MAFS522 – Quantitative and Statistical Risk Analysis

Topic One – Mixture models for modeling default correction

- Bernuolli mixture models
- Mixing distribution using Merton's structural model
- Moody's binomial expansion method
- Contagion models
- Exponential models for dependent defaults

Modelng dependent defaults

Why are we concerned about dependence between default events and between credit quality changes?

- It affects the distribution of loan portfolio losses – critical in determining quantities or other risk measures used for allocating capital for solvency purposes.

- Clustering phenomena: simultaneous defaults could affect the stability of the financial system with profound effects on the entire economy.

- Occurrence of disproportionately many joint defaults is termed "extreme credit risk".

Difficulties

- Parameter specification for default dependence models is often somewhat arbitrary.

Issues

- The "marginal" credit risk of each issuer in a pool is usually "well" determined. Modeling various correlation structures that work with the given marginal characteristics is the major challenge.

- Counterparty risk: The counterparty of a financial contract cannot honor the contractual specification. For example, in a credit default swap, the counterparty risk is high when the credit quality of the protection seller is correlated with that of the underlying reference securities.

Bernuolli mixture model

- Naturally, the discrete Bernuolli random variable with only two possible values can be used as an indicator of default of an obligor.

- The loss of a portfolio from a loss statistics $\boldsymbol{L} = (L_1, \cdots, L_m)$ with Bernuolli variables $L_i \sim B(1; P_i)$, where $B(m; p)$ denotes the binomial distribution with m independent trials and stationary probability of success p. The loss probabilities (over a given time horizon) are random variables

$$\boldsymbol{P} = (P_1, \cdots, P_m) \sim \boldsymbol{F}$$

  for some distribution function $\boldsymbol{F}$ with support in $[0, 1]^m$.

- In a mixture model, the default probability P_i of obligor i is assumed to depend on a set of common economic factors.

Conditional independence

Two events R and B are said to be conditionally independent given the third event G if and only if

$$P(R \cap B|G) = P(R(G)P(B|G)$$

or

$$P(R|B \cap G) = P(R|G).$$

Two random variable X and Y are conditionally independent given an event G if they are independent in their conditional probability distribution given G. In terms of density functions:

$$f_X(x|G)f_Y(y|G) = f_{X,Y}(x,y|G),$$

or in terms of distribution fuctions:

$$F_X(x,G)F_Y(y|G) = F_{X,Y}(x,y|G).$$

Conditional on the realization $\widehat{\boldsymbol{P}} = (\widehat{P}_1, \cdots, \widehat{P}_m)$ of $\boldsymbol{P}$, the Bernuolli random variables $L_1, \cdots, L_m$ are independent.

That is, given the default probabilities, defaults of different obligors are independent.

The (unconditional) joint distribution of L_i's is

$$\mathbb{P}(L_1 = \ell_1, \cdots, L_m = \ell_m) = \int_{[0,1]^m} \prod_{i=1}^{m} \widehat{P}_i^{\ell_i}(1 - \widehat{P}_i)^{1-\ell_i}\, d\boldsymbol{F}(\widehat{P}_1, \cdots, \widehat{P}_m),$$

where $\ell_i \in \{0, 1\}$. Here "0" denotes "no default" and "1" denotes "default". For example,

$$\mathbb{P}(L_1 = 1, L_2 = 0, L_3 = 1) = \int_{[0,1]^3} \widehat{P}_1(1 - \widehat{P}_2)\widehat{P}_3\, d\boldsymbol{F}(\widehat{P}_1, \widehat{P}_2, \widehat{P}_3).$$

The first and second moments of the single loss L_i are computed as follows. Observe that

$$E[L_i|\boldsymbol{P}] = 1 \times P_i + 0 \times (1 - P_i) = P_i,$$

so by the tower rule we obtain

$$E[L_i] = E[E[L_i|\boldsymbol{P}]] = E[P_i].$$

The dependence between defaults stems from the dependence of the default probabilities on a set of common factors.

Recall: $E[L_i^2|\boldsymbol{P}] = 1^2 \times P_i + 0^2 \times (1 - P_i) = P_i$. Also,

$$E[L_i L_j] = E[E[L_i L_j|\boldsymbol{P}]] = E[P_i P_j].$$

By the conditional variance formula, we obtain

$$\begin{aligned} \text{var}(L_i) &= \text{var}(E[L_i|\boldsymbol{P}]) + E[\text{var}(L_i|\boldsymbol{P})] \\ &= \text{var}(P_i) + E[E[L_i^2|\boldsymbol{P}] - E[L_i|\boldsymbol{P}]^2] \\ &= \text{var}(P_i) + E[P_i(1 - P_i)] = E[P_i](1 - E[P_i]). \end{aligned}$$

The covariance between a pair of losses

$$\text{cov}(L_i, L_j) = E[L_i L_j] - E[L_i]E[L_j].$$

Note that

$$\begin{aligned}
E[L_iL_j] &= \mathbb{P}(L_i = 1, L_j = 1) \times 1 \times 1 + \mathbb{P}(L_i = 1, L_j = 0) \times 1 \times 0 \\
& \quad \mathbb{P}(L_i = 0, L_j = 1) \times 0 \times 1 + \mathbb{P}(L_i = 0, L_j = 0) \times 0 \times 0 \\
&= \mathbb{P}(L_i = 1, L_j = 1) \\
&= \int_0^1 \int_0^1 \widehat{P}_i \widehat{P}_j \, d\boldsymbol{F}(\widehat{P}_i, \widehat{P}_j) \text{ by conditional independence} \\
&= E[P_iP_j].
\end{aligned}$$

Hence,

$$\text{cov}(L_i, L_j) = E[P_iP_j] - E[P_i]E[P_j] = \text{cov}(P_i, P_j),$$

so that the default correlation in a Bernuolli mixture model is

$$\text{corr}(L_i, L_j) = \frac{\text{cov}(P_i, P_j)}{\sqrt{E[P_i](1 - E[P_i])}\sqrt{E[P_j](1 - E[P_j])}}.$$

The dependence between losses in the portfolio is fully captured by the covariance structure of the multivariate distribution $\boldsymbol{F}$ of loss probabilities $\boldsymbol{P}$.

One-factor Bernuolli mixture model

Retail banking portfolios and portfolios of smaller banks are often quite homogeneous. Assuming $L_i \sim B(1; p)$ with a common random default probability $p \sim F$, where F is a distribution function with support in $[0, 1]$. As the mixture distribution is dependent on the single distribution $F(p)$, this leads to the one-factor *Bernuolli mixture model*. The joint distribution of L_i's:

$$\mathbb{P}[L_1 = \ell_1, \cdots, L_m = \ell_m] = \int_0^1 p^k (1-p)^{m-k}\, dF(p)$$

where $k = \sum_{i=1}^{m} \ell_i$ and $\ell_i \in \{0, 1\}, i = 1, 2, \cdots, m$. Write L as the random number of defaults.

- The probability that exactly k defaults occur is

$$\mathbb{P}[L = k] = \binom{m}{k} \int_0^1 p^k (1-p)^{m-k} \, dF(p).$$

This is the mixture of the binomial probabilities with the mixing distribution F.

