MAFS5250 – Computational Methods for Pricing Structured Products

Topic 5 – Exotic structured products

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5.1 Participating life insurance policies

Product nature: bonus distribution mechanism

- Annual rate of return guarantee and bonus distribution
  - interest is credited to the policy account balance according to some smoothing surplus distribution mechanism
  - specified claim to a fraction of any excess return (surplus) generated by the investments

- Surrender option held by investors
  - American style feature of early redemption
Contractual terms

- A contract of nominal value $P_0$ is issued by the insurance company at time zero.

- The contract is acquired by an investor for a single premium $V_0$, where $V_0 \leq P_0$ (sold at a discount).

- Assuming no mortality risk, there are no further payments from or to the contract prior to expiration time $T$. At expiry, the contract is settled by a single payment from the issuer to the investor.

The nominal value $P_0$ is preset while $V_0$ is determined from the pricing model as the fair value of the contract at time 0.

**Distributed reserve and buffer**

- Policy account balance, \( P(t) \) — book value of the policy

  To the insurer, \( P(t) \) is the amount set aside to cover the contract liability, considered as the distributed reserve.

- Market value of the asset base backing the contract, \( A(t) \)

- Undistributed reserve or buffer, \( B(t) \)

Mechanism in place to protect the policy reserve from unfavorable fluctuations in the asset base. Accounting rule gives

\[
A(t) = P(t) + B(t).
\]

Since pension and life insurance companies typically invest largely in highly liquid assets such as bonds and stocks for which the relevant market prices are easily observable, we can assume that \( A \) is tradeable.
Crediting mechanism of the policy value process

We write \( \{P(t)\}_{0 \leq t \leq T} \) as the account balance process of the contract. The benefit from the contract at maturity \( T \) is denoted by \( P(T) \).

The evolution of \( P(\cdot) \) between successive time points in the point set

\[ T = \{1, 2, \cdots, T\} \]

is determined by the discretely compounded policy interest rate process, \( \{r_P(t)\}_{t \in T} \). We have

\[ P(t) = [1 + r_P(t)]P(t - 1), \quad t \in T, \]

so that

\[ P(t) = P_0 \prod_{i=1}^{t} [1 + r_P(i)]. \]

Time is measured in years, \( P(\cdot) \) is updated annually, and \( r_P(\cdot) \) is annualized rate. That is, \( P(\cdot) \) is held fixed for \( (t - 1, t) \) and has a jump in value at \( t^+ \).
Dynamics of the asset side

\[ dA(t) = \mu A(t) \, dt + \sigma A(t) \, dW(t), \quad A(0) = A_0. \]

Here, \( W(t) \) is the standard Brownian process defined on the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on the finite time interval \([0, T]\), \( \sigma = \) volatility, \( \mu = \) expected rate of return.

Under the risk neutral probability measure \( Q \), all prices discounted by the continually compounded risk free interest rate \( r \) are \( Q \)-martingales. We have

\[ dA(t) = r A(t) \, dt + \sigma A(t) \, dW^Q(t), \quad A(0) = A_0, \]

where \( W^Q(t) \) is the standard Brownian process under measure \( Q \).


**Liability side of the balance sheet**

Specification of $r_P(t)$ (dynamic distribution of funds to investor’s account) based on $P(t^-)$ and $B(t^-) = A(t) - P(t^-)$:

$$r_P(t) = \max \left( r_G, \alpha \left( \frac{B(t^-)}{P(t^-)} - \gamma \right) \right),$$

$r_G$: guaranteed interest rate of the contract

$\gamma$: target buffer ratio

$\alpha$: distribution ratio

If the actual/observed buffer relative to the policy account balance at time $t^-$ exceeds the desired level $\gamma$, then the company distributes a fraction $\alpha$ of the surplus.

$P(t)$: strictly increasing process with jump at discrete time points;

$B(t)$ may be temporarily negative (insolvency with respect to the contract). This occurs when $A(s) < P(t^+)$, where $t < s < t + 1$, since $P(t^+)$ has been fixed at $t^+$ earlier than $s$. 
Evaluation of the account balance process \( P(t) \)

In the favorable scenario where the asset return is sufficiently high, specifically, \( r_G < \alpha \left( \frac{B(t^-)}{P(t^-)} - \gamma \right) \), we have

\[
P(t^+) = P(t^-) \left[ 1 + \alpha \left( \frac{B(t^-)}{P(t^-)} - \gamma \right) \right] = P(t^-) + \alpha \left[ B(t^-) - B^*(t^-) \right],
\]

where \( B^*(t^-) = \gamma P(t^-) \) is the desired level of buffer over the period \((t - 1, t)\). In general

\[
P(t^+) = P(t^-) \left[ 1 + \max \left( r_G, \alpha \left( \frac{B(t^-)}{P(t^-)} - \gamma \right) \right) \right] = P(t^-) \left[ 1 + r_G + \max \left( \alpha \left( \frac{A(t) - P(t^-)}{P(t^-)} - \gamma \right) - r_G, 0 \right) \right].
\]

Here, the policy account process \( P(\cdot) \) is highly dependent on the path followed by \( A(t) \). Note that \( P(\cdot) \) and \( B(\cdot) \) have jump at each discrete time point due to crediting mechanism while \( A(t) \) is assumed to be continuous at all times.
**Bonus option**

- The guaranteed interest rate implies a bond floor under the final payment from the contract of $P_{floor}^T = P_0(1 + r_G)^T$.

