1. Suppose the dividends and interest incomes are taxed at the rate $R$ but capital gains taxes are zero. Find the price formulas for the European put and call on an asset which pays a continuous dividend yield at the constant rate $q$, assuming that the riskless interest rate $r$ is also constant.

Hint: Explain why the riskless interest rate $r$ and dividend yield $q$ should be replaced by $r(1-R)$ and $q(1-R)$, respectively, in the Black-Scholes formulas.

2. Consider a futures on an underlying asset which pays $N$ discrete dividends between $t$ and $T$ and let $D_i$ denote the amount of the $i$th dividend paid on the ex-dividend date $t_i$. Show that the futures price is given by

$$ F(S,t) = Se^{r(T-t)} - \sum_{i=1}^{N} D_i e^{r(T-t_i)}, $$

where $S$ is the current asset price and $r$ is the riskless interest rate. Consider a European call option on the above futures. Show that the governing differential equation for the price of the call, $c_F(F,t)$, is given by (Brenner et al., 1985)

$$ \frac{\partial c_F}{\partial t} + \frac{\sigma^2}{2} \left[ F + \sum_{i=1}^{N} D_i e^{r(T-t_i)} \right] ^2 \frac{\partial^2 c_F}{\partial F^2} - rc_F = 0. $$

3. A forward start option is an option which comes into existence at some future time $T_1$ and expires at $T_2$ ($T_2 > T_1$). The strike price is set equal the asset price at $T_1$ such that the option is at-the-money at the future option’s initiation time $T_1$. Consider a forward start call option whose underlying asset has value $S$ at current time $t$ and constant dividend yield $q$, show that the value of the forward start call is given by

$$ e^{-qT_1} c(S, T_2 - T_1; S) $$

where $c(S, T_2 - T_1; S)$ is the value of an at-the-money call (strike price same as asset price) with time to expiry $T_2 - T_1$.

Hint: The value of an at-the-money call option is proportional to the asset price.

4. Consider a contingent claim whose value at maturity $T$ is given by

$$ \min(S_{T_0}, S_T), $$

where $S_{T_0}$ is the asset price at time $T_0$. Find the price formulas for this claim.
where $T_0$ is some intermediate time before maturity, $T_0 < T$, and $S_T$ and $S_{T_0}$ are the asset price at $T$ and $T_0$, respectively. Show that the value of the contingent claim at time $t$ is given by
\[
V_t = S_t[1 - N(d_1) + e^{-r(T-T_0)}N(d_2)],
\]
where $S_t$ is the asset price at time $t$ and
\[
d_1 = \frac{r(T - T_0) + \sigma^2(T - T_0)}{\sigma \sqrt{T - T_0}}, \quad d_2 = d_1 - \sigma \sqrt{T - T_0}.
\]

5. Let $Q^*$ denote the equivalent martingale measure where the asset price $S_t$ is used as the numeraire. Suppose $S_t$ follows the lognormal distribution with drift rate $r$ and volatility $\sigma$ under $Q^*$, where $r$ is the riskless interest rate. Show that
\[
\frac{dQ^*}{dQ} = \frac{S_T}{S_0} e^{-rT} = e^{-\frac{\sigma^2}{2} + \sigma Z_T},
\]
where $Q$ is the martingale measure with the money market account as the numeraire and $Z_T$ is a Brownian motion under $Q$. Using the Girsanov Theorem, show that
\[
Z_T^* = Z_T - \sigma T
\]
is a Brownian motion under $Q^*$. Explain why
\[
E_{Q^*}[1_{\{S_T \geq X\}}] = N\left(\frac{\ln \frac{S_0}{X} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma \sqrt{T}}\right),
\]
then deduce that
\[
E_Q[S_T 1_{\{S_T \geq X\}}] = e^{rT} S_0 N\left(\frac{\ln \frac{S_0}{X} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma \sqrt{T}}\right).
\]