- The uniform default probability of any obligor in the homogenous portfolio is given by

$$\overline{p} = \mathbb{P}[L_i = 1] = E[L_i] = \int_0^1 p \, dF(p).$$

- Note that $E[L_i L_j] = \mathbb{P}[L_i = 1, L_j = 1]$. The uniform default correlation of two different obligors is

$$\begin{aligned} \rho &= \text{corr}(L_i, L_j) = \frac{\mathbb{P}[L_i = 1, L_j = 1] - \overline{p}^2}{\overline{p}(1-\overline{p})} \\ &= \frac{\int_0^1 p^2\, dF(p) - \overline{p}^2}{\overline{p}(1-\overline{p})} = \frac{\text{var}(p)}{\overline{p}(1-\overline{p})}. \end{aligned}$$

- Intuitively, with a higher var(p), we have higher corr(L_i, L_j).

- Recall that the variance of the Bernuolli variable assuming values 0 and 1, and parameter $\overline{p}$ is

$$E[X^2] - E[X]^2 = \overline{p} - \overline{p}^2 = \overline{p}(1-\overline{p}).$$

  Note that $\rho = 1$ if and only if var$(p) = \overline{p}(1-\overline{p})$. This occurs when p is Bernuolli with parameter $\overline{p}$.

Remarks

1. Since var$(p) \geq 0$, so corr$(L_i, L_j) \geq 0$. The non-negativity of default correlation is obvious since L_i and L_j are dependent on the common mixture variable p. In other words, we cannot implement negative dependencies between the default events of obligors under this one-factor binomial mixture model.

2. corr$(L_i, L_j) = 0$ if and only if var$(p) = 0$, implying no randomness with regard to p. In this case, p assumes the single value $\overline{p}$. The absolute portfolio loss L follows a binomial distribution with constant default probability $\overline{p}$. Correspondingly, the default events are independent.

3. corr$(L_i, L_j) = 1$ implies a "rigid" behavior of single losses in the portfolio. This corresponds to $p = 1$ with probability $\overline{p}$ and $p = 0$ with probability $1 - \overline{p}$, where the distribution F of p is a Bernoulli distribution. Financially speaking, when an external event occurs with probability $\overline{p}$, all obligors in the portfolio default and the total portfolio is lost. Otherwise, with probability $1 - \overline{p}$, all obligors survive.

$\star$ A non-financial analogy is the death events of all passengers in an aeroplane.

Fractional losses under the one-factor binomial mixture model

Define $D_n = \sum_{i=1}^n L_i$, which is the total number of defaults in the portfolio. We then have

$$E[D_n] = n\overline{p}.$$

Recall

$$\text{var}(L_i) = E[P_i](1 - E[P_i]) = \overline{p}(1 - \overline{p}).$$

Under the assumption of uniform default probability, we have

$$\begin{aligned} \text{var}(D_n) &= \sum_{i=1}^n \text{var}(L_i) + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \text{cov}(L_i, L_j) \\ &= n\overline{p}(1 - \overline{p}) + n(n-1)(E[p^2] - E[p]^2), \end{aligned}$$

so that

$$\text{var}\left(\frac{D_n}{n}\right) = \frac{\overline{p}(1-\overline{p})}{n} + \frac{n(n-1)}{n^2}\text{var}(p) \longrightarrow \text{var}(p) \text{ as } n \to \infty.$$

When considering the fractional loss for n large, the only remaining variance is that of the distribution of p.

- One can obtain any default correlation in $[0, 1]$. Note that the correlation of default events depends only on the first and second moments of F. However, the distribution of D_n can be quite different for different distribution F.

Large portfolio approximation

By the law of large number,

$$\frac{D_n}{n} \to \widetilde{p} \text{ as } n \to \infty$$

when the realized default probability equals $\widetilde{p}$. That is,

$$\mathbb{P}\left(\frac{D_n}{n} < \theta \middle| p = \widetilde{p}\right) \xrightarrow{n\uparrow\infty} \begin{cases} 0 & \text{if } \theta < \widetilde{p} \\ 1 & \text{if } \theta > \widetilde{p} \end{cases}.$$

$$\begin{aligned} \mathbb{P}\left(\frac{D_n}{n} < \theta\right) \xrightarrow{n\uparrow\infty} & \int_0^1 \mathbf{1}_{\{\theta > \widetilde{p}\}} f(\widetilde{p})\, d\widetilde{p} \\ = & \int_0^\theta f(\widetilde{p})\, d\widetilde{p} = F(\theta). \end{aligned}$$

As $n \to \infty$, it is the probability distribution of the random default probability p that determines the fractional loss distribution.

Remarks

1. For a given fixed unconditional default probability $\overline{p}$, increasing correlation increases the probability of seeing large losses and of seeing small losses compared with a situation with small correlation.

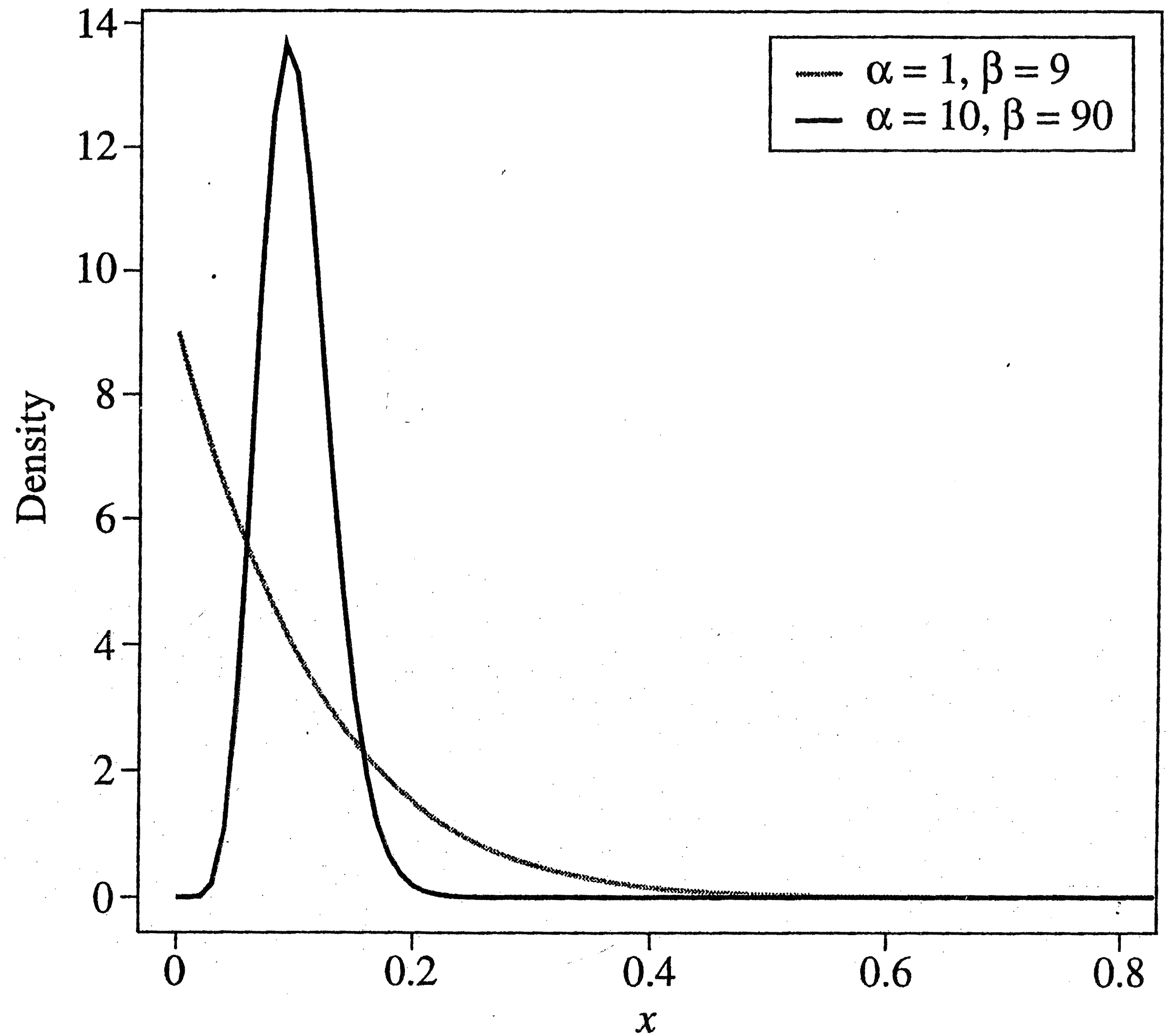
2. It is the common dependence on the mixture variable p that induces the correlation in the default events. It requires the assumption of large fluctuations in p to obtain significant correlation. The more variability that is in the mixture distribution, the more correlation of default events and more weight there in the tails of the loss distribution.

