Properties on implied survival probabilities, $P(t,T)$

1. $P(t,t) = 1$ and it is non-negative and decreasing in $T$. Also, $P(t,\infty) = 0$.

2. Normally $P(t,T)$ is continuous in its second argument, except that an important event scheduled at some time $T_1$ has direct influence on the survival of the obligor. For example, a coupon payment date if it is unclear if the coupon will be paid.

3. Viewed as a function of its first argument $t$, all survival probabilities for fixed maturity dates will tend to increase. This is obvious since $P(t_1,T) = P(t_2,T) P(t_1,t_2), \quad t_1 < t_2$.

If we want to focus on the default risk over a given time interval in the future, we should consider conditional survival probabilities.

**conditional survival probability over $[T_1,T_2]$ as seen from $t$, given that there was no default until time $T_1$**

$$P(t,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)}, \quad \text{where } t \leq T_1 < T_2;$$

$$= \frac{B(t,T_2)}{B(t,T_1)} \frac{B(t,T_1)}{B(t,T_3)}.$$

This is a direct consequence of $P[A|B] = \frac{P[AB]}{P[B]}$; $A = \text{survival until } T_2, B = \text{survival until } T_1.$
Recall odds ratio of an event: (expected number of events divided by the (expected) number of non-events. For example, "1:3 that the company will default in July" means: \( p_{\text{def}}(t, \text{June}30, \text{July}31) : p(t, \text{June}30, \text{July}31) = 1:3 \), where

\[ p_{\text{def}}(t, T, T+\Delta T) = 1 - p(t, T, T+\Delta T). \]

Discrete implied hazard rate of default over \((T, T + \Delta T]\) as seen from time \(t\)

\[
H(t, T, T + \Delta T) \Delta T = \frac{p_{\text{def}}(t, T, T+\Delta T)}{p(t, T, T+\Delta T)} = \frac{1 - \frac{p(t, T+\Delta T)}{p(t, T)}}{\frac{p(t, T+\Delta T)}{p(t, T)}} = \frac{p(t, T)}{p(t, T+\Delta T)} - 1
\]

so that

\[ p(t, T) = p(t, T + \Delta T)[1 + H(t, T, T + \Delta T)\Delta T]. \]

In the limit of \(\Delta T \to 0\), the continuous hazard rate at time \(T\) as seen at time \(t\) is given by

\[
h(t, T) = \lim_{\Delta T \to 0} H(t, T, T+\Delta T)
\]

\[
= \lim_{\Delta T \to 0} \frac{1}{\Delta T} \frac{p(t, T+\Delta T) - p(t, T)}{p(t, T+\Delta T)}
\]

\[
= -\frac{1}{p(t, T)} \frac{\partial}{\partial T} p(t, T) = -\frac{\partial}{\partial T} \ln p(t, T),
\]

provided that \(t > t\) and the term structures of survival probabilities is differentiable with respect to \(T\).
Forward spreads and implied hazard rate of default

For \( t \leq T_1 < T_2 \), the simply compounded forward rate over the period \( (T_1, T_2] \) as seen from \( t \) is given by

\[
F(t, T_1, T_2) = \frac{B(t, T_1)/B(t, T_2) - 1}{T_2 - T_1}.
\]

This is the price of the forward contract with expiration date \( T_1 \) on a unit-par zero-coupon bond maturing on \( T_2 \). To prove, we consider the compounding of interest rates over successive time intervals:

\[
\frac{1}{B(t, T_2)} = \frac{1}{B(t, T_1)} \left[ 1 + F(t, T_1, T_2)(T_2 - T_1) \right]
\]

compounding over \([t, T_2]\) compounding over \([t, T_1]\)

simply compounding over \([T_1, T_2]\)

Analogous to implied survival probabilities and implied hazard rate of default:

\[
P(t, T_1) = P(t, T + \Delta T) \left[ 1 + H(t, T, T + \Delta T) \Delta T \right].
\]
Defaultable simply compounded forward rate over \([T_1, T_2]\)

\[
\bar{F}(t, T_1, T_2) = \frac{\bar{B}(t, T_1)/\bar{B}(t, T_2) - 1}{T_2 - T_1}.
\]