- Let $V^E(s)$ denote the time-$s$ value of the contract (European style) and let $D(s)$ denote the time-$s$ value of the bond component.

\[
V^E(s) = E^Q \left[ e^{r(T-s)}P(T) | \mathcal{F}_s \right], \quad \text{for all } s \in [0, T], \quad V^E(T) = P(T) \\
D(s) = e^{-r(T-s)}P_0(1 + r_G)^T \quad \text{[annually compounded]}
\]

Value of the bonus option is given by

\[
\Gamma(s) = V^E(s) - D(s).
\]
Surrender option (American style contract)

This is the right given to the investor to terminate the contract prematurely

\[ V^A(s) = \sup_{\tau \in \mathcal{T}_{s,T}} E^Q \left[ e^{-r(\tau-s)} P(\tau) | \mathcal{F}_s \right], \]

where the contract is terminated at the investor’s discretion at time \( \tau \). Here, \( \mathcal{T}_{s,T} \) denotes the class of \( \mathcal{F}_s \)-stopping times taking values in \([s, T]\). In most contracts, the investors may have to pay a penalty charge for premature surrender. Here, we assume zero penalty charge.

- Since the surrender payoff is \( P(\tau) \), so

\[ V^A(s) \geq P(s), \quad 0 \leq s \leq T. \]

However, if there is a surrender charge, which may be either fixed or proportional or combination of both, then the surrender payoff is reduced by the surrender charge. With delay of surrender, \( V^A \) drops in value as a result.
The contract value with surrender right is the sum of the bond component, bonus option value and surrender option value. Let $\psi(s)$ denote the time-$s$ value of the surrender option. We then have

$$\psi(s) = V^A(s) - D(s) - \Gamma(s).$$

Comment on the optimal surrender policy

For an American style contract, since the value of $P(s)$, $t < s < t + 1$, has been fixed at time $t^+$, it will never be optimal to exercise the contract between two updates of $r_P(\cdot)$. Exercising the contract at time $s, t < s < t + 1$, will result in a loss of interest amounting to $[e^{r(s-t)} - 1] P(t^+) > 0$, compared to exercising at time $t^+$. This is in a similar spirit to the optimal exercise policy of an American call option on a discrete dividend paying asset, where it may be optimal to exercise such American call only at instants that are right after an ex-dividend date. Here, the investor chooses to exercise the surrender option only right after the policy account is updated.
States of the world

Note that \( P(s), t < s < t + 1 \), does not change during the time interval \((t, t + 1)\); \( P(s) \) is updated at \( t^+ \) based on \( P(t^-) \) and \( A(t) \).

All relevant information about the states of the world is summarized by \((A(s), P(t^+))\), where \( t < s < t + 1 \leq T \), observing that \( A(s) \) is continuous for all times while \( P(s) \) has jump across sampling dates. We write the time-\( s \) value of the contract as

\[
V_s = V(s, A(s), P(t^+)), \quad t < s < t + 1 \leq T, t \in T \cup \{0\}.
\]

Here, \( T \) is the set of sampling dates and the contract initiation date is time zero.

Continuity of value function across a sampling date \( t \)

\[
V_{t^-} = V_{t^+} \iff V(t^-, A(t), P(t^-)) = V(t^+, A(t), P(t^+)).
\]

Any joint realization of \((s, A_s, P_s)\) necessarily changes \( V \) in a continuous manner since there is no net cash flow associated with the contract across a sampling date.
Model formulation

- \( P(\cdot) \) is updated only discretely, so it does not change outside the set of time points \( \mathcal{T} \).

- Between these ‘sampling dates’, the Black-Scholes equation is to be solved (within the time interval between the consecutive years \( t \) and \( t + 1 \), and before the no-jump condition on \( V \) is applied):

\[
\frac{\partial V}{\partial s} + \frac{\sigma^2}{2} A^2 \frac{\partial^2 V}{\partial A^2} + r A \frac{\partial V}{\partial A} - r V = 0, \quad s \in (0, T)/\mathcal{T},
\]

with \( V_T = P(T) \).

Over the time interval \((t, t+1)\), we write the numerical option value as

\[
V_{t,k}^{i,j} = V(t + 1 - k \Delta s, i \Delta A, j \Delta P),
\]

where \( t \in \{0, 1, \cdots, T - 1\}, 0 \leq k \leq K, 0 \leq i \leq I, 0 \leq j \leq J \). As \( k \) increases from 0 to \( K \), the calendar time decreases from \( t + 1 \) to \( t \).
Computational domain

Restricted to the finite domain, \((s, A, P) = ([0, T] \times [0, \bar{A}] \times [P_0, \bar{P}])\), where \(P_0\) is the initial policy value, \(\bar{A}\) and \(\bar{P}\) are sufficiently large constants.

\[
I = \frac{\bar{A}}{\Delta A} = \text{number of equally spaced steps in the } A\text{-direction}
\]

\[
J = \frac{\bar{P}}{\Delta P} = \text{number of equally spaced steps in the } P\text{-direction}
\]

\[K = \text{number of time steps per year}\]

Discretization considerations

- Choice of the mesh size of the grid in the \((A, P)\)-space.
- Imposition of the auxiliary conditions.
- Implementation of the no-jump condition on the contract value across sampling dates.
Numerical procedure

1. Start at time $T$ and apply the terminal payoff condition on a suitable grid in the $(A,P)$-space.

2. For every value of $P$, solve the Black-Scholes equation via a finite difference scheme. This gives $V_{(T-1)+}$ everywhere in the grid.