Example – beta distribution

A beta distribution for p gives us a flexible class of distributions in the interval $[0, 1]$. The beta distribution is characterized by the two positive parameters α, β. The mean and variance of the distribution are

$$E[p] = \frac{\alpha}{\alpha + \beta}, \quad \text{var}(p) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

If we look at various combinations of the two parameters for which $\frac{\alpha}{\alpha+\beta} = \overline{p}$ for some unconditional default probability $\overline{p}$, the variance of the distribution will decrease as we increase α.



Two beta distributions both with mean 0.1 but with different variances. The density corresponds to $\alpha = 1, \beta = 9$ observes higher probability of seeing smaller losses and larger losses.

Comparison with the case of independence of defaults

- The binomial distribution for independent defaults has a very **thin tail**, thus not representing the possibility of a large number of defaults realistically.

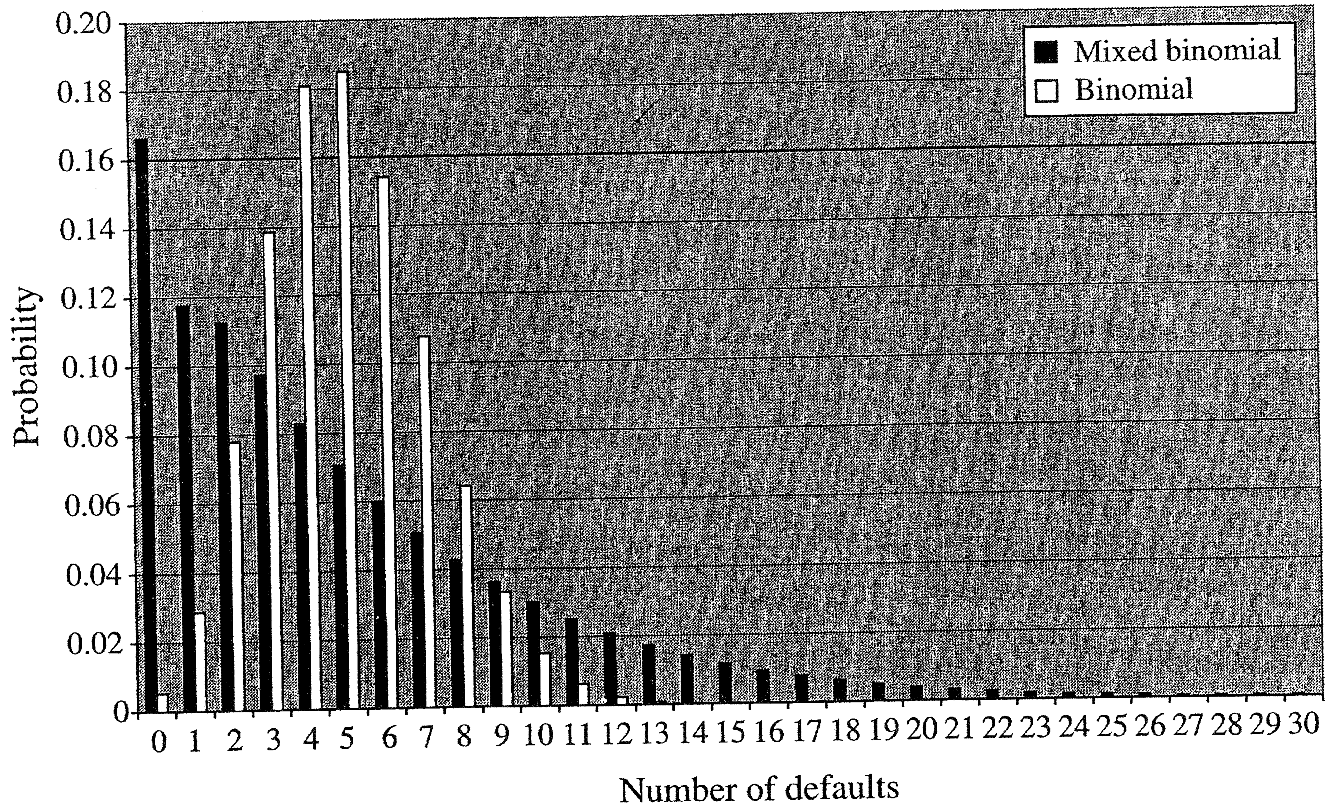
- Taking $N = 100$ obligors

Default Prob. (%)	1	2	3	4	5	6	7	8	9	10
99.0% VaR Level	5	7	9	11	13	14	16	17	19	20

What is $\mathrm{VaR}_\alpha(X)$? Maximum loss which is not exceeded with a given high probability (or confidence level).

$$VaR_\alpha(X) = \inf\{x \geq 0 | P[X \leq x] \geq \alpha\}.$$

Take $p = 5\%$, the probability with 13 defaults or less is at least 99%, that is, 99% confidence level.



The distribution of the number of defaults among 50 issuers in the case of a pure binomial model with default probability 0.1 and in cases with beta distributions ($\alpha = 1, \beta = 9$) as mixture distributions over the default probability. The later case exhibits a higher probability of seeing a large number of default losses (commonly known as the thick tail) and small number of losses also.

Mixing distribution using Merton's structural model

- Construct the random default probability (mixture variable) using the structural approach for modeling default events.

Consider n firms whose asset values V_t^i follow

$$dV_t^i = rV_t^i\,dt + \sigma V_t^i\,dB_t^i$$

with

$$B_t^i = \rho \widetilde{B}_t^0 + \sqrt{1-\rho^2}\,\widetilde{B}_t^i.$$

The GBM driving V_t^i can be decomposed into a common factor $\widetilde{B}_t^0$ and the firm-specific factor $\widetilde{B}_t^i$. Also, $\widetilde{B}^0, \widetilde{B}^1, \widetilde{B}^2, \cdots$ are independent standard Brownian motions. The firms are assumed to be identical in terms of drift rate and volatility.