Instantaneous continuously compounded forward rates

\[
f(t, T) = \lim_{\Delta T \to 0} F(t, T, T + \Delta T) = -\frac{\partial}{\partial T} \ln B(t, T)
\]

\[
\bar{f}(t, T) = \lim_{\Delta T \to 0} \bar{F}(t, T, T + \Delta T) = -\frac{\partial}{\partial T} \ln \bar{B}(t, T).
\]

The implied hazard rate of default at time \(T > t\) as seen from time \(t\) is the spread between the forward rates:

\[
h(t, T) = \bar{f}(t, T) - f(t, T)
\]

obtained using

\[
\bar{f}(t, T) - f(t, T) = -\frac{\partial}{\partial T} \ln \frac{\bar{B}(t, T)}{B(t, T)}
\]

\[
= -\frac{\partial}{\partial T} \ln P(t, T) = h(t, T).
\]
Implied hazard rate of default

Recall

\[
P(t, T_1, T_2) = \frac{\overline{B}(t, T_2) B(t, T_1)}{B(t, T_2) \overline{B}(t, T_1)}
\]

\[
= \frac{1 + F(t, T_1, T_2)(T_2 - T_1)}{1 + \overline{F}(t, T_1, T_2)(T_2 - T_1)} = 1 - P_{def}(t, T_1, T_2),
\]

and upon expanding, we obtain

\[
P_{def}(t, T_1, T_2) \left[1 + \overline{F}(t, T_1, T_2)(T_2 - T_1)\right] = \left[F(t, T_1, T_2) - F(t, T_1, T_2)\right](T_2 - T_1).
\]

If we take \( T_1 = T \) and \( T_2 = T + \Delta T \), then

\[
\frac{P_{def}(t, T, T + \Delta T)}{P(t, T, T + \Delta T)} = \frac{F(t, T, T + \Delta T) \Delta T}{\overline{F}(t, T, T + \Delta T) - F(t, T, T + \Delta T)}.
\]


**Instantaneous short rates and hazard rate**

The *local default probability* at time $t$ over the next small time step $\Delta t$

$$
\frac{1}{\Delta t} Q[\tau \leq t + \Delta t | F_t \wedge \{\tau > t\}] \approx \bar{r}(t) - r(t) = \lambda(t)
$$

where $r(t) = f(t, t)$ is the riskfree short rate and $\bar{r}(t) = \bar{f}(t, t)$ is the defaultable short rate.

**Recovery value**

View an asset with positive recovery as an asset with an additional positive payoff at *default*. The recovery value is the *expected* value of the recovery shortly after the occurrence of a default.

*This definition ignores the difficulties that are involved in the real world determination of recovery in credit derivatives, like time delays, dealer polls or delivery options.*
**Payment upon default**

Define $e(t, T, T + \Delta T)$ to be the value at time $t < T$ of a deterministic payoff of $1$ paid at $T + \Delta T$ if and only if a default happens in $[T, T + \Delta T]$.

$$e(t, T, T + \Delta T) = E_Q [\beta(t, T + \Delta T)[I(T) - I(T + \Delta T)] | \mathcal{F}_t].$$

Note that

$$I(T) - I(T + \Delta T) = \begin{cases} 
1 & \text{if default occurs in } [T, T + \Delta T] \\
0 & \text{otherwise} 
\end{cases}.$$

$$E_Q[\beta(t, T + \Delta T)I(T)] = E_Q[\beta(t, T + \Delta T)]E_Q[I(T)] \overset{\text{(independence assumption)}}{=} B(t, T + \Delta T)P(t, T),$$

$$E_Q[\beta(t, T + \Delta T)I(T + \Delta T)] = \overline{B}(t, T + \Delta T),$$

and

$$B(t, T + \Delta T) = \overline{B}(t, T + \Delta T)/P(t, T + \Delta T).$$
It is seen that
\[ e(t, T, T + \Delta T) = B(t, T + \Delta T)P(t, T) - \overline{B}(t, T + \Delta T) \]
\[ = \overline{B}(t, T + \Delta T) \left[ \frac{P(t, T)}{P(t, T + \Delta T)} - 1 \right] \]
\[ = \Delta T \overline{B}(t, T + \Delta T) \frac{P_{def}(t, T, T + \Delta T)}{P(t, T, T + \Delta T)} \]
\[ = \Delta T \overline{B}(t, T + \Delta T) P(t, T) P_{def}(t, T, T + \Delta T) \frac{B(t, T, T + \Delta T)}{\overline{B}(t, T, T + \Delta T)} \]
\[ = P(t, T) \left[ P_{def}(t, T, T + \Delta T) \Delta T \right] B(t, T + \Delta T). \]