3. Apply the no-jump condition on the contract value to obtain $V_{(T-1)-}$ everywhere in the grid. Actually, we require the procedure of implementing the jump condition on $P(\cdot)$ [jumping from $P(t^-)$ to $P(t^+)$ at time $t$].

4. Repeat steps 2 and 3 to obtain $V_{t-}$ from $V_{(t+1)-}$ everywhere in the grid working backwards from $t = T - 1$ to $t = 0$. Minor remark: It is not necessary to apply the no-jump condition at $t = 0$. 
We adopt the fully implicit scheme ($A$ as the independent state variable and $P$ is fixed):

\[
\frac{V_{i,j}^{t,k} - V_{i,j}^{t,k+1}}{\Delta s} + \frac{1}{2}\sigma^2 (i \Delta A)^2 \left( \frac{V_{i,j}^{t,k+1} - 2V_{i,j}^{t,k} + V_{i,j}^{t,k-1}}{\Delta A^2} \right) \\
+ r(i \Delta A) \left( \frac{V_{i,j}^{t,k+1} - V_{i,j}^{t,k}}{2 \Delta A} \right) - rV_{i,j}^{t,k+1} = 0,
\]

which can be simplified to the following implicit relation:

\[
E^i V_{i,j}^{t,k-1} + H^i V_{i,j}^{t,k} + G^i V_{i,j}^{t,k+1} = V_{i,j}^{t,k},
\]

where

\[
E^i = \frac{r(i \Delta A)}{2} \frac{\Delta s}{\Delta A} - \frac{\sigma^2 (i \Delta A)^2}{2} \frac{\Delta s}{(\Delta A)^2},
\]

\[
H^i = 1 + r \Delta s + \sigma^2 (i \Delta A)^2 \frac{\Delta s}{(\Delta A)^2},
\]

\[
G^i = - \frac{r(i \Delta A)}{2} \frac{\Delta s}{\Delta A} - \frac{\sigma^2 (i \Delta A)^2}{2} \frac{\Delta s}{(\Delta A)^2}.
\]
Boundary conditions

At $A = 0, i = 0$, the governing differential equation reduces to

$$\frac{\partial V}{\partial s} - rV = 0.$$ 

There is no equity participation, so the option value is only influenced by discounting. The corresponding finite difference relation is

$$V_{t+1,0,j}^0 = \frac{1}{1 - r\Delta s}V_{t,0,j}^0.$$ 

Here, $1 - r\Delta s$ can be visualized as the discrete discount factor over one time step.

For $A = \bar{A}, i = I$, the value function is approximately linear

$$\frac{\partial^2 V}{\partial A^2} = 0 \quad \text{for} \quad A \rightarrow \infty.$$ 

Applying this condition at $I - 1$, this yields the relation:

$$V_{t,k+1}^{I,j} = 2V_{t,k+1}^{I-1,j} - V_{t,k+1}^{I-2,j}.$$ 

In other words, the numerical boundary value $V_{t,k+1}^{I,j}$ can be computed based on the value functions at the two neighboring interior points.
The matrix representation of the implicit scheme is given by
\[
\begin{pmatrix}
H^1 & G^1 & 0 & \cdots & \cdots & 0 \\
E^2 & H^2 & G^2 & 0 & & \vdots \\
0 & E^3 & H^3 & G^3 & & \vdots \\
& \vdots & \vdots & \ddots & \ddots & \vdots \\
& 0 & E^{I-2} & H^{I-2} & G^{I-2} & \vdots \\
0 & \cdots & \cdots & 0 & (E^{I-1} - G^{I-1}) & (H^{I-1} + 2G^{I-1})
\end{pmatrix}
\begin{pmatrix}
V_{1,j}^{t,k+1} \\
V_{2,j}^{t,k+1} \\
\vdots \\
V_{I-1,j}^{t,k+1}
\end{pmatrix}
\]

\[=\]
\[
\begin{pmatrix}
V_{1,j}^{t,k} - E^1 (1 - r \Delta s) V_{0,j}^{t,k} \\
V_{2,j}^{t,k} \\
\vdots \\
V_{I-1,j}^{t,k}
\end{pmatrix}
\]

The last row in the coefficient matrix reflects the incorporation of the numerical boundary condition in the far field.
Implementation of the no-jump condition on the contract value

We need to code the relationship between \( V_{t,K}^{i,j} \) and \( V_{t-1,0}^{i,j} \) according to the no-jump condition. Recall

\[
P(t^+) = P(t^-) + \max(r_GP(t^-), \alpha\{[A(t) - P(t^-)] - \gamma P(t^-)\}).
\]