The logarithm of the asset value of the firms are all correlated with the common correlation coefficient ρ. The dependence of stochastic movements of firm values among different firms is exhibited through the dependence on $\widetilde{B}_t^0$.

Default mechanism

Let D_i denote the default threshold of Firm i. Firm i defaults when

$$V_T^i = V_0^i \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T^i\right) < D^i$$

or

$$\ln V_0^i - \ln D^i + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\left(\rho \widetilde{B}_T^0 + \sqrt{1-\rho^2}\,\widetilde{B}_T^i\right) < 0.$$

We write formally $\widetilde{B}_T^i = \epsilon_i\sqrt{T}$, where ϵ_i is a standard normal random variable.

In terms of ϵ_i and ϵ_0, firm i defaults when

$$\frac{\ln V_0^i - \ln D^i + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} + \rho\epsilon_0 + \sqrt{1-\rho^2}\,\epsilon_i < 0.$$

Conditional on a realization of the common factor, say, $\epsilon_0 = u$, firm i defaults when

$$\epsilon_i < -\frac{c_i + \rho u}{\sqrt{1-\rho^2}}$$

where

$$c_i = \frac{\ln \frac{V_0^i}{D_i} + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}.$$

Assume V_0^i/D_i to be the same for all i so that $c_i = c$ for all i. Given $\epsilon_0 = u$, the probability of default is

$$p(u) = N\left(-\frac{c + \rho u}{\sqrt{1-\rho^2}}\right).$$

Given $\epsilon_0 = u$, defaults of the firms are independent. The mixing distribution is that of the common factor ϵ_0, and the normal cumulative distribution function N transforms ϵ_0 into a distribution on $[0, 1]$.

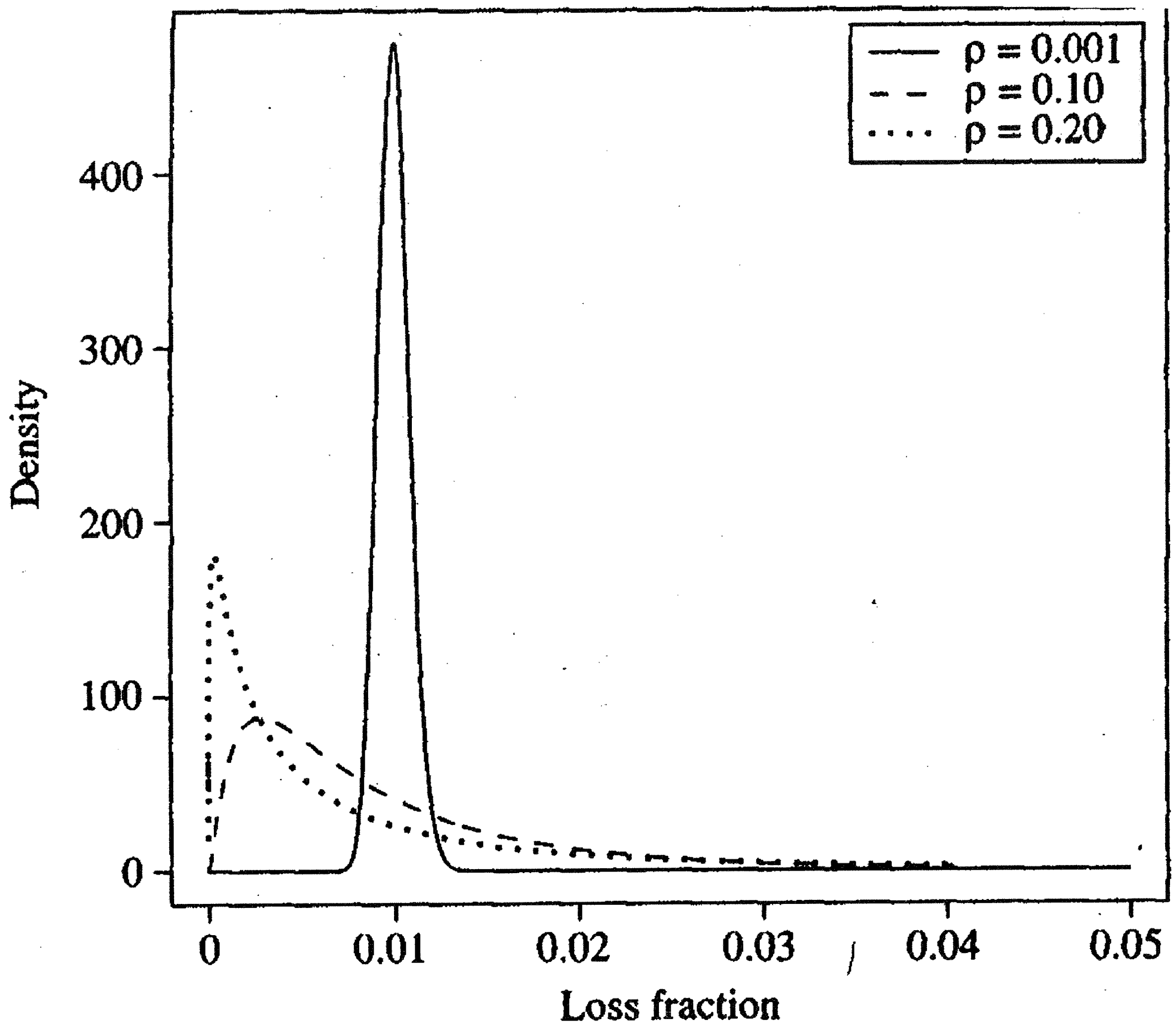
The distribution function $F(\theta)$ corresponding to the distribution of the mixing variable $p = p(\epsilon_0)$ is

$$\begin{aligned} F(\theta) &= P[p(\epsilon_0) \leq \theta] = P\left[N\left(-\frac{c+\rho\epsilon_0}{\sqrt{1-\rho^2}}\right) \leq \theta\right] \\ &= P\left[-\epsilon_0 \leq \frac{1}{\rho}\left(\sqrt{1-\rho^2}N^{-1}(\theta)+c\right)\right] \\ &= N\left(\frac{1}{\rho}\left(\sqrt{1-\rho^2}N^{-1}(\theta)-N^{-1}(\overline{p})\right)\right) \quad \text{where} \quad \overline{p} = N(-c). \end{aligned}$$

Here, $\overline{p}$ is the unconditional default probability corresponding to $\rho = 0$.

Note that $F(\theta)$ has the appealing feature that it has dependence on ρ and $\overline{p}$. The probability that the fraction of defaults being less than or equal to θ is

$$P\left(\frac{D_n}{n} \leq \theta\right) = \int_0^1 \sum_{k=0}^{n\theta} {}_nC_k p(u)^k [1-p(u)]^{n-k} f(u)\,du.$$



The figure shows the loss distribution in an infinitely diversified loan portfolio consisting of loans of equal size and with one common factor of default risk. The unconditional default probability is fixed at 1% but the correlation in asset values varies from nearly 0 to 0.2.

Moody's Binomial Expansion Method

- For a binomial distribution with independent obligors, the tail with fewer (but larger) obligors is "fatter" than the tail with many (but smaller) independent obligors. Actually, variance of fractional loss decreases as $O\left(\frac{1}{n}\right)$ as n increases since $\mathrm{var}\left(\frac{D_n}{n}\right) = \frac{p(1-p)}{n}$.

