MAFS525 — Computational Methods for Pricing Structural Products

Computer Assignment Two

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Target Redemption Note

In this assignment, you are asked to construct the finite difference scheme for pricing the target redemption note and replore the pricing behavior of the product.

Product description

The target redemption note is an index linked note that provides a guaranteed sum of coupons (target cap) with the possibility of early termination. In a typical structure, the coupons are calculated based on an inverse floating LIBOR / Euribor formula. Once the accumulated amount of coupons has reached the pre-specified target cap, the note will be terminated with final payment of the par. The knock-out criterion depends on a path dependent state variable defined by the running accumulated coupon sum.
In summary, a target redemption note is similar to an inverse floating rate note, embedded with additional features like the possibility of early termination and a guaranteed sum of coupon payments.

Example

As an example, let us consider the 5-year target redemption note issued by Credit Suisse First Boston on 10 November, 2003.

- The first year coupon rate is fixed at 9%.
- The coupon rates in subsequent years are calculated based on an inverse floating formula, $\max(8.65\% - 2L, 0)$, where the index L denotes the 12-month Euribor on the coupon date.
- The note will be terminated prematurely on a coupon date when the accumulated coupon rate meets the target cap of 15%.
- The date of the par payment is uncertain, which is taken to be the earlier date among the pre-specified note's maturity date and the coupon date when the accumulated coupon amount meets the target cap.
- The lure of a handsome initial coupon combined with the perception that the par may be received within a short span of time has made these notes attractive to Asian retail investors in the early 2000's when the interest rates were at a low level.
- The investor has a higher gain on the time value of the cash flow stream when the Euribor decreases since a shorter time is required to collect the coupon payments and par.

- At the other extreme, the investor faces the worst scenario when the 12-month Euribor trades above 4.325% one year later and never comes down again. In this case, he then has to hold the note for 5 years and receive the par and the remaining coupon on the maturity date.

Pricing considerations

- The note value is given by the sum of present values of the par and coupon payments and this sum depends on the times at which the payments are received by the note holder. The interest rate fluctuation leads to uncertainty in the coupon payments received on the coupon dates, and also results in uncertainty in the redemption date of the note (knock-out).
- The uncertainty regarding the termination date is governed by a path dependent variable, which is the running accumulated coupon sum. The note value thus depends on the two stochastic state variables, namely, the interest rate and the path dependent variable of running coupon sum.
- In summary, the pricing model with that of an Asian barrier option, whose knock-out barrier is governed by the average asset value (Zvan *et al.*, 1999). When there is only one coupon payment date, we manage to obtain a closed form valuation formula for the note value.

Formulation of pricing model

- The target redemption note is subject to randomness appearing in the cash flow stream due to interest rate fluctuation and potential pre-mature knock-out. The note holder is not certain on the coupon amount received on each coupon date and the termination date at which the par is received.
- The target redemption note can be considered as a contingent claim with interest rate and path dependent accumulated coupons as the underlying stochastic state variables. We use the short rate $r(t)$ as the interest rate state variable and consider the one-factor model of the short rate that follows a stochastic process of the form

  $$dr = \mu(r,t)\,dt + \overline{\sigma} r^{\beta}\,dZ,$$

  where $\mu(r,t)$ is the instantaneous drift, $\overline{\sigma} r^{\beta}$ is the instantaneous volatility, and dZ is the differential of a Wiener process. Here, $\overline{\sigma}$ and β are constants. When $\mu(r,t)$ is specified to be mean-reverting, then $\beta = 0$ and $\beta = 1/2$ correspond to the well known Vasicek model and Cox-Ingersoll-Ross model, respectively.
- Let t_k denote the k^{th} coupon date, $k = 1, 2, \cdots, K$, where $t_K = T$ is the note's maturity date. Let $\widehat{N}$ denote the notional value of the note and $\widehat{\tau}$ denote the constant time period between consecutive coupon dates. In the contractual design of the commonly traded target redemption notes, conditional on non-termination of the note at t_k, the inverse floater formula for the coupon amount received at t_k takes the form: $c(r, t_k) = \widehat{N}\widehat{\tau}\max(f - mL(r, t_k; \widehat{\tau}), 0)$. Here, f is a constant cap value, m is a constant positive

multiplier, $L(r, t_k; \widehat{\tau})$ is the $\widehat{\tau}$-period interest rate index (say, LIBOR) at time t_k. We use $A(t)$ to denote the running sum of coupons received by the holder up to time t. Provided that the note survives up to the coupon date t_j, we have

$$\begin{aligned} A(t) &= \widehat{N}\widehat{\tau}\sum_{k=1}^{j} \max(f - mL(r, t_k; \widehat{\tau}), 0) \quad \text{for } t_j^+ < t < t_{j+1}^-, j = 1, 2, \cdots, K-1, \\ A(t) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad t < t_1. \end{aligned}$$

