1. Suppose we choose the mixture distribution in the Bernoulli mixture model to be the beta distribution whose density function is given by

$$f(\tilde{p}) = \frac{1}{\beta(a, b)} \tilde{p}^{a-1} (1 - \tilde{p})^{b-1}, \quad a, b > 0, \quad 0 < \tilde{p} < 1,$$

where the beta function \( \beta(a, b) \) is defined by

$$\beta(a, b) = \int_0^1 x^{a-1} (1 - x)^{b-1} dx = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)}, \quad \Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx.$$

(a) Based on the Bernoulli mixture model, show that the probability of \( k \) defaults out of \( m \) obligors is given by

$$P[M = k] = \binom{m}{k} \int_0^1 \tilde{p}^k (1 - \tilde{p})^{m-k} f(\tilde{p}) d\tilde{p} = \binom{m}{k} \frac{\beta(a + k, b + m - k)}{\beta(a, b)}.$$

Find the corresponding default-event correlation coefficient \( \rho(X_i, X_j) \).

(b) Recall that the mean and variance of the beta distribution are given by

$$\text{mean} = \frac{a}{a + b} \quad \text{and} \quad \text{variance} = \frac{ab}{(a + b)^2 (a + b + 1)}.$$

As \( a \) increases, do we have higher or lower default-event correlation?

Hint: Look at various combinations of the two parameters for which \( \frac{a}{a + b} = \bar{p} \) for some default probability \( \bar{p} \).

2. Consider the Davis-Lo contagion model, show that

$$E[D_n] = n[1 - (1 - p)(1 - pq)^{n-1}]$$

$$\text{cov}(Z_i, Z_j) = \beta_n pq - (E[D_n/n])^2,$$

where

$$\beta_n pq = p^2 + 2p(1 - p)[1 - (1 - q)(1 - pq)^{n-2}] + (1 - p)^2[1 - 2(1 - pq)^{n-2} + (1 - 2pq + pq^2)^{n-2}].$$

Hint: The event \((Z_1 = 1, \ldots, Z_k = 1, Z_{k+1} = 0, \ldots, Z_n = 0)\) can be achieved in various disjoint combinations. Firstly, we may have \((X_1 = 1, X_k = 1, X_{k+1} = 0, X_n = 0, Y_{ij} = 0, i = 1, \ldots, k, j = k + 1, \ldots, n)\) i.e. bonds 1 to \( k \) default directly and do not infect bonds \( k + 1 \) to \( n \). On the other hand, bonds 1 to \( i \) (for some \( i < k \)) may default directly and infect the other bonds of the first \( k \) but none of the remaining.
3. This problem is an extension of the mixture approach to the Poisson model of default. Consider the Poisson mixture model where the loss statistic $S$ is a random vector $L = (L_1, \cdots, L_m)$ of Poisson random variables $L_i \sim \text{Pois}(\Lambda_i)$, where $\Lambda = (\Lambda_1, \cdots, \Lambda_m)$ is a random vector with some distribution function $F$ with support in $[0, \infty)^m$. Note that the default probability of obligor $i$ is given by $p_i = \mathbb{P}[L_i \geq 1]$ (apparently allowing multiple defaults). We assume that conditional on a realization $\lambda = (\lambda_1, \cdots, \lambda_m)$ of $\Lambda$, the variables $L_1, L_2, \cdots, L_m$ are independent:

$\left. L_i \right|_{\lambda_i = \lambda_i} \sim \text{Pois}(\lambda_i), \quad (\left. L_i \right|_{\lambda = \lambda})_{i=1,\cdots,m}$ are independent.

The (unconditional) joint distribution of the variables $L_i$ is given by

$$\mathbb{P}[L_1 = \ell_1, \cdots, L_m = \ell_m] = \int_{[0, \infty)^m} e^{-\sum \lambda_i} \prod \lambda_i^{\ell_i} \ell_i! \, dF(\lambda_1, \cdots, \lambda_m).$$

Show that the correlation coefficient between pairwise default events is given by

$$\rho(L_i, L_j) = \frac{\text{cov}(\Lambda_i, \Lambda_j)}{\sqrt{\text{var}(\Lambda_i) + \mathbb{E}[\Lambda_i] \sqrt{\text{var}(\Lambda_j) + \mathbb{E}[\Lambda_j]}}}.$$

Hint:

$$\mathbb{E}[L_i] = \mathbb{E}[\Lambda_i]$$

$$\text{var}(L_i) = \text{var}(\mathbb{E}[L_i|\Lambda]) + \mathbb{E}[\text{var}(L_i|\Lambda)]$$

$$= \text{var}(\Lambda_i) + \mathbb{E}[\Lambda_i].$$

4. Consider a portfolio of $m$ risky bonds (of equal face value) with uniform default probability $p$. Let $L_i$ denote the default event indicator of bond $i$, where $L_i \sim \text{B}(1; p)$. Let $\rho$ be the uniform correlation coefficient between pairwise defaults of any two bonds. In this problem, we use the Binomial Expansion Technique where the defaults in a comparison portfolio are assumed to be independent. By matching the second order moment of the original portfolio and the comparison portfolio consisting of $n(\rho)$ independent bonds, show that the diversification score $n(\rho)$ is given

$$n(\rho) = \frac{m}{1 + \rho(m - 1)}.$$

Explain why the above diversification score is bounded from above by $1/\rho$. 