6. Suppose the dynamics of $X_t$ and $S_t$ under the risk neutral measure $Q$ are governed by
\[
\frac{dX_t}{X_t} = r dt + \sigma_X dZ_{X,t}^Q \quad \text{and} \quad \frac{dS_t}{S_t} = r dt + \sigma_S dZ_{S,t}^Q,
\]
where $Z_{X,t}^Q$ and $Z_{S,t}^Q$ are $Q$-Brownian. Note that both $X_t/M_t$ and $S_t/M_t$ are martingales under $Q$, where $M_t = e^{rt}$. Show that $X_t/S_t$ is a martingale under $Q^*$, where $Q^*$ is the measure defined by
\[
L_t = \frac{dQ^*}{dQ} |_{F_t} = \frac{S_t}{S_0} \left/ \frac{M_t}{M_0} \right., \quad t \in (0, T].
\]
HINT: $Z_{X,t}^Q - \rho \sigma S$ and $Z_{S,t}^Q - \sigma S$ are $Q^*$-Brownian.

7. Consider the exchange option which entities the holder the right but not the obligation to exchange risky asset $S_2$ for another risky asset $S_1$. Let the price dynamics of $S_1$ and $S_2$ under the risk neutral measure be governed by
\[
\frac{dS_i}{S_i} = (r - q_i) dt + \sigma_i dZ_i, \quad i = 1, 2,
\]
where \(dZ_1 \, dZ_2 = \rho \, dt\). Let \(V(S_1, S_2, \tau)\) denote the price function of the exchange option, whose terminal payoff takes the form
\[
V(S_1, S_2, 0) = \max(S_1 - S_2, 0).
\]
Show that the governing equation for \(V(S_1, S_2, \tau)\) is given by
\[
\frac{\partial V}{\partial \tau} = \frac{\sigma_1^2 S_1^2 \partial^2 V}{2 \partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{\sigma_2^2 S_2^2 \partial^2 V}{2 \partial S_2^2} \\
+ (r - q_1) S_1 \frac{\partial V}{\partial S_1} + (r - q_2) S_2 \frac{\partial V}{\partial S_2} - rV.
\]
By taking \(S_2\) as the numeraire and defining the similarity variables
\[
x = \frac{S_1}{S_2} \quad \text{and} \quad W(x, \tau) = \frac{V(S_1, S_2, \tau)}{S_2},
\]
show that the governing equation for \(W(x, \tau)\) becomes
\[
\frac{\partial W}{\partial \tau} = \frac{\sigma_2^2 x^2 \partial^2 W}{2 \partial x^2} + (q_2 - q_1) x \frac{\partial W}{\partial x} - q_2 W.
\]
Verify that the solution to \(W(x, \tau)\) is given by
\[
W(x, \tau) = e^{-q_2 \tau} x N(d_1) - e^{-q_2 \tau} N(d_2)
\]
where
\[
d_1 = \ln \left(\frac{S_1}{S_2}\right) + \left(q_2 - q_1 + \frac{\sigma_1^2}{2}\right) \tau \quad \sigma_2 = \sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2.
\]
In summary, the price function \(V(S_1, S_2, \tau)\) can be expressed as
\[
V(S_1, S_2, \tau) = e^{-r \tau} \left[e^{(r-q_1)\tau} N(d_1) - e^{(r-q_2)\tau} N(d_2)\right],
\]
where
\[
d_1 = \frac{\ln \left(\frac{S_1}{S_2}\right) + \left[(r - q_1) - (r - q_2) + \frac{\sigma_2^2}{2}\right] \tau}{\sigma \sqrt{\tau}}, \quad \sigma^2 = \sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2, \\
d_2 = -\frac{\ln \left(\frac{S_2}{S_1}\right) + \left[(r - q_2) - (r - q_1) + \frac{\sigma_2^2}{2}\right] \tau}{\sigma \sqrt{\tau}} = d_1 - \sigma \sqrt{\tau}.
\]
8. Suppose the terminal payoff of an exchange rate option is $F_T \mathbb{1}_{\{F_T > X\}}$. Let $V_d(F, t)$ denote the value of the option in the domestic currency world, show that

$$V_d(F, t) = F e^{-r_f (T-t)} E_{Q_f} \left[ \mathbb{1}_{\{F_T > X\}} \mid F_t = F \right] = F e^{-r_f \tau} N(d)$$

where

$$d = \frac{\ln \frac{F}{X} + (r_d - r_f + \frac{\sigma_d^2}{2}) \tau}{\sigma_F \sqrt{\tau}}, \quad \tau = T - t.$$