Here, t_j^+ and t_{j+1}^- denote the time immediately after t_j and immediately before t_{j+1}, respectively. Let c_{cap} denote the target cap of the coupon (as percentage of the notional). The guaranteed total sum of coupons received by the holder is $\widehat{N}c_{cap}$, so $A(t)$ is bounded from above by $\widehat{N}c_{cap}$. The note is terminated when the total sum of coupons received reaches $\widehat{N}c_{cap}$. Suppose the note is terminated at t_k, then the coupon amount received on the termination date is $\widehat{N}c_{cap} - A(t_k^-)$. Let $c(r, t_k)$ denote the coupon amount received at $t_k, k = 1, 2, \cdots, K$, then

$$c(r, t_k) = \begin{cases} \widehat{N}\widehat{\tau}\max(f - mL(r, t_k; \widehat{\tau}), 0) & \text{if the note is not terminated at } t_k \\ \widehat{N}c_{cap} - A(t_k^-) & \text{if the note is terminated at } t_k \end{cases}.$$

Partial differential equation formulation

- Let $V(r, t)$ denote the value of the target redemption note, with dependence on short rate r and calendar time t. Since the running sum of coupons A changes only on the coupon dates, so there is no dependence on A in the governing differential equation.

- Apart from the coupon dates, the note value V is obtained by solving the following partial differential equation.

  $$\frac{\partial V}{\partial t} + \frac{\bar{\sigma}^2}{2} r^{2\beta} \frac{\partial^2 V}{\partial r^2} + \widehat{\mu}(r, t)\frac{\partial V}{\partial r} - rV = 0, \quad t_j^+ < t < t_{j+1}^-,$$

  where

  $$\widehat{\mu}(r, t) = \mu(r, t) + \overline{\sigma} r^{\beta}\lambda(r, t).$$

  Here, $\lambda(r, t)$ denotes the market price of interest rate risk.

- On the coupon date t_k, A changes according to the updating rule

  $$A(t_k^+) = A(t_k^-) + c(r, t_k),$$

  where $A(t_k^+)$ and $A(t_k^-)$ denote the value of A immediately after and before the coupon date t_k, respectively.

- When $A(t_k^+) = c_{cap}$, the note is terminated at t_k^+. The final payout is the sum of the par and the remaining coupon of amount $c_{cap} - A(t_k^-)$. Assuming no premature knock-out at t_k^+, the note value then changes according to the no-arbitrage jump condition

  $$V(r, t_k^+) = V(r, t_k^-) - c(r, t_k) \quad \text{provided that } A(t_k^+) < c_{cap}.$$

- In summary, the note value is evaluated by solving a coupled set of partial differential equations indexed by different values of the path dependent variable A and applying the jump conditions as specified by Eq. (8) on those coupon dates.

Auxiliary conditons

- The terminal payoff depends on $A(T^-)$, since the remaining portion of the total guaranteed coupon amount is paid at maturity if no prior knock out occurs. Hence, at time right before expiry, the note value is given by

  $$V(r, T^-) = (1 + c_{cap}) - A(T^-), \quad \text{if} \quad A(T^-) < c_{cap}.$$

- Knock-out occurs on a coupon payment date t_j when $A(t_j^+)$ reaches c_{cap} so that the note value right before t_j is given by

  $$V(r, t_j^-) = (1 + c_{cap}) - A(t_j^-), \quad j = 1, 2, \cdots, K - 1.$$

  Here, the knock-out barrier in the pricing model of the target redemption note is associated with the path dependent state variable A.

- As $r \to \infty, V(r, t) \to 0$, like those of usual bond pricing models.

