1. When the underlying asset pays a continuous dividend yield at the rate $q$, the expected rate of return of the asset is $r - q$ under the risk neutral measure. Under the continuous Geometric Brownian process model, the logarithm of the asset price ratio over $\Delta t$ time interval is normally distributed with mean $\left(r - q - \frac{\sigma^2}{2}\right)\Delta t$ and variance $\sigma^2\Delta t$. Accordingly, the mean and variance of $\frac{S_{t+\Delta t}}{S_t}$ are $e^{(r-q)\Delta t}$ and $e^{2(r-q)\Delta t}(e^{\sigma^2\Delta t} - 1)$. By equating the mean and variance of the discrete binomial model and the continuous Geometric Brownian process model, we obtain

$$pu + (1-p)d = e^{(r-q)\Delta t}$$
$$pu^2 + (1-p)d^2 = e^{2(r-q)\Delta t}e^{\sigma^2\Delta t}.$$ 

Also, we use the usual tree-symmetry condition: $u = 1/d$. Solving the system of 3 equations, we obtain

$$u = \frac{1}{d} = \frac{\sigma^2 + 1 + \sqrt{(\sigma^2 + 1)^2 - 4R^2}}{2R}, \quad p = \frac{R - d}{u - d},$$

where $R = e^{(r-q)\Delta t}$ and $\sigma^2 = 2e^{\sigma^2\Delta t}$. As an analytic approximation to $u$ and $d$ up to order $\Delta t$ accuracy, we take

$$u = e^{\sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{-\sigma\sqrt{\Delta t}}.$$ 

There is only one modification that occurs in the binomial parameter $p$, where

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d},$$

while $u$ and $d$ remain the same. The binomial pricing formula takes a similar form (discounted expectation of the terminal payoff):

$$V = [pV_u e^{\Delta t} + (1-p)V_d e^{-r\Delta t}].$$

The discount factor $e^{-r\Delta t}$ remains the same while the risk neutral probability of up-move $p$ is modified.

2. (a) With the usual notation

$$p = \frac{R - d}{u - d} \quad \text{and} \quad 1-p = \frac{u - R}{u - d}.$$ 

If $R < d$ or $R > u$, then one of the above two probabilities becomes negative. This happens when either $e^{(r-q)\Delta t} < e^{-\sigma\sqrt{\Delta t}}$. 


\[ e^{(r-q)\Delta t} > e^{\sigma \sqrt{\Delta t}}. \]

The above two inequalities are equivalent to 
\[ (q - r)\sqrt{\Delta t} > \sigma \text{ or } (r - q)\sqrt{\Delta t} > \sigma. \]
Hence, negative probabilities occur when
\[ \sigma < |(r - q)\sqrt{\Delta t}|. \]

To avoid the occurrence of negative probability values, the time step must be chosen to be sufficiently small such that
\[ \Delta t < \frac{\sigma^2}{(r-q)^2}. \]

(b) We approximate \( \ln \frac{S_t+\Delta t}{S_t} \) by a discrete random variable \( \zeta^a \), where
\[
\zeta^a = \begin{cases} 
  v_1 & \text{with probability equals 0.5} \\
  v_2 & \text{with probability equals 0.5} 
\end{cases}
\]

Matching the mean and variance of the discrete and continuous distributions, we obtain
\[
E[\zeta^a] = \frac{v_1 + v_2}{2} = \left( r - q - \frac{\sigma^2}{2} \right) \Delta t
\]
\[
\text{var}(\zeta^a) = \frac{v_1^2 + v_2^2}{2} = \sigma^2 \Delta t \text{ [dropping } O((\Delta t)^2) \text{ term]}. 
\]
Solving the pair of equations [up to \( O(\Delta t) \) accuracy], we obtain
\[
v_1 = \left( r - q - \frac{\sigma^2}{2} \right) \Delta t + \sigma \sqrt{\Delta t} \quad \text{and} \quad v_2 = \left( r - q - \frac{\sigma^2}{2} \right) \Delta t - \sigma \sqrt{\Delta t}.
\]
As a check, we consider
\[
v_1^2 + v_2^2 = 2 \left[ \left( r - q - \frac{\sigma^2}{2} \right) \Delta t \right]^2 + 2\sigma^2 \Delta t
\]
so that
\[
\frac{v_1^2 + v_2^2}{2} = \sigma^2 \Delta t + O((\Delta t)^2).
\]