The above result indicates that
\[ e(t, T, T + \Delta T) = E_{\mathcal{F}_t} \left[ \beta(t, T + \Delta T) \left( I(T) - I(T + \Delta T) \right) \right] \]
\[ = E_{\mathcal{F}_t} \left[ \beta(t, T + \Delta T) \mid \mathcal{F}_t \right] E_{\mathcal{F}_t} \left[ I(T) - I(T + \Delta T) \mid \mathcal{F}_t \right]. \]

\[ = E_{\mathcal{F}_t} \left[ \beta(t, T + \Delta T) \mid \mathcal{F}_t \right] \overline{B}(t, T + \Delta T) \]
\[ P(t, T) P_{def}(t, T, T + \Delta T) \Delta T \]
On taking the limit $\Delta T \to 0$, we obtain

rate of default compensation  

\[ e(t, T) = \lim_{\Delta T \to 0} \frac{e(t, T, T + \Delta T)}{\Delta T} \]

\[ = \bar{B}(t, T)h(t, T) = B(t, T)P(t, T)h(t, T). \]

The value of a security that pays $\pi(s)$ if a default occurs at time $s$ for all $t < s < T$ is given by

\[ \int_t^T \pi(s)e(t, s) \, ds = \int_t^T \pi(s)\bar{B}(t, s)h(t, s) \, ds. \]

This result holds for deterministic recovery rates.

**Stochastic recovery**

Let $\Pi'$ denote the random recovery amount received at $T + \Delta T$ if a default occurs in $(T, T + \Delta T)$. The value of this payoff is

\[ \Pi'(t, T, T + \Delta T)e(t, T, T + \Delta T), \]

where $\Pi'(t, T, T + \Delta T)$ is the conditional expectation of $\Pi'$ conditional on a default occurring in $(T, T + \Delta T)$. 
Building blocks for credit derivatives pricing

Tenor structure

\[
\begin{align*}
0 & \quad \text{maturity date} \\
T_0 & \quad T_1 & \quad T_2 & \quad T_K \\
\end{align*}
\]

\[\delta_k = T_{k+1} - T_k, \quad 0 \leq k \leq K - 1\]

Coupon and repayment dates for bonds, fixing dates for rates, payment and settlement dates for credit derivatives all fall on \(T_k, 0 \leq k \leq K\).

At every date \(T_k\), a default can occur, or the obligor can continue until the next date \(T_{k+1}\). After default, the default-free world goes on but the defaultable assets only earn their recovery payoff and cease to exist after default.
Fundamental quantities of the model

- Term structure of default-free interest rates $F(0,T)$
- Term structure of implied hazard rates $H(0,T)$
- Expected recovery rate $\pi$ (rate of recovery as percentage of par)

From $B(0,T_i) = \frac{B(0,T_{i-1})}{1 + \delta_{i-1}F(0,T_{i-1},T_i)}$, $i = 1, 2, \ldots, k$, and $B(0,T_0) = B(0,0) = 1$, we obtain

$$B(0,T_k) = \prod_{i=1}^{k} \frac{1}{1 + \delta_{i-1}F(0,T_{i-1},T_i)}.$$
Similarly, from \( P(0, T_i) = \frac{P(0, T_{i-1})}{1 + \delta_{i-1} H(0, T_{i-1}, T_i)} \), we deduce that

\[
\overline{B}(0, T_k) = B(0, T_k) P(0, T_k) = B(0, T_k) \prod_{i=1}^{k} \frac{1}{1 + \delta_{i-1} H(0, T_{i-1}, T_i)}.
\]

\( e(0, T_k, T_{k+1}) = \delta_k H(0, T_k, T_{k+1}) \overline{B}(0, T_k+1) \)

= value of $1 at \( T_{k+1} \) if a default has occurred in \( (T_k, T_{k+1}] \).