- The boundary condition at the other extreme of the short rate can be quite tricky. For the Vasieck model, r is allowed to assume value from $-\infty$ to ∞.

- We are mostly interested in the domain where $r > 0$. We are not supposed to impose any boundary condition at $r = 0$. Rather, the differential equation for the note value remains valid at $r = 0$.

- Ironically, the note value tends to infinity as $r \to -\infty$. As for theoretical interest, we postulate that under the Vasicek model

  $$V(r, t) \sim (1 + c_{cap}) P_t^{t_1} \quad \text{as} \quad r \to -\infty, \quad t < t_1.$$

  where $P_t^{t_1}$ denotes the value of the unit par discount bond at time t with maturity date t_1. This is because the target redemption note will be terminated on the first coupon date t_1, thus it behaves like a discount bond with par $1 + c_{cap}$ and maturity date t_1.

- For the CIR model, a boundary condition at $r = 0$ may be required for certain parameter values.

Finite difference scheme

- The proposed finite volume scheme consists of the following three major components:
  1. Use of the finite volume approach to derive the discretized equation. Here, the discretized equation at a grid point is obtained by numerical approximation of the integral form of the governing equation over a finite volume domain.
  2. Adoption of the upstream weighting technique so that numerical solutions are free of oscillations.
  3. Design of the linear interpolation procedure for handling the jump conditions at coupon dates.

- Suppose the one-factor CIR interest rate model is used, that is, the risk neutral drift $\widehat{\mu}(r,t) = \alpha - \gamma r + \overline{\sigma} r^{\beta}\lambda(r,t)$ and $\beta = 1/2$. Here, we use time to expiry $\tau = T - t$ as the temporal variable so that the note value function is written as $V(r,\tau;A)$.
- For $T - t_k < \tau < T - t_{k-1}, k = 1, 2, \cdots, K$, we have

  $$\frac{\partial V}{\partial \tau} = \frac{\overline{\sigma}^2 r}{2}\frac{\partial^2 V}{\partial r^2} + \widehat{\mu}(r,t)\frac{\partial V}{\partial r} - rV, \quad r > 0 \text{ and } 0 \leq A < c_{cap}, \tag{1}$$

  with auxiliary conditions:

  $$V(r, 0^+; A) = (1 + c_{cap}) - A;$$

  and at $A = c_{cap}$

  $$V(r,\tau) = 1.$$

- For a given value of A, each one-dimensional sub-problem within the time period between two successive coupon payment dates can be solved independently.
- The computational domain is characterized by the grid point (τ_n, A_i, r_j), for $n = 0, 1, \cdots, n_{max}, i = 0, 1, \cdots, i_{max}, j = 0, 1, \cdots, j_{max}$. For notational convenience, we set $\tau_0 = 0, r_0 = 0, A_0 = 0$ and $A_{i_{max}} = c_{cap}$.
- We adopt non-uniform grid sizes so that finer grids are used near the maturity time $\tau = 0$ and near the current values and boundary values of A and r.
- Let $V^n_{i,j}$ denote the numerical solution to the note value at the grid point (τ_n, A_i, r_j).
- For each fixed i, the initial condition is posed as $V^0_{i,j} = (1 + c_{cap}) - A_i$, for all j.
- Consider the control volume over the interval $[r_{j-1/2}, r_{j+1/2}]$, where $r_{j-1/2}$ is the mid-point between r_{j-1} and r_j, and similar notation for $r_{j+\frac{1}{2}}$. Integrating Eq. (1) over the finite volume (corresponding to the interval $[r_{j-1/2}, r_{j+1/2}]$) and approximating the corresponding integrals, we obtain the following implicit scheme:

  $$\begin{aligned}\frac{V^{n+1}_{i,j} - V^n_{i,j}}{\Delta\tau_n} &= \frac{\bar{\sigma}^2 r_j}{2\ell_j}\left(\frac{V^{n+1}_{i,j+1} - V^{n+1}_{i,j}}{\Delta r_j} - \frac{V^{n+1}_{i,j} - V^{n+1}_{i,j-1}}{\Delta r_{j-1}}\right) \\ &\quad + \frac{\widehat{\mu}(r_j)}{\ell_j}\left(V^{n+1}_{i,j+1/2} - V^{n+1}_{i,j-1/2}\right) - r_j V^{n+1}_{i,j} \end{aligned}\tag{2}$$

  where $\ell_j = r_{j+1/2} - r_{j-1/2}$ denotes the length of the control volume and $\Delta r_j = r_{j+1} - r_j$ is the step width.

