1. Consider the Brownian motion with drift defined by 
\[ X(t) = \mu t + \sigma Z(t), \quad X(0) = 0, \] 
where \( Z(t) \) is the standard Brownian motion. Find \( E[X(t)|X(t_0)], \) \( \text{var}(X(t)|X(t_0)) \) and \( \text{cov}(X(t_1), X(t_2)) \).

2. Show that 
\[ \sigma \int_t^T [Z(u) - Z(t)] \, du \]
has zero mean and variance \( \sigma^2(T - t)^3/3 \).

*Hint:* Consider 
\[
\text{var} \left( \int_t^T [Z(u) - Z(t)] \, du \right) 
= E \left[ \int_t^T \int_t^T [Z(u) - Z(t)][Z(v) - Z(t)] \, dudv \right] 
= \int_t^T \int_t^T E\{[Z(u) - Z(t)][Z(v) - Z(t)]\} \, dudv 
= \int_t^T \int_t^T \min(u, v) - t \, dudv.
\]

3. Suppose the stochastic variables \( S_1 \) and \( S_2 \) follow the Geometric Brownian motions where 
\[
\frac{dS_i}{S_i} = \mu_i \, dt + \sigma_i \, dZ_i, \quad i = 1, 2.
\]
Let \( \rho_{12} \) denote the correlation coefficient between the Wiener processes \( dZ_1 \) and \( dZ_2 \). Let \( f = S_1 S_2 \), show that \( f \) also follows the Geometric Brownian process of the form 
\[
\frac{df}{f} = \mu \, dt + \sigma \, dZ_f
\]
where \( \mu = \mu_1 + \mu_2 + \rho_{12} \sigma_1 \sigma_2 \) and \( \sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho_{12} \sigma_1 \sigma_2 \). Similarly, we let \( g = \frac{S_1}{S_2} \). Show that 
\[
\frac{dg}{g} = \tilde{\mu} \, dt + \tilde{\sigma} \, dZ_g
\]
where \( \tilde{\mu} = \mu_1 - \mu_2 - \rho_{12} \sigma_1 \sigma_2 + \sigma_2^2 \) and \( \tilde{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12} \sigma_1 \sigma_2 \).
Hint: Note that
\[ \frac{d}{dt} \left( \frac{1}{S_2} \right) = -\mu_2 \, dt + \sigma_2^2 \, dt - \sigma_2 \, dZ_2. \]

Treat \( S_1/S_2 \) as the product of \( S_1 \) and \( 1/S_2 \) and use the result obtained for the product of Geometric Brownian processes.

4. Define the discrete random variable \( X \) by
\[
X(\omega) = \begin{cases} 
2 & \text{if } \omega = \omega_1 \\
3 & \text{if } \omega = \omega_2 \\
4 & \text{if } \omega = \omega_3
\end{cases}
\]
where the sample space \( \Omega = \{\omega_1, \omega_2, \omega_3\} \), \( P[\omega_1] = P[\omega_2] = P[\omega_3] = 1/3 \). Find a new probability measure \( \tilde{P} \) such that the mean becomes \( E_{\tilde{P}}[X] = 3.5 \) while the variance remains unchanged. Is \( \tilde{P} \) unique?

5. Given that \( S_t \) is a Geometric Brownian motion which follows
\[
\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dZ_t
\]
where \( Z_t \) is \( P \)-Brownian motion. Find another measure \( \tilde{P} \) by specifying the Radon-Nikodym derivative \( \frac{d\tilde{P}}{dP} \) such that \( S_t \) is governed by
\[
\frac{dS_t}{S_t} = \mu' \, dt + \sigma \, d\tilde{Z}_t
\]
under the measure \( \tilde{P} \), where \( \tilde{Z}_t \) is \( \tilde{P} \)-Brownian motion and \( \mu' \) is the new drift rate.

