1. The CreditRisk+ Model is designed to incorporate the effects of variability in the average rates of default. In this problem, we would like to show that the CreditRisk+ Model behaves as if the default rates were fixed when the standard deviation of the mean default rate for each sector tends to zero. Recall that the CreditRisk+ has the following probability generating function for the default losses:

\[
G(z) = \prod_{k=1}^{n} G_k(z) = \prod_{k=1}^{n} \left( \frac{1 - p_k}{1 - \frac{p_k}{\mu_k} \sum_{j=1}^{m(k)} \epsilon_j^{(k)} v_j^{(k)} z^{v_j^{(k)}}} \right)^{\alpha_k}
\]

where

\[
\alpha_k = \frac{\mu_k^2}{\sigma_k^2}, \quad \beta_k = \frac{\sigma_k^2}{\mu_k}, \quad p_k = \frac{\beta_k}{1 + \beta_k} \quad \text{and} \quad \mu_k = \sum_{k=1}^{m(k)} \epsilon_j^{(k)} v_j^{(k)}.
\]

Now, we consider the limit where \(\sigma_k \to 0\). Show that as \(\sigma_k \to 0\), we recover the result in the static case (deterministic default probabilities) where

\[
G(z) \to e^{\sum_j \epsilon_j^{(k)} v_j^{(k)} (z^{v_j^{(k)}} - 1)}.
\]

**Hint:** As \(\sigma_k \to 0\), we have

\[
\beta_k \to 0, \quad p_k \to 0 \quad \text{and} \quad \alpha_k \to \frac{\mu_k}{p_k}.
\]

2. This problem is related to the calculation of the risk contributions of the obligors. Let \(\sigma_P^2\) be the portfolio variance and \(RC_A\) be the risk contribution of obligor \(A\). Show that

\[
\sigma_P^2 = \sum_{k=1}^{n} \epsilon_k^2 \left( \frac{\sigma_k}{\mu_k} \right)^2 + \sum_A \epsilon_A v_A
\]

and

\[
RC_A = \frac{E_A \mu_A}{\sigma_P} \left[ E_A + \sum_k \left( \frac{\sigma_k}{\mu_k} \right)^2 \epsilon_k \theta_{A_k} \right].
\]

Based on the above formula of \(RC_A\), check whether

\[
\sum_A RC_A = \sigma_P.
\]

**Hint:**

\[
RC_A = \frac{E_A \frac{\partial \sigma_P^2}{\partial \sigma_P}}{2\sigma_P \partial E_A}.
\]
3. We consider the derivation of an approximation formula of the pairwise correlation. Under the assumption of small default probabilities so that

$$\mu_A = 1 - e^{-p_A \Delta t} \approx p_A \Delta t,$$

show that the pairwise correlation is approximately given by

$$\rho_{AB} = \sqrt{\mu_A \mu_B} \sum_{k=1}^{n} \theta_A k \theta_B k \left( \frac{\sigma_k}{\mu_k} \right)^2.$$ 

4. The CreditRisk$^+$ model is a Poisson mixture model. How does the CreditRisk$^+$ approach model the default correlation among obligors in the credit portfolio using the sector analysis. How does it model the stochastic probability of default of an individual obligor?

5. Let $K_j$ denote the aggregate loss in band $j$ (in units of $L$), $j = 1, 2 \cdots, m$. We assume $K_j$ to be a discrete random variable taking integer values $0, v_j, 2v_j, \cdots$. Given $n$ defaults in band $j$, the loss will be $nv_j$ (in units of $L$). Let $G_j(z)$ denote the probability generating function of $K_j$. We then have

$$G_j(z) = \sum_{n=0}^{\infty} P(n \text{ defaults}) z^{nv_j} = \sum_{n=0}^{\infty} \frac{e^{-\mu_j} \mu_j^n z^{nv_j}}{n!} = e^{-\mu_j} e^{\mu_j z + z^{v_j}}.$$ 

Since default events are independent, $K_1, \cdots, K_m$ are independent discrete random variables. Suppose we define

$$G(z) = \sum_{n=0}^{\infty} P(\text{aggregate loss } = nL) z^n,$$

then

$$G(z) = \prod_{j=1}^{m} G_j(z).$$

One can show easily that

$$G(z) = e^{\mu [p(z) - 1]}$$

where

$$\mu = \sum_{j=1}^{m} \mu_j \quad \text{and} \quad p(z) = \frac{\sum_{j=1}^{m} \left( \frac{\epsilon_j}{v_j} \right) z^{v_j}}{\sum_{j=1}^{m} \frac{\epsilon_j}{v_j}}.$$ 

Let $A_n$ denote

$$P(\text{loss of } nL) = \frac{1}{n!} \frac{d^n G(z)}{dz^n} \bigg|_{z=0},$$

show that

$$A_n = \sum_{j: v_j \leq n} \frac{\epsilon_j}{n} A_{n-v_j},$$

Give a financial interpretation of the result.