- The values of $V^{n+1}_{i,j+1/2}$ and $V^{n+1}_{i,j-1/2}$ in the drift term in Eq. (2) are computed according to the upstream weighting scheme presented below [see Eq. (20) below].
- When the first-order hyperbolic convective term is large compared to the parabolic diffusion term, the equation is said to be convection dominated. Under dominated convection condition, the numerical solution may suffer from spurious oscillations. In this case, it becomes highly inaccurate to compute the comparative statics such as delta and gamma. To avoid spurious oscillations in the numerical calculations, the upstream weighting scheme is used to compute the value of $V^{n+1}_{i,j+1/2}$, which is defined by

  $$V^{n+1}_{i,j+1/2} = \begin{cases} V^{n+1}_{i,j} & \text{if } \widehat{\mu}(r_j) < 0 \\ V^{n+1}_{i,j+1} & \text{if } \widehat{\mu}(r_j) \geq 0 \end{cases}.$$

- The initial and boundary conditions are given by

$$\begin{aligned}
V_{i,j}^0 &= 1 + c_{cap} - A_i && \text{for all } i \text{ and } j\\
V_{i_{max},j}^n &= 1 && \text{for all } n \text{ and } j\\
V_{i,j_{max}}^n &= 0 && \text{for } n \neq 0, i \neq i_{max}.
\end{aligned}$$

- One may assign a temporal weighting factor θ, $0 \leq \theta \leq 1$, to the spatial discretization terms evaluated at the new and old time levels. This leads to the following weighted finite volume scheme:

$$\begin{aligned}
\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta\tau_n} &= \theta\left[\frac{\bar{\sigma}^2 r_j}{2\ell_j}\left(\frac{V_{i,j+1}^{n+1} - V_{i,j}^{n+1}}{\Delta r_j} - \frac{V_{i,j}^{n+1} - V_{i,j-1}^{n+1}}{\Delta r_{j-1}}\right)\right.\\
&\quad \left.+ \frac{\widehat{\mu}(r_j)}{\ell_j}\left(V_{i,j+1/2}^{n+1} - V_{i,j-1/2}^{n+1}\right) - r_j V_{i,j}^{n+1}\right]\\
&\quad + (1-\theta)\left[\frac{\bar{\sigma}^2 r_j}{2\ell_j}\left(\frac{V_{i,j+1}^n - V_{i,j}^n}{\Delta r_j} - \frac{V_{i,j}^n - V_{i,j-1}^n}{\Delta r_{j-1}}\right)\right.\\
&\quad \left.+ \frac{\widehat{\mu}(r_j)}{\ell_j}\left(V_{i,j+1/2}^n - V_{i,j-1/2}^n\right) - r_j V_{i,j}^n\right].
\end{aligned}$$

  By setting $\theta = 0, 1/2$ and 1, we obtain the fully explicit scheme, Crank-Nicolson scheme and fully implicit scheme, respectively.

Stability and spurious oscillations

- With regard to stability, both the Crank-Nicolson and fully implicit schemes are unconditionally stable. The fully explicit scheme is stable only if the time step is sufficiently small relative to the spatial grid size. The advantage of the Crank-Nicolson scheme is that it is second order accurate in time while both implicit and explicit schemes exhibit only linear rate of convergence.
- With regard to oscillations, fully implicit scheme is not susceptible to spurious oscillations while the Crank-Nicolson and explicit schemes require certain time step constraint in order to avoid numerical oscillations.