6. Consider a forward contract on an underlying commodity, find the portfolio consisting of the underlying commodity and bond (bond’s maturity coincides with forward’s maturity) that replicates the forward contract. Show that the hedge ratio \( \Delta \) is always equal to one. Give the financial argument to justify why the hedge ratio is one. Let \( B(t, T) \) denote the price at current time \( t \) of the unit-par zero-coupon bond maturing at time \( T \) and \( S \) denote the price of commodity at time \( t \). Show that the forward price \( F(S, \tau) \) is given by
\[
F(S, \tau) = S/B(t, T), \quad \tau = T - t.
\]

7. Consider a portfolio containing \( \Delta \) units of asset and \( M \) dollars of riskless asset in the form of money market account. The portfolio is dynamically adjusted so as to replicate an option. Let \( S \) and \( V(S, t) \) denote the value of the underlying asset and the option, respectively. Let \( r \) denote the riskless interest rate and \( \Pi \) denote the value of the self-financing replicating portfolio. When the self-financing trading strategy is adopted, explain why
\[
\Pi = \Delta S + M \quad \text{and} \quad d\Pi = \Delta \, dS + rM \, dt,
\]
where \( r \) is the riskless interest rate. Here, the differential term \( S \, d\Delta \) does not enter into \( d\Pi \). Assume that the asset price dynamics follows the Geometric Brownian process:

\[
\frac{dS}{S} = \rho \, dt + \sigma \, dZ.
\]

Using the condition that the option value and the value of the replicating portfolio should match at all times, show that the number of units of asset held must be given by

\[
\Delta = \frac{\partial V}{\partial S}.
\]

How to proceed further in order to obtain the Black-Scholes equation for \( V \)?

8. When a European option is currently out-of-the-money, show that a higher volatility of the asset price or a longer time to expiry makes it more likely for the option to expire in-the-money. What would be the impact on the value of delta? Do we have the same effect or opposite effect when the option is currently in-the-money?

9. Show that when the European call price is a convex function of the asset price, the elasticity of the call price is always greater than or equal to one. Give the financial argument to explain why the elasticity of the price of a European option increases in absolute value when the option becomes more out-of-the-money and closer to expiry. Can you think of a situation where the European put’s elasticity has absolute value less than one, that is, the European put option is less riskier than the underlying asset?

10. Suppose the greeks of the value of a derivative security are defined by

\[
\Theta = \frac{\partial f}{\partial t}, \quad \Delta = \frac{\partial f}{\partial S}, \quad \Gamma = \frac{\partial^2 f}{\partial S^2}.
\]

(a) Find the relation between \( \Theta \) and \( \Gamma \) for a delta-neutral portfolio where \( \Delta = 0 \).

(b) Show that the theta may become positive for an in-the-money European call option on a continuous dividend paying asset when the dividend yield is sufficiently high.

(c) Explain by financial argument why the theta value tends asymptotically to \(-rXe^{-rt}\) from below when the asset value is sufficiently high.

11. Consider a European capped call option whose terminal payoff function is given by

\[
c_M(S, \tau; X, M) = \min(\max(S - X, 0), M),
\]

where \( X \) is the strike price and \( M \) is the cap. Show that the value of the European capped call is given by

\[
c_M(S, \tau; X, M) = c(S, \tau; X) - c(S, \tau; X + M),
\]

where \( c(S, \tau; X + M) \) is the value of a European vanilla call with strike price \( X + M \).
12. Consider the value of a European call option written by an issuer whose only asset is \( \alpha \) \((\alpha < 1)\) units of the underlying asset. At expiration, the terminal payoff of this call is then given by

\[
S_T - X \quad \text{if} \quad \alpha S_T \geq S_T - X \geq 0 \alpha S_T \quad \text{if} \quad S_T - X > \alpha S_T \quad (1)
\]

and zero otherwise. Show that the value of this European call option is given by

\[
c_L(S, \tau; X, \alpha) = c(S, \tau; X) - (1 - \alpha)c \left( S, \tau; \frac{X}{1 - \alpha} \right), \quad \alpha < 1,
\]

where \( c \left( S, \tau; \frac{X}{1 - \alpha} \right) \) is the value of a European vanilla call with strike price \( \frac{X}{1 - \alpha} \).