- Moody's are aware that a pure binomial distribution with independent defaults is unrealistic. They make the tails of distribution fatter by assuming fewer obligors (the *diversity score*). For example, adjustment is made for:

  - industry concentration

The idea is to approximate the loss on a portfolio of n positively correlated loans with the loss on a smaller number of independent loans with larger face value.

Moody's Diversity Scores

Number of Firms in the Same Industry	Diversity Score
1	1.0
2	1.5
3	2.0
4	2.3
5	2.6
6	3.0
7	3.2
8	3.5
9	3.7
10	4.0
10 or more	Evaluated case by case

When there are 3 firms in the same industry, there is less diversification so the diversity score is only 2.

- Based on the idea that issuers in the same industry sector are related, while issuers in different industry sectors can be treated as independent.

Consider a portfolio of 51 bonds $= 1 \times 2 + 2 \times 7 + 3 \times 3 + 4 \times 4 + 5 \times 2$.

No of issuers in sector	1	2	3	4	5
No of incidences	2	7	3	4	2
Diversity	2	10.5	6	9.2	5.2

- For the example, there are 2 cases in which issuers are the only representatives of their industry sector, 7 cases in which pairs of issuers are in the same sector, etc. The total diversity score $= 2 + 10.5 + 6 + 9.2 + 5.2 = 32.9$, which is rounded to 33. The original portfolio of 51 bonds is treated to be equivalent to a portfolio of 33 independent bonds with the same default probability but with notional value 51/33 times the original notional.

Moody's binomial expansion technique (BET)

The two parameters in a binomial experiment are n and p.

- Diversity score, weighted average rating factor and binomial expansion technique.
- Generate the loss distribution.

To build a hypothetical pool of uncorrelated and homogeneous assets that mimic the default behavior of the original pool of correlated and inhomogeneous assets.

Criteria of an idealized comparison portfolio

The diversity score of a given pool of participations is the number n of bonds in an idealized comparison portfolio that meets the following criteria:

- Comparison portfolio and collateral pool have the same face value.
- Bonds in the comparison portfolio have equal face values.
- Comparison bonds are equally likely to default, and their defaults are independent.
- Comparison bonds are of the same average default probability as the participations of the collateral pool.
- According to some measure of risk, the comparison portfolio has the same total risk as does the collateral pool.

Improved version of the binomial approximation using diversity scores

Seek the reduction of the problem of multiple defaults to binomial distributions.

If the n loans each with equal face value are independent and they have the same default probability, then the distribution of the portfolio loss is a binomial distribution with n as the number of trials.

Let F_i be the face value of each bond, p_i be the probability of default within the relevant time horizon, and ρ_{ij} between the linear correlation coefficient of default events.

Assuming zero recovery, the loss variable L_i (in dollar amount) associated with bond i with face value F_i is given by

$$L_i = F_i \mathbf{1}_{\{D_i\}},$$

where D_i is the default event of bond i.

With n bonds, the total notional principal of the portfolio is $\sum_{i=1}^{n} F_i$. The mean and variance of the loss principal $\widehat{P}$ is

$$\begin{aligned}
E[\widehat{P}] &= E[L_1 + \cdots + L_n] = \sum_{i=1}^{n} F_i E[\mathbf{1}_{\{D_i\}}] = \sum_{i=1}^{n} p_i F_i \\
\mathrm{var}(\widehat{P}) &= \sum_{i=1}^{n} \sum_{j=1}^{n} E[L_i L_j] - E[L_i]E[L_j] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} F_i F_j (E[\mathbf{1}_{\{D_i\}} \mathbf{1}_{\{D_j\}}] - E[\mathbf{1}_{\{D_j\}}]E[\mathbf{1}_{\{D_j\}}]) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} F_i F_j \rho_{ij} \sqrt{p_i(1-p_i)p_j(1-p_j)}.
\end{aligned}$$

Here, ρ_{ij} is specified rather than as a quantity calculated based on information on joint defaults.

- We construct an approximating portfolio consisting D independent loans, each with the same face value F and the same default probability p.

To determine the binomial distribution of the loss amount of the comparison portfolio, we need to specify the constant default probability p, number of obligors D and the common face value of the bonds. These lead to the following system of equations:

$$\sum_{i=1}^{n} F_i = DF$$
$$\sum_{i=1}^{n} p_i F_i = DFp$$
$$\text{var}(\widehat{P}) = F^2 Dp(1-p).$$

Solving the equations, we obtain

$$
\begin{aligned}
p &= \frac{\sum_{i=1}^{n} p_i F_i}{\sum_{i=1}^{n} F_i} \\
D &= \frac{\sum_{i=1}^{n} p_i F_i \sum_{i=1}^{n} (1-p_i) F_i}{\sum_{i=1}^{n} \sum_{j=1}^{n} F_i F_j \rho_{ij} \sqrt{\rho_i (1-p_i) \rho_j (1-p_j)}} \\
F &= \sum_{i=1}^{n} F_i \Big/ D.
\end{aligned}
$$

Here, D is called the *diversity score*.

Contagion models

Contagion means that once a firm defaults, it may bring down other firms with it. Define Y_{ij} to be an "infection" variable. Both X_i and Y_{ij} are Bernuolli variables assuming values 0 and 1.

X_i is the default indicator of firm i due to its firm specific causes（自身原因）

$$Y_{ij} = \begin{cases} 1 & \text{if default of firm } i \text{ brings down firm } j \\ 0 & \text{if default of firm } i \text{ does not bring down firm } j \end{cases}.$$

Assuming homogeneous property on the parameters:

$$E[X_i] = p \quad \text{and} \quad E[Y_{ij}] = q.$$

The default indicator of firm i is

$$Z_i = X_i + (1 - X_i)\left[1 - \prod_{j \neq i}(1 - X_j Y_{ji})\right].$$

Note that Z_i equals one either when there is a direct default of firm i or if there is no direct default and $\prod_{j \neq i}(1 - X_j Y_{ji}) = 0$. The latter case occurs when at least one of the factor $X_j Y_{ji}$ is 1, which happens when firm j defaults and infects firm i. Define $D_n = Z_1 + \cdots + Z_n$, Davis and Lo (2001) find that

$$\begin{aligned}
E[D_n] &= n[1 - (1 - p)(1 - pq)^{n-1}] \\
\text{var}(D_n) &= n(n - 1)\beta_n^{pq} - (E[D_n])^2 \\
\text{cov}(Z_i, Z_j) &= \beta_n^{pq} - (E[D_n/n])^2,
\end{aligned}$$

where

$$\begin{aligned}
\beta_n^{pq} &= p^2 + 2p(1 - p)[1 - (1 - q)(1 - pq)^{n-2}] \\
&\quad + (1 - p)^2[1 - 2(1 - pq)^{n-2} + (1 - 2pq + pq^2)^{n-2}].
\end{aligned}$$

1. When there is no infection, $q = 0$, zero contagion gives a pure binomial model. In this case, $E[D_n]$ becomes np, which is the same as that of the binomial distribution.