For each \( i \) and \( j \) in the grid, we write \( P(t^+) = \tilde{j}\Delta P, P(t^-) = j\Delta P \), where \( j \) is some integer index. In terms of \( \tilde{j} \) and \( j \), the crediting mechanism across a sampling date is modeled by

\[
\tilde{j} = \frac{j\Delta P + \max\{r_G(j\Delta P), \alpha[(i\Delta A - j\Delta P) - \gamma(j\Delta P)]\}}{\Delta P} = j + \max\left\{r_Gj, \alpha \left[\left(\frac{i\Delta A}{\Delta P} - j\right) - \gamma j\right]\right\}. \tag{i}
\]

Remark  Though there is an upward jump in \( P \), the contract value remains continuous across the date of interest crediting.
We solve for the contract value at finite number of preset values of $P$, $P = j\Delta P$, $j = j_0, j_0 + 1, \ldots, J$, with $P_0 = j_0\Delta P$ and $P = J\Delta P$. The updated $P$ would not fall onto one of these preset values. Linear interpolation between neighboring contract values is then applied.

- Denote the integer part of $\tilde{j}$ as $\underline{j}$. If $\underline{j} + 1 \leq J$, compute $V_{t-1,0}^{i,j}$ by using the linear interpolation
  \[ V_{t-1,0}^{i,j} = [1 - (\tilde{j} - \underline{j})]V_{t,K}^{i,j} + (\tilde{j} - \underline{j})V_{t,K}^{i,j+1}. \]  

- At time $t^-$, $P(t^-)$ assumes the value $J\Delta P$. Subsequently, $P(t^+)$ jumps to $\tilde{j}\Delta P > J\Delta P$. The updated value of $P$ falls outside the preset upper bound of $P$, we apply extrapolation beyond the computational domain. If $\underline{j} + 1 > J$ and hence lies outside the grid, then (ii) cannot be used. Instead, since for large values of $P$, the contract value $V$ is approximately linear in $P$, we can apply the linear extrapolation as follows:
  \[ V_{t-1,0}^{i,J} = V_{t,K}^{i,J} + (\tilde{j} - J)\left(V_{t,K}^{i,J} - V_{t,K}^{i,J-1}\right). \]
The no-jump condition implements the transformation of $V_{t_+}$ to $V_{t_-}$, where $V_t$ and $A(t)$ are continuous and $P(t)$ has a jump across a sampling date. The finite difference scheme can be iterated for another year.
Summary

- $P$ stays at the same constant value as $P((t-1)^+)$, for $t-1 < s < t$.

- For a given fixed value of $P$, solve the Black Scholes equation numerically between $t^+$ and $(t+1)^-$. 

- $V_{t,K}^{i,j} = V_{t-1,0}^{i,j}$ across the sampling date $t$.

Note that $i$ stays at the same value since the underlying asset price is continuous across the sampling date $t$. However, $j$ jumps to $\tilde{j}$ across $t$. 
Numerical results and pricing behavior

A large number of experiments have resulted in the choices of values shown in the table below in the numerical implementation.

<table>
<thead>
<tr>
<th>Choice of parameter values</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{A}$</td>
<td>1000</td>
</tr>
<tr>
<td>$\bar{P}$</td>
<td>2000</td>
</tr>
<tr>
<td>$I$</td>
<td>800</td>
</tr>
<tr>
<td>$J$</td>
<td>200</td>
</tr>
<tr>
<td>$K$</td>
<td>100</td>
</tr>
</tbody>
</table>

- Setting $\bar{A}$ and $\bar{P}$ involves a tradeoff between covering as much probability mass as possible and avoiding to enlarge the solution region unnecessarily.

- It is relatively more important to operate with a fine grid in the $A$-direction as $A$ is the uncertainty generating factor in the model.
Convergence and computation time

Basic set of parameter values: \( A_0 = P_0 = 100, r = 5\%, r_G = 4\%, T = 20 \text{ years}, \alpha = 0.3, \gamma = 0.1, \sigma = 15\%, K = 100. \)

<table>
<thead>
<tr>
<th>((I,J))</th>
<th>(100,100)</th>
<th>(200,200)</th>
<th>(400,400)</th>
<th>(800,800)</th>
<th>(1600,1600)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contract value</td>
<td>113.45</td>
<td>111.96</td>
<td>111.36</td>
<td>111.30</td>
<td>111.28</td>
</tr>
<tr>
<td>Relative error</td>
<td>1.95%</td>
<td>0.62%</td>
<td>0.08%</td>
<td>0.02%</td>
<td>0.00%</td>
</tr>
<tr>
<td>CPU time (sec.)</td>
<td>11</td>
<td>48</td>
<td>202</td>
<td>828</td>
<td>3858</td>
</tr>
</tbody>
</table>

The number of time steps per year \( K \) has been fixed at 100. The numerical tests show that gain in accuracy from further increasing \( K \) is negligible.

Operation counts (CPU time)

Doubling \( I \) and \( J \) increases CPU time roughly by four-folds.
\[ r_G < r \]

- Value of bond element (decreases with increasing maturity since \( r_G < r \))
- Value of the European contract
- Value of the American contract

\[
\text{Value of sum of option elements} = \text{value of contract} - \text{value of bond element. Value of each option element increases as the time to maturity increases.}
\]
The guaranteed interest rate is raised from being smaller than the riskless interest rate \( r \) to being larger than \( r \) \([r_G = 0.04, r = 0.03]\).