3. For a \( n \)-step trinomial tree, the number of nodes at which we need to perform backward induction calculations is given by
\[
\sum_{i=0}^{n-1} (2i + 1) = n + \frac{n(n-1)}{2} \times 2 = n^2.
\]
Each backward induction calculation involves 2 additions and 3 multiplications. Therefore, the number of multiplications is \( 3n^2 \) and the number of additions is \( 2n^2 \). In a similar
manner, the number of nodes in a $n$-step binomial tree at which we need to perform backward induction calculations is given by

$$
\sum_{i=0}^{n-1} (i + 1) = n + \frac{n(n-1)}{2} = \frac{n^2 + n}{2}.
$$

Each backward induction calculation involves 1 addition and 2 multiplications. Therefore, the number of multiplications is $n^2 + n$ and the number of additions is $\frac{n^2 + n}{2}$.

4. Unlike the derivation in the lecture note, we now keep all the terms that are $O((\Delta t)^2)$.

From the second equation, we obtain

$$
\sqrt{(r - \frac{\sigma^2}{2})^2 \Delta t^2 + \sigma^2 \Delta t}.
$$

Substituting $v$ into the first equation: $(2p - 1)v = \left(r - \frac{\sigma^2}{2}\right) \Delta t$, we have

$$
p = \frac{1}{2} \left[ 1 + \frac{\left(r - \frac{\sigma^2}{2}\right) \Delta t}{\sqrt{\sigma^2 \Delta t + \left(r - \frac{\sigma^2}{2}\right)^2 \Delta t^2}} \right].
$$

5. Consider the system of equations for $p_1$, $p_2$ and $p_3$:

$$
\begin{pmatrix}
1 & 1 & 1 \\
u & 1 & d \\
u^2 & 1 & d^2
\end{pmatrix}
\begin{pmatrix}
p_1 

p_2 

p_3
\end{pmatrix}
= 
\begin{pmatrix}
1 

R 

W
\end{pmatrix}.
$$

Eliminating $p_2$ from the equations, we obtain

$$(u - 1)p_1 + (d - 1)p_3 = R - 1$$

$$(u^2 - 1)p_1 + (d^2 - 1)p_3 = W - 1.$$ 

Solving for $p_1$ and $p_3$ gives

$$p_1 = \frac{(W - R)u - (R - 1)}{(u - 1)(u^2 - 1)}$$

and

$$p_3 = \frac{(W - R)u^2 - (R - 1)u^3}{(u - 1)(u^2 - 1)}.$$ 

When $\lambda = 1$, the parameter $u$ becomes $e^{\sqrt{\Delta t}}$, which agrees with that of the Cox-Rubinstein-Ross binomial scheme. One can show that

$$p_1 + p_3 = 1 + O(\Delta t),$$

or equivalently,

$$p_2 = O(\Delta t).$$

If we consider order of accuracy up to $O(\Delta t)$, then $p_2$ vanishes. As a result, the trinomial scheme reduces to a binomial scheme.

6. By equating the corresponding mean, variances and covariances [up to $O(\Delta t)$ accuracy], we have
Unlike the floating strike lookback call normalized by the asset price, where the exercise payoff is expressible as $M_{(t_j)} / S(t_j) - 1 = Y_j - 1$, the exercise payoff in the fixed strike lookback

\[
E[\zeta_1] = v_1(p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8) = \left( r - \frac{\sigma_1^2}{2} \right) \Delta t \quad (i)
\]

\[
E[\zeta_2] = v_2(p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8) = \left( r - \frac{\sigma_2^2}{2} \right) \Delta t \quad (ii)
\]

\[
E[\zeta_3] = v_3(p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8) = \left( r - \frac{\sigma_3^2}{2} \right) \Delta t \quad (iii)
\]

\[
\text{var}[\zeta_1] = v_1^2(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8) = \sigma_1^2 \Delta t \quad (iv)
\]

\[
\text{var}[\zeta_2] = v_2^2(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8) = \sigma_2^2 \Delta t \quad (v)
\]

\[
\text{var}[\zeta_3] = v_3^2(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8) = \sigma_3^2 \Delta t \quad (vi)
\]

