fixed-rate payer, assuming no default risk of the swap counterparty.

Hint: The discount factors over 3 months and 9 months from now are given by the zero coupon bond prices, and from which the implied 6-month LIBOR that is reset 3 months from now can be determined. The 6-month LIBOR payment to be paid 3 months from now has been fixed, whose value should be reflected in the price of the floating rate bond maturing 3 month from now.

3. We let $P_t(t_1, t_2)$ denote the time-$t$ price of a bond bought at $t_1 \geq t$ and maturing at $t_2 \geq t_1$. The forward rate quoted at time $t$ for a loan commencing at time $T$ with repayment at time $T + s$ is

$$R_t(T, T + s) = \frac{P_t(t, T)}{P_t(t, T + s)} - 1.$$ 

For $T$-maturity call and put options on an $s$-period zero-coupon bond, suppose the strike price is $K$, their payments at maturity are

$$c(P_T(T, T + s), K, T) = \max(P_T(T, T + s) - KP_T(T, T), 0)$$

$$p(P_T(T, T + s), K, T) = \max(KP_T(T, T) - P_T(T, T + s), 0).$$

Note that $P_T(T, T) = 1$. An interest rate caplet makes a payment if the interest rate is above the strike rate $K_R$. While the caplet payment is made at time $T + s$, the payoff at time $T$ is given by

$$\text{caplet}(P_T(T, T + s), T) = \frac{1}{1 + R_T(T, T + s)} \max(R_T(T, T + s) - K_R P_T(T, T), 0).$$
Note that $R_T(T, T + s) = \frac{1}{P_T(T, T + s)} - 1$ so that $\frac{1}{1 + R_T(T, T + s)} = P_T(T, T + s)$. The payoff to a forward rate agreement at time $T + s$ is $R_T(T, T + s) - R_b(T, T + s)$ (see the analogy between a caplet and a forward rate agreement). Show that an interest rate caplet is equivalent to a bond put option.

4. Suppose the strike prices $X_1$ and $X_2$ satisfy $X_1 > X_2$, show that for European calls on a non-dividend paying asset, the difference in call values satisfies

$$B(\tau)(X_2 - X_1) \leq c(S, \tau; X_1) - c(S, \tau; X_2) \leq 0,$$

where $B(\tau)$ is the value of a pure discount bond with par value of unity and time to maturity $\tau$. Furthermore, deduce that

$$-B(\tau) \leq \frac{\partial c}{\partial X}(S, \tau; X) \leq 0.$$

In other words, suppose the call price can be expressed as a differentiable function of the strike price, then the derivative must be non-positive and no greater in absolute value than the price of a pure discount bond of the same maturity. Do the above results also hold for European/American calls on a dividend paying asset?

5. Show that the put prices (European and American) are convex functions of the asset price, that is,

$$p(\lambda S_1 + (1 - \lambda)S_2, X) \leq \lambda p(S_1, X) + (1 - \lambda)p(S_2, X), \quad 0 \leq \lambda \leq 1,$$

where $S_1$ and $S_2$ denote the asset prices and $X$ denotes the strike price.

*Hint:* Let $S_1 = h_1 X$ and $S_2 = h_2 X$, and note that the put price function is homogeneous of degree one in the asset price and the strike price, the above inequality can be expressed as

$$[\lambda h_1 + (1 - \lambda)h_2]p\left(X, \frac{X}{\lambda h_1 + (1 - \lambda)h_2}\right) \leq \lambda h_1 p\left(X, \frac{X}{h_1}\right) + (1 - \lambda)h_2 p\left(X, \frac{X}{h_2}\right).$$

Use the result that the put prices are convex functions of the strike price.

6. Consider the following two portfolios:

**Portfolio A:** One European call option plus cash of the amount $X$.

**Portfolio B:** One American put option, one unit of the underlying asset minus cash of the amount $D$. The loan is in the form of a portfolio of bonds whose par values and dates of maturity match with the sizes and dates of the dividends.
Assume the underlying asset pays dividends and \( D \) denotes the present value of the dividends paid by the underlying asset during the life of the option. Show that if the American put is not exercised early, Portfolio \( B \) is worth \( \max(S_T, X) \), which is less than the value of Portfolio \( A \). Even when the American put is exercised prior to expiry, show that Portfolio \( A \) is always worth more than Portfolio \( B \) at the moment of exercise. Hence, deduce that

\[ S - D - X < C - P. \]

**Hint:** \( c < C \) for calls on a dividend paying asset and the cash amount in Portfolio \( A \) grows with time.

7. Show that the lower and upper bounds on the difference between the prices of American call and put options on a foreign currency are given by

\[ SB_f(\tau) - X < C - P < S - XB(\tau), \]

where \( B_f(\tau) \) and \( B(\tau) \) are bond prices in the foreign and domestic currencies, respectively, both with par value of unity in the respective currency and time to maturity \( \tau \), and \( S \) is the spot domestic currency price of one unit of foreign currency.

**Hint:** To show the left inequality, consider the values of the following two portfolios: the first one contains a European currency call option plus \( X \) dollars of domestic currency, the second portfolio contains an American currency put option plus \( B_f(\tau) \) units of foreign currency. To show the right inequality, we choose the first portfolio to contain an American currency call option plus \( XB(\tau) \) dollars of domestic currency, and the second portfolio to contain a European currency put option plus one unit of foreign currency.

8. Suppose the strike price is growing at the riskless interest rate, show that the price of an American put option is the same as that of its European counterpart. Find the value of an American put option when (i) strike price \( X = 0 \), (ii) asset price \( S = 0 \).

9. Suppose we have an option to exchange one asset for another. Let the underlying asset \( A \) have price \( S_t \) and the strike asset \( B \) have the price \( Q_t \). Let \( F^P_{t,T}(S) \) denote the time-\( t \) price of a prepaid forward on the underlying asset, paying \( S_T \) at time \( T \); and similar definition for \( F^P_{t,T}(Q) \). For European exchange options, the time-\( T \) call and put payoffs are

\[ c(S_T, Q_T, 0) = \max(S_T - Q_T, 0) \quad \text{and} \quad p(S_T, Q_T, 0) = \max(Q_T - S_T, 0). \]

respectively. Show the following put-call parity relation:

\[ c(S_T, Q_T, T - t) - p(S_T, Q_T, T - t) = F^P_{t,T}(S) - F^P_{t,T}(Q). \]