- The surrender option has virtually no value since the guaranteed interest rate is higher than the riskless interest rate. There is very little incentive to terminate the contract prematurely.
- The bonus option has dropped in value (lower value is attached to the equity participation) and the contract value mainly consists of the bond element.
Effect on contract value with varying values of riskless rate $r$.

- A higher interest rate $r$ implies higher values of the bonus and surrender options. The value of the bond component declines.
- There is a critical interest rate above which the American contracts should be immediately exercised (as evidenced by values of American contracts staying the same when $r$ is above some threshold value). At sufficiently high $r$, the funds are better reinvested in riskless bonds to realize the higher return.
More discussion on the pricing behavior

- Raising $r_G$ implies smaller values of the bonus and surrender options.

- An increase in $T$ yields a larger bonus option value.

- $\sigma = 0$ is not sufficient to make the values of the option elements equal zero. (i) When $r < r_G$ and $\sigma = 0$, the issuer will never be able to build reserves for bonus payments and the contracts are in effect above par riskless bonds. (ii) When $r_G < r$ and $\sigma = 0$, the company will surely be able to build bonus reserves and be forced to distribute part of this in the form of bonus.
5.2 Variance swaps

- Finite difference approach for valuing a discretely sampled variance swap within an extended Black-Scholes framework – incorporation of observed volatility skew using a local volatility function $\sigma(S,t)$.

- Decomposing into a set of one-dimensional PDE problems.

- Capable of handling various definitions of the variance.

**Product nature of variance swaps**

They are forward contracts on discrete realized variance.

- provide an easy way for investors to gain exposure to future level of variance; speculate on future variance levels; allow directional trading of volatility level

- trade the spread between the implied and realized variance levels

- hedge the variance exposures of other positions
Payoff at expiration = $L(\sigma^2_R - K)$,

$\sigma^2_R = \text{discrete realized stock variance (quoted in annualized terms) over the life of the contract}$

$K = \text{strike price for variance,}$

$L = \text{notational amount per annualized volatility point squared.}$

**Remark**

For any time $t \geq t_0$, $\sigma^2_R$ of an in-progress variance swap can be decomposed into a weighted sum consisting of known fixed amount (the realized variance up to that time) and an unknown amount (the variance over the remaining life of the swap).

Without loss of generality, we take $t = t_0 = 0$ and $\sigma^2_R$ is the unrealized variance over the remaining life of the swap.
Formulas for the calculation of the discrete realized variance

• The contractual specification should provide details on the source and observation frequency of the underlying asset price.

• An annualization factor is used in transforming into an annualized volatility. We use the notation:
  \( N = \) number of observations;
  \( F_A = \) annualization factor (taken to be 260 for daily sampling)

The discrete realized variance can be based on the daily return of asset price

\[
\sigma^2_R = \frac{F_A}{N} \sum_{i=1}^{N} \left( \frac{S_{i+1} - S_i}{S_i} \right)^2
\]

or based on logarithm of the daily return

\[
\sigma^2_R = \frac{F_A}{N} \sum_{i=1}^{N} \left( \ln \frac{S_{i+1}}{S_i} \right)^2.
\]
Remark

Corresponding to the statistical definition of an estimate of the discrete realized variance, the annualized realized discrete variance $\sigma^2_R$ is measured by a discrete sample of $N$ returns, $R_i$, $i = 1, 2, \ldots, N$ as follows

$$\sigma^2_{R, \text{stat}} = FA \frac{N}{N - 1} \left[ \frac{1}{N} \left( \sum_{i=1}^{N} R_i^2 \right) - \left( \frac{1}{N} \sum_{i=1}^{N} R_i \right)^2 \right],$$

where $R_i$ denotes either the actual return $\frac{S_{i+1} - S_i}{S_i}$ or the log return $\ln \frac{S_{i+1}}{S_i}$.

Since the mean daily returns are typically quite small, so the variance swap contract often defines the discrete realized variance as

$$\sigma^2_R = \frac{FA}{N} \sum_{i=1}^{N} R_i^2.$$

The factor $N/(N - 1)$ has been removed since it was used to account for the loss of one degree of freedom used to determine the mean return.
No-arbitrage pricing

Using the risk neutral valuation principle, the fair value $P$ of the variance swap at initiation is given by

$$P = e^{-rT}L \left\{ EQ[\sigma^2_R] - K \right\},$$

where $L$ is the notional and $EQ[\cdot]$ denotes the expectation under the $Q$ measure conditional on information available at time zero.

Most of the times, we are interested to find the fair strike price $K$ such that the value of the variance swap is zero at initiation, where

$$K = EQ[\sigma^2_R] = \frac{FA}{N} \sum_{i=1}^{N} EQ \left[ \left( \frac{S_{i+1} - S_i}{S_i} \right)^2 \right].$$
Approximate replication of discrete realized variance