\[
E[\zeta_1^2] = v_1v_2(p_1 + p_2 - p_3 - p_4 - p_5 - p_6 + p_7 + p_8) = \sigma_1\sigma_2\rho_{12} \Delta t \quad (vii)
\]

\[
E[\zeta_1^2] = v_1v_3(p_1 - p_2 - p_3 - p_4 - p_5 - p_6 + p_7 + p_8) = \sigma_1\sigma_3\rho_{13} \Delta t \quad (viii)
\]

\[
E[\zeta_2^2] = v_2v_3(p_1 - p_2 - p_3 + p_4 + p_5 - p_6 - p_7 + p_8) = \sigma_2\sigma_3\rho_{23} \Delta t \quad (ix)
\]

Lastly, the sum of probabilities must be one so that

\[
p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 = 1. \quad (x)
\]

Recall that $v_1 = \lambda_1 \sigma \sqrt{\Delta t}, v_2 = \lambda_2 \sigma \sqrt{\Delta t}$ and $v_3 = \lambda_3 \sigma \sqrt{\Delta t}$. In order that Eqs (iv), (v) and (vi) are consistent, we must set $\lambda_1 = \lambda_2 = \lambda_3$. We write the common value as $\lambda$. These 3 equations then reduce to single equation:

\[
p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = \frac{1}{\lambda^2}.
\]

There are only 8 equations and 9 unknowns. We impose the last condition: $E[\zeta_1^a \zeta_2^a \zeta_3^a] = 0$ [up to $O(\Delta t)$ accuracy], which gives one additional equation:

\[
E[\zeta_1^a \zeta_2^a \zeta_3^a] = v_1v_2v_3(p_1 - p_2 - p_3 + p_4 + p_5 - p_6 + p_7 - p_8) = 0.
\]

The probability values are obtained as follows:

\[
p_2 = \frac{1}{8} \left[ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{r - \sigma_1^2}{2} + \frac{r - \sigma_2^2}{2} - \frac{r - \sigma_3^2}{2} \right) \right] \left( \rho_{12} - \rho_{13} - \rho_{23} \right) \frac{1}{\lambda^2}
\]

\[
p_3 = \frac{1}{8} \left[ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{r - \sigma_1^2}{2} - \frac{r - \sigma_2^2}{2} + \frac{r - \sigma_3^2}{2} \right) \right] \left( \rho_{13} - \rho_{12} - \rho_{23} \right) \frac{1}{\lambda^2}
\]

\[
p_4 = \frac{1}{8} \left[ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{r - \sigma_2^2}{2} - \frac{r - \sigma_3^2}{2} + \frac{r - \sigma_3^2}{2} \right) \right] \left( \rho_{23} - \rho_{13} - \rho_{12} \right) \frac{1}{\lambda^2}, \text{ etc.}
\]

7. Unlike the floating strike lookback call normalized by the asset price, where the exercise payoff is expressible as $M_{(t_j)} / S(t_j) - 1 = Y_j - 1$, the exercise payoff in the fixed strike lookback
8. (a) When $t_{j+1} \neq iZ$, for all $i$, there will be no monitoring of the realized maximum of the asset price in the next time step, so the usual backward induction procedure prevails. The usual discounted expectation procedure gives

$$C_X(S(t_j), K', t_j) = [pC_X(uS(t_j), K', t_{j+1}) + (1 - p)C_X(dS(t_j), K', t_{j+1})]e^{-r\Delta t}.$$ 

Since $K'$ does not change, so $k$ increases (decreases) by one when $S(t_j)$ moves up (down). Upon normalizing $C_X(S(t_j), K', t_j)$ by $S(t_j)$, we obtain

$$X^Z(k, t_j) = [pX^Z(k + 1, t_{j+1})u + (1 - p)X^Z(k - 1, t_{j+1})d]e^{-r\Delta t}.$$ 