9. Let $F_{S\text{\textdialed}}$ denote the Singaporean currency price of one unit of US currency and $F_{H\text{\textdialed}}$ denote the Hong Kong currency price of one unit of Singaporean currency. Suppose we assume $F_{S\text{\textdialed}}$ to be governed by the following dynamics under the risk neutral measure $Q_S$ in the Singaporean currency world:

$$\frac{dF_{S\text{\textdialed}}}{F_{S\text{\textdialed}}} = (r_{SGD} - r_{USD}) dt + \sigma_{F_{S\text{\textdialed}}} dZ_{F_{S\text{\textdialed}}},$$

where $r_{SGD}$ and $r_{USD}$ are the Singaporean and US riskless interest rates, respectively. Similar Geometric Brownian process assumption is made for other exchange rate processes. The digital quanto option pays one US dollar at maturity if $F_{S\text{\textdialed}}$ is above $\alpha F_{H\text{\textdialed}}$ for some constant value $\alpha$. Find the value of the digital quanto option in Hong Kong dollar in terms of the exchange rates, the riskless interest rates of the different currency worlds and volatility values.

10. In the Merton model of risky debt, suppose we define

$$\sigma_V(\tau; d) = \frac{\sigma A \partial V}{V \partial A},$$

which gives the volatility of the value of the risky debt. Also, we denote the credit spread by $s(\tau; d)$, where $s(\tau; d) = Y(\tau) - r$. Show that

(a) $\frac{\partial s}{\partial d} = \frac{1}{\tau d} \sigma_V(\tau; d) > 0$;

(b) $\frac{\partial s}{\partial \sigma^2} = \frac{1}{2 \sqrt{\tau} N(d_1)} \sigma_V(\tau; d) > 0$, where $d_1 = \frac{\ln d}{\sigma \sqrt{\tau}} - \frac{\sigma \sqrt{\tau}}{2}$;

(c) $\frac{\partial s}{\partial r} = -\sigma_V(\tau; d) < 0$.

Give financial interpretation to each of the above results.

11. A firm is an entity consisting of its assets and let $A_t$ denote the market value of firm’s assets. Assume that the total asset value follows a stochastic process modeled by

$$\frac{dA_t}{A_t} = \mu \ dt + \sigma \ dZ_t,$$

where $\mu$ and $\sigma^2$ (assumed to be constant) are the instantaneous mean and variance, respectively, of the rate of return on $A_t$. Let $C$ and $D$ denote the market value of the
current liabilities and market value of debt, respectively. Let $T$ be the maturity date of the debt with face value $D_T$. Suppose the current liabilities of amount $C_T$ are also payable at time $T$, and it constitutes a claim senior to the debt. Also, let $F$ denote the present value of total amount of interest and dividends paid over the term $T$. For simplicity, $F$ is assumed to be prepaid at time $t = 0$.

The debt is in default if $A_T$ is less than the total amount payable at maturity date $T$, that is,

$$A_T < D_T + C_T.$$  

(a) Show that the probability of default is given by

$$p = N\left(\frac{\ln \frac{D_T + C_T}{A_T - F} - \mu T + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}}\right).$$

(b) Explain why the expected loan loss $L$ on the debt is given by

$$EL = \int_{C_T}^{D_T + C_T} (D_T + C_T - a) f(a) \, da + \int_0^{C_T} D_T f(a) \, da,$$

where $f$ is the density function of $A_T$. Give the financial interpretation to each integral.