Approximation One

We approximate the discretely sampled variance $\sigma_R^2$ by the continuous integral:

$$\sigma_1^2 = \frac{1}{T} \int_0^T \sigma^2(S_t, t) \, dt.$$ 

The dynamics is assumed to be governed by

$$\frac{dS_t}{S_t} = (r - q) \, dt + \sigma_t \, dW_t,$$

where $\sigma_t$ denotes the local volatility function with both state and time dependence. In terms of $\ln S_t$, we obtain

$$d \ln S_t = \left( r - q - \frac{\sigma_t^2}{2} \right) \, dt + \sigma_t \, dW_t$$

$$\ln S_T - \ln S_0 = \int_0^T \left( r - q - \frac{\sigma_t^2}{2} \right) \, dt + \int_0^T \sigma_t \, dW_t.$$
The expectation of the stochastic integral term \( E_Q \left[ \int_0^T \sigma_t \, dW_t \right] \) vanishes, so we obtain

\[
\frac{1}{2} E_Q \left[ \int_0^T \sigma_t^2 \, dt \right] = (r - q)T + \ln S_0 - E_Q[\ln S_T].
\]

It then follows that

\[
E_Q[\sigma_R^2] \approx E_Q[\sigma_1^2] = 2(r - q) + \frac{2}{T} \ln S_0 - \frac{2}{T} E_Q[\ln S_T].
\]

The variance contract is seen to be related to the log contract.

The stock price dynamics under \( Q \) is assumed to have state-dependent volatility \( \sigma_t = \sigma(S_t, t) \). Here, we assume no dynamic evolution of volatility.
Approximation Two

Recall the mathematical approximation formula

\[ \ln(1 + x) \approx x - \frac{x^2}{2} + O(x^3) \]

and let \( x = \frac{S_{i+1} - S_i}{S_i} \). We then have the following expansion:

\[
\ln S_{i+1} - \ln S_i = \frac{1}{S_i} (S_{i+1} - S_i) - \frac{1}{2} \left( \frac{S_{i+1} - S_i}{S_i} \right)^2 + O \left( \frac{S_{i+1} - S_i}{S_i} \right)^3 .
\]

We approximate \( \sigma_R^2 \) by dropping the third order term to obtain

\[
\sigma_R^2 = \frac{2FA}{N} \left[ \sum_{i=0}^{N-1} \left( \frac{S_{i+1} - S_i}{S_i} \right) + \ln S_0 - \ln S_N \right] .
\]

The difficulty of dealing with expectation calculation of the squared term \( \left( \frac{S_{i+1} - S_i}{S_i} \right)^2 \) is resolved by the above approximation. Interestingly, the log term appears again.
It is seen that

\[ E_Q \left( \frac{S_{i+1}}{S_i} \right) = e^{(r-q)\tilde{\Delta}}, \]

where \( \tilde{\Delta} \) is the time interval between dates \( i \) and \( i + 1 \).

Taking expectation with respect to \( Q \), we obtain

\[ E_Q[\sigma_R^2] \approx E_Q[\sigma_2^2] = \frac{2F_A}{N} \left\{ \sum_{i=0}^{N-1} \left[ e^{(r-q)\tilde{\Delta}} - 1 \right] + \ln S_0 - E_Q[\ln S_T] \right\}. \]

Note that \( -\ln \frac{S_T}{S_0} \) is the short position in a log contract with reference price \( S_0 \). Also, \( F_A = 1/\tilde{\Delta} \) and \( N\tilde{\Delta} = T \) so that \( \frac{F_A}{N} = \frac{1}{T} \). When \( \tilde{\Delta} \) is small, \( e^{(r-q)\tilde{\Delta}} - 1 \approx (r - q)\tilde{\Delta} \) so that

\[ \frac{2F_A}{N} \sum_{i=1}^{N} \left[ e^{(r-q)\tilde{\Delta}} - 1 \right] \approx 2(r - q). \]
Calculation of the expected discrete realized variance

Observe that

\[ E_Q[\sigma^2_R] = E_Q \left[ \frac{FA}{N} \sum_{i=1}^{N} \left( \frac{S_{i+1} - S_i}{S_i} \right)^2 \right] = \frac{FA}{N} \sum_{i=1}^{N} E_Q \left[ \left( \frac{S_{i+1} - S_i}{S_i} \right)^2 \right]. \]

- The procedure reduces to evaluating \( N \) expectations of squared returns of the form

\[ E_Q \left[ \left( \frac{S_T - S_{T - \tilde{\Delta}}}{S_T - \tilde{\Delta}} \right)^2 \right] \]

for some fixed time interval \( \tilde{\Delta} \) and \( N \) different values of tensor \( T, T = i\tilde{\Delta}, i = 1, 2, \cdots, N \).

- For \( T > \tilde{\Delta} \), \( S_{T - \tilde{\Delta}} \) is also an unknown at time zero.

- The pricing model thus becomes a two-state option model.
Mathematical structure of the pricing formulation

The Dirac function $\delta(x)$ is defined by

$$
\delta(x) = \begin{cases} 
0 & x \neq 0 \\
\infty & x = 0 
\end{cases}
$$

and

$$
\int_{-\infty}^{\infty} \delta(x) \, dx = 1.
$$

Another useful property is

$$
\int_{-\infty}^{\infty} \delta(x-x_0)f(x) \, dx = f(x_0).
$$

Define $I_t$ as a path dependent state variable containing information of the asset price $S_{T-\Delta}$ when time $t$ goes beyond $T-\Delta$, which is appropriately given by

$$
I_t = \int_{0}^{t} \delta(T-\Delta-\tau)S_{\tau} \, d\tau = \begin{cases} 
0 & \text{if } t < T-\Delta \\
S_{T-\Delta} & \text{if } t \geq T-\Delta
\end{cases}
$$
• We consider a contingent claim whose payoff at time $T$ depends on $S_T$ and the path dependent state variable $I_T = S_{T - \Delta}$.