(b) When $k \geq 1$ and $t_{j+1} = iZ$, the next time step is a monitoring instant for the new realized maximum. Since $k \geq 1$, an updated realized $K'$ will be recorded at $t_{j+1} = iZ$, so the index $k$ becomes zero at $t_iZ$, irrespective of either an up-move or down-move of the asset price. Recall that $S(t_j) = K'u^k$, $k \geq 1$, given that an up-move of the asset price occurs in the next time step, the guaranteed payment at maturity is $uS(t_j) - K'$. When normalized by $S(t_j)$, it becomes $u - u^{-k}$. This occurs with probability $p$ and it is paid $N - (iz - 1)$ time steps later. Similarly, for a down-move, the normalized guaranteed payment at maturity is $[dS(t_j) - K']/S(t_j) = u^{-1} - u^{-k}$. Combining all these results together, we obtain

$$X^Z(k, t_{iZ-1}) = X^Z(0, t_{iZ}) + [p(u - u^{-k}) + (1 - p)(u^{-1} - u^{-k})]e^{-(N-iz+1)\Delta t}.$$ 

(c) When $k = 0$ and $t_{j+1} = iz$, we have $S(t_j) = K'$. A newly realized maximum is recorded if the asset price has an up-move in the next time step and $k$ becomes zero. Otherwise, $k$ becomes $-1$ for a down-move of the asset price. For an up-move with probability $p$, since $S(t_j) = K'$, the normalized guaranteed payment at maturity is $[uS(t_j) - K']/S(t_j) = u^{-1}$. Combining all these results, we obtain

$$X^Z(0, T_{iZ-1}) = \left[pX^Z(0, t_{iZ})u + (1 - p)X^Z(-1, t_{iZ})d\right]e^{-r\Delta t} + p(u - 1)e^{-(N-iz+1)\Delta t}.$$ 

9. If $m$ is set equal to $\widehat{m}$, then the window Parisian feature reduces to the consecutive Parisian feature. We define a binary string $A = (a_1, a_2, \cdots, a_{N_w})$ of size $N_w$ to represent the history of the asset price path falling inside or outside the knock-out region at the previous $N_w$ consecutive monitoring instants prior to the current time. By convention, the value of $a_p$ is set to be 1 if the asset price falls on or below the down barrier $B$ at the $p$-th monitoring instant counting backward from the current time; and it is set to be 0 if otherwise.
There are altogether $2^{N_w}$ different binary strings to represent all possible breaching history of asset price path at the previous $N_w$ monitoring instants. The number of states that have to be recorded is $C_0^{N_w} + C_1^{N_w} + \cdots + C_{N-1}^{N_w}$, where $C_i^{N_w}$ denotes the combination of $N_w$ strings taken $i$ strings at a time. We sum from $i = 0$ to $i = N - 1$ since the window Parisian option value becomes zero when the number of breaches reaches $N$, so those states with $N$ or more “1” in the string are irrelevant.

Let $V_{\text{win}}[m, j; A]$ denote the value of a window Parisian option at the $(m, j)$-th node, augmented with the asset price path history represented by the binary string $A$. The binary string $A$ has to be modified according to the event of either breaching or no breaching at a monitoring instant. The corresponding numerical scheme can be succinctly represented by

$$V_{\text{win}}[m - 1, j; A] = \begin{cases} p_{A}V_{\text{win}}[m, j + 1; A] + p_0V_{\text{win}}[m, j; A] + p_dV_{\text{win}}[m, j - 1; A]e^{-r\Delta t} & \text{if } m\Delta t \neq t^*_l \\ \end{cases}$$

where

$$g_{\text{win}}(A, j) = \begin{cases} (1, a_1, a_2, \cdots, a_{N_w-1}) & \text{if } x_j \leq \ln B \\ (0, a_1, a_2, \cdots, a_{N_w-1}) & \text{if } x_j > \ln B \\ \end{cases}$$

Note that $V_{\text{win}}[m, j; A] = 0$ at a monitoring instant when the string $A$ has $N$ or more “1”. Due to the higher level of path dependence exhibited by the window feature, the operation counts of the window Parisian option calculations are roughly $C_0^{N_w} + C_1^{N_w} + \cdots + C_{N-1}^{N_w}$ times of those of the plain vanilla option calculations.