2. Increasing the contagion brings more mass to high and low default numbers. To preserve the mean, we must compensate for an increase in the infection parameter by decreasing the probability of direct default. Note that

$$\begin{aligned} E[Z_i] &= E\left[1 - (1 - X_i)\prod_{j \neq i}(1 - X_j Y_{ji})\right] \\ &= (1 - p)(1 - pq)^{n-1}. \end{aligned}$$

   By equating the probability of no default with and without infection effect, we find $\widehat{p}(q)$ such that

$$[1 - \widehat{p}(q)][1 - \widehat{p}(q)q]^{n-1} = 1 - p.$$

   We have $\widehat{p}(0) = p$ and $\widehat{p}(q) < p$ for $q > 0$. While the mean is preserved, the variance is increased.

Recall the following results:

- $(1-q)^i =$ probability that all i self-defaulting bonds do not affect a particular non-defaulting bond
- $1-(1-q)^i =$ probability that at least one of the self-defaulting bonds affect a particular non-defaulting bond

- The k bonds that are defaulting can be chosen in C_k^n combinations.

Write α_{nk}^{pg} as the probability that out of the n bonds, k ($\leq n$) particular bonds default. Consider the two cases:

(i) All these k bonds are self-defaulting and they do not infect any of the remaining $n-k$ bonds;

(ii) Of the k bonds defaulting, i of them are self-defaulting while $k-i$ of them are infected by the first of these i self-defaulting bonds, $i = 1, 2, \cdots, k$.

Probability mass function of D_n

Let $F(k; n, p, q)$ denote the probability mass function of D_n, where

$$F(k; n, p, q) = \mathbb{P}[D_n = k],$$

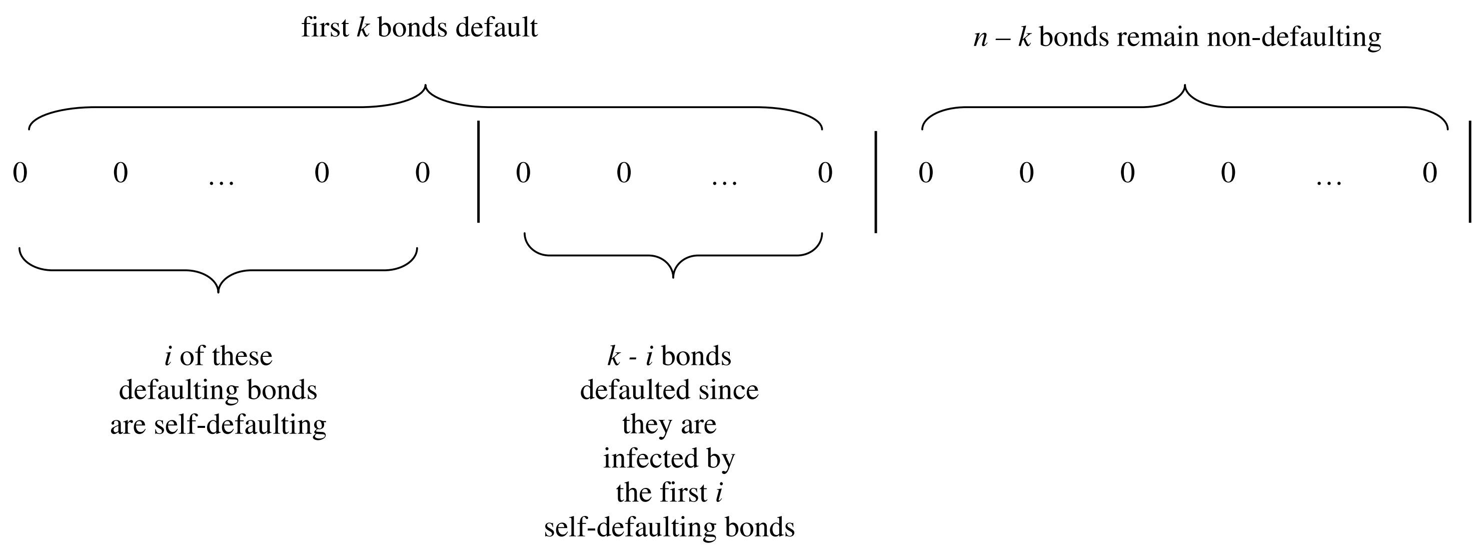
then $F(k; n, p, q) = C_k^n \alpha_{nk}^{pq}$, where

$$\begin{aligned} \alpha_{nk}^{pq} &= p^k (1-p)^{n-k} (1-q)^{k(n-k)} \\ &\quad + \sum_{i=1}^{k-1} C_i^k p^i (1-p)^{n-i} [1-(1-q)^i]^{k-i} (1-q)^{i(n-k)}. \end{aligned}$$

- It is necessary to isolate the special case where all of the k defaulting bonds do not infect any other non-defaulting bonds in the portfolio. This is captured by the first term in α_{nk}^{pq}.

- With k bonds that are defaulting, i of them are self-defaulting and $k-i$ of them are infected by the first i of these self-defaulting bonds.

For $i = 1, 2, \cdots, k-1$, consider



Probability of occurrence is $p^i(1-p)^{n-i}[1-(1-q)^i]^{k-i}(1-q)^{i(n-k)}$.

Exponential model for dependent defaults

Reference

Kay Giesecke, "A simple exponential model for dependent defaults," (2003) *Journal of Fixed Income*, vol.13(3), p.74-83.

Model setup

- A firm's default is driven by idiosyncratic as well as other regional, sectoral or economy-wide shocks, whose arrivals are modeled by independent Poisson processes.
- Default times are assumed to be jointly exponentially distributed. In this case, the exponential copula arises naturally.

Advantages

1. All relevant results are given in closed form.
2. Efficient simulation of dependent default times is straightforward.

Bivariate version of the exponential models

Suppose there are independent Poisson processes N_1, N_2 and N with respective intensity λ_1, λ_2 and λ. Here, λ_i is the idiosyncratic shock intensity of firm i and λ is the intensity of a macro-economic shock affecting all firms simultaneously.

Define the default time τ_i of firm i by

$$\tau_i = \inf\{t \geq 0 : N_i(t) + N(t) > 0\}.$$

That is, a default occurs completely unexpectedly if either an idiosyncratic or a systematic shock strikes the firm for the first time. Since N_i and N are independent, firm i defaults with intensity $\lambda_i + \lambda$ so that the survival function is

$$S_i(t) = P[\tau_i > t] = P[N_i(t) + N(t) = 0] = e^{-(\lambda_i+\lambda)t}.$$

The expected default time and variance are

$$E[\tau_i] = \frac{1}{\lambda_i + \lambda} \quad \text{and} \quad \text{var}(\tau_i) = \frac{1}{(\lambda_i + \lambda)^2}.$$

Define

$$t \vee u = \max(u,t), \quad t \wedge u = \min(u,t),$$

so that

$$\begin{aligned} t + u &= \max(u,t) + \min(u,t) \\ e^{\lambda(t\wedge u)} &= \min(e^{\lambda t}, e^{\lambda u}). \end{aligned}$$

The joint survival probability is found to be

$$\begin{aligned} S(t,u) &= P[\tau_1 > t, \tau_2 > u] \\ &= P[N_1(t) = 0, N_2(u) = 0, N(t \vee u) = 0] \\ &= e^{-\lambda_1 t - \lambda_2 u - \lambda(t\vee u)} \\ &= e^{-(\lambda_1+\lambda)t - (\lambda_2+\lambda)u + \lambda(t\wedge u)} \\ &= S_1(t)S_2(u)\min(e^{\lambda t}, e^{\lambda u}). \end{aligned}$$

All random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, P)$. Depending on the specific application, P is the physical probability (risk management setting) or some risk neutral probability (valuation setting).