Write the value of the contingent claim as $P = P(S, I, t)$. The governing equation is

$$\frac{\partial P}{\partial t} + \delta(T - \Delta - t)S\frac{\partial P}{\partial I} + \frac{1}{2}\sigma^2(S, t)S^2\frac{\partial^2 P}{\partial S^2} + (r - q)S\frac{\partial P}{\partial S} - rP = 0,$$

with terminal condition:

$$P(T, S, I) = \left(\frac{S}{I} - 1\right)^2.$$

The governing equation contains the extra term

$$\frac{\partial P}{\partial I} \frac{dI}{dt} = \delta(T - \Delta - t)S\frac{\partial P}{\partial I}$$

due to the additional path dependent state variable $I_t$. 
• The path dependent variable $I$ jumps in value across time $T - \Delta$ since

$$I_t = \begin{cases} 
0 & t < T - \Delta \\
S_{T - \Delta} & t \geq T - \Delta 
\end{cases}.$$

$I$ is constant away from time $T - \Delta$.

• The option value remains continuous across $T - \Delta$ even though $I$ has a jump. This is because there is no cash flow arising from the contract at $T - \Delta$

• At time away from $T - \Delta$, the governing pde reduces to the usual Black-Scholes equation. However, the terminal payoff still contains $I$.

On the other hand, from Feynman-Kac Theorem, we have

$$P(S_0, I_0, 0) = e^{-rT}E_Q \left[ \left( \frac{S_T - S_{T - \Delta}}{S_{T - \Delta}} \right)^2 \right].$$
Transformation of the governing system of equations

Set $z = \ln S, x = \ln I$ and $H(z, x, t) = P(S, I, t)$. The governing equation becomes

$$\frac{\partial H}{\partial t} + v(z, t) \frac{\partial^2 H}{\partial z^2} + b(z, t) \frac{\partial H}{\partial z} - r H = 0$$

with terminal condition:

$$H(z, x, T) = (e^{z-x} - 1)^2.$$

Here, we have

$$b(z, t) = r - q - \frac{v(z, t)}{2}$$

and

$$v(z, t) = \sigma^2(S, t) = \sigma^2(e^z, t).$$

The governing equation does not contain $x$, so $x$ is seen as a parameter in the terminal payoff.

- We specify a system of one-dimensional PDEs, indexed by $x$, that share information only at time $T - \Delta$. 
• At $t = 0$, $I_0 = 0$ so $x_0 = -\infty$. We would like to solve for $H(z_0, -\infty, 0)$.

• Write $f(z, t) = H(z, -\infty, t)$, then $f(z, t)$ is governed by

$$
\frac{\partial f(z, t)}{\partial t} + \frac{v(z, t) \partial f(z, t)}{2} + b(t, z) \frac{\partial f(z, t)}{\partial z} - rf(z, t) = 0, \quad 0 < t < T - \Delta,
$$

with terminal condition:

$$
f(z, T - \Delta) = H(z, -\infty, T - \Delta).
$$
Two-step finite difference solution approach

(i) \(0 \leq t < T - \tilde{\Delta},\)

(ii) \(T - \tilde{\Delta} \leq t \leq T.\)

The path dependent function records the realization of the stock price at \(T - \tilde{\Delta}.\)

The two-stage procedure is equivalent to applying the tower rule in iterated expectation:

\[
E_0^Q \left[ \left( \frac{S_T - S_{T-\tilde{\Delta}}}{S_{T-\tilde{\Delta}}} \right)^2 \right] = E_0^Q \left[ E_{T-\tilde{\Delta}}^Q \left( \frac{S_T - S_{T-\tilde{\Delta}}}{S_{T-\tilde{\Delta}}} \right)^2 \right].
\]
Discretization of the computational domain

Restrict to a region: \(0 \leq t \leq T - \Delta, z_{\text{min}} \leq z \leq z_{\text{max}}\) in the \((z,t)\)-plane.

- We take \(M + 1\) nodes along the \(t\)-axis; \(2\tilde{M} + 1\) nodes along the \(z\)-axis.

\[
z_i = z_{\text{min}} + i\Delta z, \quad i = 0, 1, \ldots, 2\tilde{M}
\]
\[
t_j = j\Delta t, \quad j = 0, 1, \ldots, M.
\]

where \(\Delta z = \frac{z_{\text{max}} - z_{\text{min}}}{2\tilde{M}}\) and \(\Delta t = \frac{T - \Delta}{M}\).
• In order to provide the terminal condition for the numerical computation in the time interval: $0 \leq t < T - \Delta$, it is necessary to find the numerical approximation of

$$H(z, -\infty, (T - \Delta)^-)$$

at $z = z_i, i = 0, 1, \cdots, 2M$.

• At $(T - \Delta)^+$, the prevailing stock price should be identical to the path dependent state variable, so $S_{(T-\Delta)^+} = I_{(T-\Delta)^+}$. This gives $z = x$.

• There is no jump on the value function, so

$$f(z, T - \Delta) = H(z, -\infty, (T - \Delta^-)) = H(z, z, (T - \Delta)^+)$$

It therefore suffices to obtain values for $H(z_i, z_i, (T - \Delta)^+), i = 0, \cdots, 2M$.