10. The payoff of a floating strike lookback call at time $t$, $t \in [0, T]$, is given by

$$\max_{\tau \in [0, t]} S_{\tau} - S_t,$$

where $\max_{\tau \in [0, t]} S_{\tau}$ denotes the realized maximum of the asset price over $[0, t]$. The corresponding grid function at the $(n, j)^{th}$ node with asset price $S^n_j = S^u_j$ is given by

$$g_{\text{lookback}}(k, j) = \max(k, j).$$

Here, $k$ is the numbering index for the lookback state variable and $S^k$ is the terminal payoff dictates

$$V^{n}_{j, k} = [p_{A}V^{n+1}_{j+1, \text{lookback}(k, j+1)} + p_0V^{n+1}_{j, \text{lookback}(k, j)} + p_dV^{n+1}_{j-1, \text{lookback}(k, j-1)}]e^{-r\Delta t}.$$

At the terminal nodes in the $N$-step trinomial tree, the terminal payoff dictates

$$V^{N}_{j, k} = S^k - S^u_j,$$

where $j = -N, -N + 1, \cdots, N - 1, N$, and $k = -N, -N + 1, \cdots, N - 1, N$.

To incorporate the American early exercise feature, we simply incorporate the dynamic programming procedure at each node and for each number index:

$$V^{n}_{j, k} = \max \left\{ [p_{A}V^{n+1}_{j+1, \text{lookback}(k, j+1)} + p_0V^{n+1}_{j, \text{lookback}(k, j)} + p_dV^{n+1}_{j-1, \text{lookback}(k, j-1)}] e^{-r\Delta t}, S^k - S^u_j \right\}.$$
11. We let the current time be time 0 for convenience, $t_i$ be the $i^{th}$ observation date, $T_i$ be the settlement date of stocks based on the $i^{th}$ observation, $T_i > t_i$. Let $S$ denote the asset value at the current time and define $M = \ln \frac{H}{S}$. We write $X_i$ as the log asset price ratio $\ln \frac{S_i}{S}$ on the $i^{th}$ observation date. According to eq.(4.1.27) in Kwok’s text, the restricted density function of the log asset price ratio with an upstream barrier $M$ is given by

$$f_{up}(x, t_i; M) = \frac{1}{\sigma \sqrt{t_i}} \left[ n \left( \frac{x - \mu t_i}{\sigma \sqrt{t_i}} \right) - e^{-2M} n \left( \frac{x - 2M - \mu t_i}{\sigma \sqrt{t_i}} \right) \right],$$

where $\mu = r - q - \frac{\sigma^2}{2}$. The up-and-out call option value is given by

$$c_{uo}(S, t_i; K, H) = e^{-r t_i} \int_{\ln \frac{H}{S}}^{\infty} (Se^x - K) f_{up}(x, t_i; M) \, dx.$$ 

We take advantage of the well known down-and-out call option value function $c_{do}(S, t_i; H)$, where the strike price $K$ is larger than the down barrier $H$. From eq.(4.1.40) in Kwok’s text [also refer to eq.(4.1.10)], we obtain

$$c_{do}(S, t_i; K, H) = e^{-r t_i} \int_{\ln K}^{\infty} (Se^x - K) f_{down}(x, t_i; M) \, dx$$

where $\lambda = \frac{2(r - q)}{\sigma^2}$ and $f_{down}(x, t_i; M)$ has the same analytic form as that of $f_{up}(x, t_i; M)$, $M = \ln \frac{H}{S}$. Here, $c_{E}(S, t_i; K)$ is the value function of the vanilla European call option as given by

$$c_{E}(S, t_i; K) = Se^{-q t_i} N(d_1^{(i)}) - Ke^{-r t_i} N(d_2^{(i)}),$$

where

$$d_1^{(i)} = \frac{\ln \frac{S}{K} + (r - q + \frac{\sigma^2}{2}) t_i}{\sigma \sqrt{t_i}} \quad \text{and} \quad d_2^{(i)} = d_1^{(i)} - \sigma \sqrt{t_i}.$$ 

Since $f_{up}(x, t_i; M)$ and $f_{down}(x, t_i; M)$ share the same analytic form, we deduce that [see eq.(4.1.41) in Kwok’s text]

$$c_{uo}(S, t_i; K, H) = c_{do}(S, t_i; K, H) - c_{do}(S, t_i; H, H)$$

$$= \left[ c_{E}(S, t_i; K) - \left( \frac{H}{S} \right)^{\lambda-1} c_{E}\left( \frac{H^2}{S}, t_i; K \right) \right]$$

$$- \left[ c_{E}(S, t_i; H) - \left( \frac{H}{S} \right)^{\lambda-1} c_{E}\left( \frac{H^2}{S}, t_i; H \right) \right].$$