Survival copula

There exists a unique solution $C^\tau : [0,1]^2 \to [0,1]$, called the survival copula of the default time vector (τ_1, τ_2) such that the joint survival probabilities can be represented by

$$S(t,u) = C^\tau(S_1(t), S_2(u)).$$

The copula C^τ describes the complete non-linear default time dependence structure. C^τ marries the marginal survival probabilities into joint survival probabilities.

Define $\theta_i = \dfrac{\lambda}{\lambda_i + \lambda}$, we obtain

$$C^\tau(w,v) = S(S_1^{-1}(w), S_2^{-1}(v)) = \min(vw^{1-\theta_1}, wv^{1-\theta_2}).$$

Write $w = S_1(t)$ and $v = S_2(u)$ so that $t = S_1^{-1}(w)$ and $u = S_2^{-1}(v)$.

$$S(t,u) = C^{\tau}(w,v) = S_1(t)S_2(u)\min(e^{\lambda t}, e^{\lambda u}).$$

Now, $S_1(t) = e^{-(\lambda_1+\lambda)t}$ and $S_2(t) = e^{-(\lambda_2+\lambda)t}$, so that

$$\begin{aligned} w^{-\theta_1} &= \left[e^{-(\lambda_1+\lambda)t}\right]^{\left(-\frac{\lambda}{\lambda_1+\lambda}\right)} = e^{\lambda t}, \\ v^{-\theta_2} &= \left[e^{-(\lambda_2+\lambda)u}\right]^{\left(-\frac{\lambda}{\lambda_2+\lambda}\right)} = e^{\lambda u}. \end{aligned}$$

Lastly, we obtain

$$\begin{aligned} S(t,u) = C^{\tau}(w,v) &= wv\min(w^{-\theta_1}, v^{-\theta_2}) \\ &= \min(vw^{1-\theta_1}, wv^{1-\theta_2}). \end{aligned}$$

The parameter vector $\theta = (\theta_1, \theta_2)$ controls the degree of dependence between the default times.

1. Firms default independently of each other ($\lambda = 0$ or $\lambda_1, \lambda_2 \to \infty$)

$$\theta_1 = \theta_2 = 0, \quad C_\theta^\tau(u,v) = uv \text{ (product copula)}$$

2. Firms are perfectly correlated (firms default simultaneously, $\lambda \to \infty$ or $\lambda_1 = \lambda_2 = 0$)

$$\theta_1 = \theta_2 = 1 \quad \text{and} \quad C_\theta^\tau(u,v) = u \wedge v.$$

It can be shown that

$$uv \leq C_\theta^\tau(u,v) \leq u \wedge v, \theta \in [0,1]^2, u,v \in [0,1].$$

Also, the defaults can only be positively correlated.

Joint default probabilities and default copula

Similarly, define K^τ by

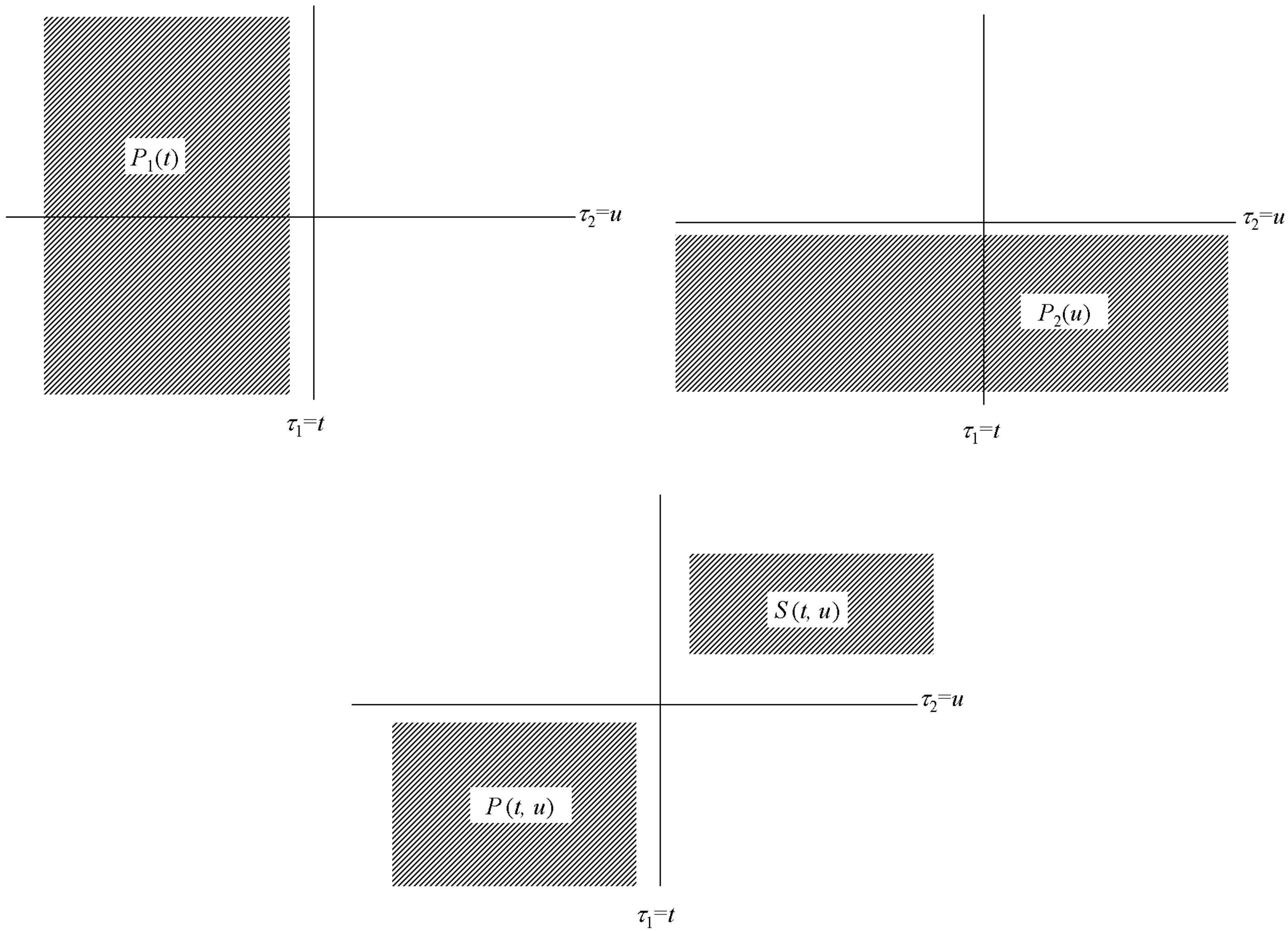
$$K^\tau(P_1(t), P_2(t)) = P[\tau_1 \leq t, \tau_2 \leq u] = P(t,u)$$

where $P_i(t) = P[\tau_i \leq t] = 1 - S_i(t)$. Since

$$S(t,u) = 1 - P_1(t) - P_2(u) + P(t,u)$$

so that these copulas are related by

$$\begin{aligned} K^\tau(u,v) &= C^\tau(1-u, 1-v) + u + v - 1 \\ &= \min([1-v][1-u]^{1-\theta_1}; [1-u][1-v]^{1-\theta_2}) + u + v - 1. \end{aligned}$$


$P_1(t)$
$\tau_2=u$
$\tau_1=t$
$P_2(u)$
$\tau_2=u$
$\tau_1=t$
$S(t, u)$
$\tau_2=u$
$P(t, u)$
$\tau_1=t$

Multi-variate extension

Assume that there are $n \geq 2$ firms. The default of an individual firm is driven by some idiosyncratic shock as well as other sectoral, industry, country-specific or economy-wide shocks.