Jump conditions on coupon dates

- A jump condition is applied on each coupon date t_k. In terms of time to expiry, suppose the time to expiry corresponding to the coupon date t_k falls between the two time levels τ_n and τ_{n+1}, where $\tau_n \leq T - t_k < \tau_{n+1}$, the finite volume scheme is modified as shown below

$$V_{i,j}^{n+1} = \widehat{V}_{i,j}^n + \min\{c_{cap} - A_i, \widehat{\tau}\max(f - mL(r_j, t_k; \widehat{\tau}), 0)\}.$$

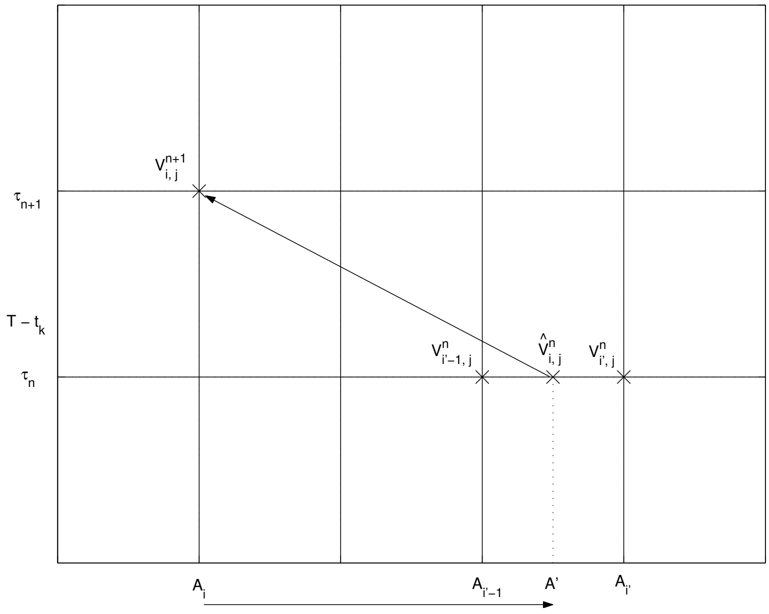
- Here, $\widehat{V}_{i,j}^n$ is the approximate value of $V(r_j, \tau_n; A')$ computed by using the following linear interpolation procedure

$$\widehat{V}_{i,j}^n = \frac{A' - A_{i'-1}}{A_{i'} - A_{i'-1}} V_{i',j}^n + \frac{A_{i'} - A'}{A_{i'} - A_{i'-1}} V_{i'-1,j}^n, \qquad (a)$$

  where

$$A' = A_i + \min\{c_{cap} - A_i, \widehat{\tau}\max(f - mL(r_j, t_k; \widehat{\tau}), 0)\}. \qquad (b)$$

- Normally, A' does not fall onto one of the computational grid points and so linear interpolation is performed to estimate $V(r_j, \tau_n; A')$ using $V^n_{i',j}$ and $V^n_{i'-1,j}$, where A' lies between $A_{i'-1}$ and $A_{i'}$. The second term in A' is the actual coupon payment at time $t = t_k$. Note that A' is capped by c_{cap} since the note holder receives at most the accumulated coupon c_{cap}. In this way, A' always falls between 0 and c_{cap}.



A schematic representation of the linear interpolation procedure at τ_n. Here, A_i jumps to A' according to Eq. (b). Normally, A' does not fall onto one of the computational grid points so that linear interpolation is performed to estimate $V(r_j, \tau_n; A')$ using $V^n_{i',j}$ and $V^n_{i'-1,j}$ where A' lies between $A_{i'-1}$ and $A_{i'}$ [see Eq. (a)].

Work elements in this computer assignment

- Perform sample calculations to illustrate the numerical performance of the proposed finite volume scheme. Test the convergence of the numerical results by varying the time steps and step widths in the computational domain.
- Examine the pricing behavior of the target redemption note subject to varying values of interest rate level, volatility of interest rates, reversion level and reversion speed in the intereswt rate dynamcis. Plot the note value against varying values of the parameters.
- Calculate the probability of premature termination of the note at difference coupon dates.

The contract of the target redemption note used in your sample calculations are specified as follows:

Notional amount	100
Target cap rate	15%
First year coupon rate (fixed)	9%
Inverse floater formula	$\max(8.5\% - 2L, 0)$, $L =$ 3-month LIBOR
Coupon payment frequency	quarterly
Maturity date	5 years from now

The parameter values in the CIR interest rate model are taken to be: $\alpha = 0.02$, $\gamma = 0.5$, $\lambda = 0.01$ and $\bar{\sigma} = 0.2$. For the construction of the discretized computational domain, we use 60 grids along the r-direction and 15 grids along the A-direction, and r_{max} is taken to be 0.9.