In a similar manner, one can show that the value of the up-and-out put option with $K < H$ is given by (see Problem 4.5 in Kwok’s text)

$$p_{uo}(S, t_i; K, H) = Xe^{-r t_i} N(-d_2^{(i)}) - Se^{-q t_i} N(-d_1^{(i)})$$

$$- \left[ \left( \frac{H}{S} \right)^{\lambda-1} Xe^{-r t_i} N(-d_4^{(i)}) - \left( \frac{H}{S} \right)^{\lambda+1} Se^{-q t_i} N(-d_3^{(i)}) \right].$$
where
\[ d_3^{(i)} = \frac{2 \ln \frac{H}{S}}{\sigma \sqrt{t_i}} + d_1^{(i)} \quad \text{and} \quad d_4^{(i)} = d_3^{(i)} - \sigma \sqrt{t_i}. \]

According to Lam et al. (2009), we may express \( c_{uo}(S, t_i; K, H) \) and \( p_{uo}(S, t_i; K, H) \) as
\[
\begin{align*}
\text{\( c_{uo}(S, t_i; K, H) = e^{-r t_i} \left\{ E_Q[e^{X_{t_i}} 1_A] - KE_Q[1_A] \right\} \)} \\
\text{\( p_{uo}(S, t_i; K, H) = e^{-r t_i} \left\{ KE_Q[1_B] - E_Q[e^{X_{t_i}} 1_B] \right\} \)}.
\end{align*}
\]

where \( Q \) is a risk neutral measure, and
\[
\begin{align*}
A &= \left\{ \omega \in \Omega \mid X_{t_i} \geq \ln \frac{K}{S}, M_t < \ln \frac{H}{S} \right\}, \\
B &= \left\{ \omega \in \Omega \mid X_{t_i} < \ln \frac{K}{S}, M_t < \ln \frac{H}{S} \right\}, \\
M_t &= \max_{0 \leq u \leq t_i} X_u.
\end{align*}
\]

With the delay settlement on \( T_{t_i} \), the value function of the up-and-out-call and up-and-out-put with delay settlement are given by
\[
\begin{align*}
\text{\( c_{uo}(S, T_{t_i}; K, H) = e^{-r T_{t_i}} \left\{ E_Q[e^{X_{T_{t_i}}} 1_A]E_Q[e^{X_{T_{t_i}} - X_{t_i}}] - KE_Q[1_A] \right\} \)} \\
\text{\( p_{uo}(S, T_{t_i}; K, H) = e^{-r T_{t_i}} \left\{ KE_Q[1_B] - E_Q[e^{X_{T_{t_i}}} 1_B]E_Q[e^{X_{T_{t_i}} - X_{t_i}}] \right\} \)}.
\end{align*}
\]

Since \( E_Q[e^{X_{T_{t_i}} - X_{t_i}}] = e^{-q(T_{t_i} - t_i)} \), we can deduce easily that-
\[
\begin{align*}
\text{\( p_{uo}(S, T_{t_i}; K, H) = X e^{-r T_{t_i}} N(-d_3^{(i)}) - S e^{-q T_{t_i}} N(-d_4^{(i)}) \)} \\
&\quad - \left[ \left( \frac{H}{S} \right)^{\lambda-1} X e^{-r T_{t_i}} N(-d_3^{(i)}) - \left( \frac{H}{S} \right)^{\lambda+1} S e^{-q T_{t_i}} N(-d_3^{(i)}) \right].
\end{align*}
\]

Comparing \( p_{uo}(S, T_{t_i}; K, H) \) and \( p_u(S, t_i; K, H) \), the up-and-out put price function with and without delay settlement, we observe that the discount factors differ while the expectation terms \( N(-d_j^{(i)}) \), \( j = 1, 2, 3, 4 \), stay the same. This is not surprising since the stocks and strike price are delivered on \( T_{t_i} \) after the \( t_i \)th observation date \( t_i \) while the deciding criterion on the delivery of one or two stocks remains to be determined by \( S_{t_i} \geq K \) or \( S_{t_i} < K \). In a similar manner, we can deduce \( c_{uo}(S, T_{t_i}; K, H) \) from \( c_{uo}(S, t_i; K, H) \) by modifying the discount factors while the expectation terms \( N(d_j^{(i)}) \), \( j = 1, 2, 3, 4 \), stay the same.