Define a matrix $(a_{ij})_{n\times m}$, when $a_{ij} = 1$ if shock $j \in \{1, 2, \cdots, m\}$ modeled through the Poisson process N_j with intensity λ_j, leads to the default of firm $i \in \{1, 2, \cdots, n\}$ and $a_{ij} = 0$ otherwise. For example, when $n = 3$

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Note that $m = \sum_{k=1}^{n} {}_nC_k = 2^n - 1$. Suppose the economy-wide shock events are excluded, one then set $a_{i7} = 0$ for $i = 1, 2, 3$. This corresponds to bivariate dependence only.

Take $n = 3$ firms, there are $2^3 - 1 = 7$ possible shocks.

1st shock affects Firm 1 only
2nd shock affects Firm 2 only
3rd shock affects Firm 3 only
4th shock affects both Firm 1 and Firm 2
5th shock affects both Firm 1 and Firm 3
6th shock affects both Firm 2 and Firm 3
7th shock affects both Firm 1, Firm 2 and Firm 3

Firm 1 defaults if

$$N_1(t) + N_4(t) + N_5(t) + N_7(t) = \sum_{k=1}^{7} a_{1k} N_k(t) > 0.$$

Firm 2 defaults if

$$N_2(t) + N_4(t) + N_6(t) + N_7(t) = \sum_{k=1}^{7} a_{2k} N_k(t) > 0.$$

Joint survival function

$$\tau_i = \inf \left\{ t \geq 0 : \sum_{k=1}^{m} a_{ik} N_k(t) > 0 \right\}$$

meaning that firm i defaults with intensity $\sum_{k=1}^{m} a_{ik}\lambda_k$ and

$$S_i(t) = \exp\left(- \sum_{k=1}^{m} a_{ik}\lambda_k t \right).$$

The joint survival function

$$\begin{aligned} S(t_1, t_2, \cdots, t_n) &= P[\tau_1 > t_1, \cdots, \tau_n > t_n] \\ &= \exp\left(- \sum_{k=1}^{m} \lambda_k \max(a_{1k}t_1, \cdots, a_{nk}t_n) \right). \end{aligned}$$

Take $n = 3$, consider

$$\begin{aligned} S(t_1, t_2, t_3) &= \exp(-\lambda_1 \max(t_1) - \lambda_2 \max(t_2) - \lambda_3 \max(t_3) \\ &\quad -\lambda_4 \max(t_1, t_2) - \lambda_5 \max(t_1, t_3) \\ &\quad -\lambda_6 \max(t_2, t_3) - \lambda_7 \max(t_1, t_2, t_3)) \end{aligned}$$

- The first term $e^{-\lambda_1 \max(t_1)} = e^{-\lambda_1 t_1}$ means that the first shock has not arrived up to time t_1.

- The fourth term $e^{-\lambda_4 \max(t_1, t_2)}$ means that the fourth shock has not arrived up to $\max(t_1, t_2)$, so both the first and second firms have not been affected by the fourth shock up to $\max(t_1, t_2)$.

- The seventh term $e^{-\lambda_7 \max(t_1, t_2, t_3)}$ means that the seventh shock has not arrived up to $\max(t_1, t_2, t_3)$, so all firms have not been affected by the seventh shock up to $\max(t_1, t_2, t_3)$.

Survival copula function

The exponential survival copula associated with S can be found via $C^\tau(u_1, \cdots, u_n) = S(S_1^{-1}(u_1), \cdots, S_n^{-1}(u_n))$. Fixing some $i, j \in \{1, 2, \cdots, n\}$ with $i \neq j$, the two-dimensional marginal copula is given by

$$\begin{aligned} C^\tau(u_i, u_j) &= C^\tau(1, \cdots, 1, u_i, 1, \cdots, 1, u_j, 1, \cdots, 1) \\ &= \min(u_j u_i^{1-\theta_i}, u_i u_j^{1-\theta_j}) \end{aligned}$$

where we define, analogously to the bivariate case,

$$\theta_i = \frac{\sum_{k=1}^m a_{ik} a_{jk} \lambda_k}{\sum_{k=1}^m a_{ik} \lambda_k}, \quad \theta_j = \frac{\sum_{k=1}^m a_{ik} a_{jk} \lambda_k}{\sum_{k=1}^m a_{jk} \lambda_k}$$

as the ratio of joint default intensity of firms i and j to default intensity of firm i or j, respectively.

Mathematical Appendices

Understanding "conditional independence"

Example Hemophilia （血友症） is a hereditary disease. If a mother has it, then with probability $\frac{1}{2}$, any of her sons independently will inherit it.

Let H, H_1 and H_2 denote the events that the mother, the first son, and the second son are hemophilic, respectively. Note that H_1 and H_2 are *conditionally independent* given H. However H_1 and H_2 are not independent. This is because if we know that one son is hemophilic, the mother is hemophilic. With probability $1/2$, the other son is also hemophilic. Note that

$$\begin{aligned} P(H_1|H) &= P(H_1|H \cap H_2) = \frac{1}{2} \\ P(H_2|H) &= P(H_2|H \cap H_1) = \frac{1}{2}. \end{aligned}$$

Conditional Variance

Just as we have defined the conditional expectation of X given the value of Y, we can also define the conditional variance of X, given that $Y = y$, as follows:

$$\text{var}(X|Y) = E[(X - E[X|Y])^2|Y].$$

That is, $\text{var}(X|Y)$ is equal to the (conditional) expected square of the difference between X and its (conditional) mean when the value of Y is given. In other words, $\text{var}(X|Y)$ is exactly analogous to the usual definition of variance, but now all expectations are conditional on the fact that Y is known.

Formula

$$\text{var}(X) = E[\text{var}(X|Y)] + \text{var}(E[X|Y]).$$

- To obtain the unconditional variance of X, we add the expectation of conditional variance of X given Y to the variance of expectation of X given Y.

Proof

We start from

$$\text{var}(X|Y) = E[X^2|Y] - [E[X|Y])^2$$

so

$$E[\text{var}(X|Y)] = E[E[X^2|Y]] - E[E[X|Y])^2]$$

$$= E[X^2] - E[(E[X|Y])^2]. \qquad (i)$$

Treating $E[X|Y]$ as a random variable, and observing $E[E[X|Y]] = E[X]$,

$$\text{var}(E[X|Y]) = E[(E[X|Y])^2] - (E[X])^2. \qquad (ii)$$

By adding eqs. (i) and (ii), we arrive at the formula.