- Forward hazard rates have a simple intuitive interpretation both in the economic sense (via forward zero bond spreads) as well as in the probability sense (via local conditional default probabilities).
Continuous limits

Taking the limit $\delta_i \to 0$, for all $i = 0, 1, \ldots, k$

$$B(0, T_k) = \exp \left( - \int_0^{T_k} f(0, s) \, ds \right)$$

$$\overline{B}(0, T_k) = \exp \left( - \int_0^{T_k} [h(0, s) + f(0, s)] \, ds \right)$$

$$e(0, T_k) = h(0, T_k) \overline{B}(0, T_k).$$

Alternatively, the above relations can be obtained by integrating

$$f(0, T) = -\frac{\partial}{\partial T} \ln B(0, T) \quad \text{with} \quad B(0, 0) = 1$$

$$\overline{f}(0, T) = h(0, T) + f(0, T) = -\frac{\partial}{\partial T} \ln \overline{B}(0, T) \quad \text{with} \quad \overline{B}(0, 0) = 1.$$
Defaultable fixed coupon bond

\[
\bar{c}(0) = \sum_{n=1}^{K} \bar{c}_n \bar{B}(0, T_n) + \bar{B}(0, T_K) + \pi \sum_{k=1}^{K} e(0, T_{k-1}, T_k)
\]

(coupon) \hspace{1cm} \text{(principal)}

The recovery payment can be written as

\[
\pi \sum_{k=1}^{K} e(0, T_{k-1}, T_k) = \sum_{k=1}^{K} \pi \delta_{k-1} H(0, T_{k-1}, T_k) \bar{B}(0, T_k).
\]

The recovery payments can be considered as an additional coupon payment stream of \(\pi \delta_{k-1} H(0, T_{k-1}, T_k)\).
Defaultable floater

Recall that $L(T_{n-1}, T_n)$ is the reference LIBOR rate applied over $[T_{n-1}, T_n]$ at $T_{n-1}$ so that $1 + L(T_{n-1}, T_n)\delta_{n-1}$ is the compounding factor over $[T_{n-1}, T_n]$. Application of no-arbitrage argument gives

$$B(T_{n-1}, T_n) = \frac{1}{1 + L(T_{n-1}, T_n)\delta_{n-1}}.$$

- The coupon payment at $T_n$ equals LIBOR plus a spread

$$\delta_{n-1} [L(T_{n-1}, T_n) + s^{par}] = \left[ \frac{1}{B(T_{n-1}, T_n)} - 1 \right] + s^{par} \delta_{n-1}.$$

- Consider the payment of $\frac{1}{B(T_{n-1}, T_n)}$ at $T_n$, its value at $T_{n-1}$ is $\frac{\overline{B}(T_{n-1}, T_n)}{B(T_{n-1}, T_n)} = P(T_{n-1}, T_n)$. Why? We use the defaultable discount factor $\overline{B}(T_{n-1}, T_n)$ since the coupon payment may be defaultable over $[T_{n-1}, T_n]$. 


• Seen at $t = 0$, the value becomes

$$E_Q \left[ b^{\left(0, T_{n+1}\right)} I\left(T_{n+1}\right) P\left(T_{n+1}, T_n\right) \right]$$

$$= b^{\left(0, T_{n+1}\right)} E_Q \left[ I\left(T_{n+1}\right) P\left(T_{n+1}, T_n\right) \right] \quad \text{(independence assumption)}$$

$$= b^{\left(0, T_{n+1}\right)} P\left(0, T_n\right).$$

Combining with the fixed part of the coupon payment and observing the relation

$$[B(0, T_{n-1}) - B(0, T_n)] P(0, T_n) = \left[ \frac{B(0, T_{n-1})}{B(0, T_n)} - 1 \right] \overline{B}(0, T_n)$$

$$= \delta_{n-1} F(0, T_{n-1}, T_n) \overline{B}(0, T_n),$$

the model price of the defaultable floating rate bond is

$$\overline{c}(0) = \sum_{n=1}^{K} \delta_{n-1} F(0, T_{n-1}, T_n) \overline{B}(0, T_n) + s^{\text{par}} \sum_{n=1}^{K} \delta_{n-1} \overline{B}(0, T_n)$$

$$+ \overline{B}(0, T_K) + \pi \sum_{k=1}^{K} e(0, T_{k-1}, T_k).$$