We solve $2M + 1$ Black-Scholes PDEs to obtain all $2M + 1$ values for $H(z, -\infty, (T - \Delta)^-)$ at the spatial grid points $z_i, i = 0, 1, \cdots, 2M$, then solve one Black-Scholes equation with these terminal conditions back to time zero.
• Define $H^i(z,t) = H(z,z_i,t)$, where the parameter $x$ is set to be $z_i = z_{\text{min}} + i\Delta z$. Solve for $H^i(z,t)$ over the time interval $(T - \Delta, T)$, where

$$\frac{\partial H^i(z,t)}{\partial t} + \frac{1}{2}v(z,t)\frac{\partial^2 H^i(z,t)}{\partial z^2} + b(z,t)\frac{\partial H^i(z,t)}{\partial z} - r H^i(z,t) = 0$$

with terminal condition:

$$H^i(z,T) = (e^{z-z_i} - 1)^2.$$

This is just an one-dimensional problem indexed by $x = z_i$. Since these indexed problems do not share any information after time $T - \Delta$, each $H^i(z,t)$ is solved individually. At $t = (T - \Delta)^+$, we extract the numerical solution of $H^i(z,t)$ at $z = z_i$. It then follows that

$$H(z_i,z_i,(T - \Delta)^+) = H^i(z_i,(T - \Delta)^+), \quad i = 0, 1, \cdots, 2\bar{M}.$$
Terminal condition for $H^i(z, t)$:

$$H^i(z, T) = (e^{z_i - z} - 1)^2, \quad i = 0, 1, \ldots, 2\tilde{M}.$$  

Terminal condition for $f(z, t)$:

$$f(z_i, T - \tilde{\Delta}) = H_i(z_i, T - \tilde{\Delta}), \quad i = 0, 1, \ldots, 2\tilde{M}.$$
Special case: $T = \Delta$. In this case, $S_{T-\Delta} = S_0$; we only need to find

$$\frac{1}{S_0^2} E_Q[(S_T - S_0)^2].$$

- We solve for $P(S, t)$ such that

$$\frac{\partial P}{\partial t} + \frac{\sigma^2(S, t)}{2} S^2 \frac{\partial^2 P}{\partial S^2} + (r - q) S \frac{\partial P}{\partial S} - r P = 0, \quad 0 < t < \Delta,$$

with terminal condition: $P(S_{\Delta}, \Delta) = \frac{1}{S_0^2} (S_{\Delta} - S_0)^2$.

**Total number of one-dimensional problems to be solved**

Given $N$ sampling dates and $2\tilde{M} + 1$ grids in the $z$-direction, we have to solve a total of

$$(N - 1) \times (2\tilde{M} + 2) + 1 = 2(N - 1)\tilde{M} + 2N - 1$$

one-dimensional problems.
Numerical experiments

We benchmark the method by computing expected values of the unrealized variance for different tenors and sampling frequencies. Also, we compare these numerical results to values obtained using Monte Carlo simulation as well as the two approximations: $\sigma_1^2$ and $\sigma_2^2$.

- Taking $\sigma(S,t)$ to be the constant value $\sigma$, then $\ln \frac{S_T}{S_{T-\Delta}}$ is normally distributed with mean $(r - q - \frac{\sigma^2}{2})\Delta$ and variance $\sigma^2\Delta$. We have

$$
E_Q \left[ \frac{S_T}{S_{T-\Delta}} \right] = e^{(r-q)\Delta} \quad \text{and} \quad E_Q \left[ \left( \frac{S_T}{S_{T-\Delta}} \right)^2 \right] = e^{2(r-q)\Delta + \sigma^2\Delta}.
$$
It then follows that
\[
E_Q \left[ \left( \frac{S_T}{S_{T-\Delta}} - 1 \right)^2 \right] = 1 + e^{2(r-q)\Delta + \sigma^2\Delta} - 2e^{(r-q)\Delta},
\]
so
\[
E_Q \left[ \sigma_R^2 \right] = FA \left[ 1 + e^{2(r-q)\Delta + \sigma^2\Delta} - 2e^{(r-q)\Delta} \right].
\]
We also note that since
\[
E_Q[\ln S_T] = \ln S_0 + \left( r - q - \frac{1}{2} \sigma^2 \right) T
\]
when the volatility is constant, we have
\[
E_Q[\sigma^2_{1}] = \sigma^2
\]
and
\[
E_Q \left[ \sigma^2_{2} \right] = 2\frac{FA}{N} \left\{ \sum_{i=0}^{N-1} \left[ e^{(r-q)\Delta} - 1 \right] - (r - q)T + \frac{1}{2} \sigma^2 T \right\}.
\]
Table 1 Expected value of unrealized variance when volatility in constant
table.

<table>
<thead>
<tr>
<th>$T$</th>
<th>Analytic results</th>
<th>Little-Pant</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exact</td>
<td>App 1</td>
</tr>
<tr>
<td>Daily sampling</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>625.412</td>
<td>625.000</td>
</tr>
<tr>
<td>1.00</td>
<td>625.412</td>
<td>625.000</td>
</tr>
<tr>
<td>2.00</td>
<td>625.412</td>
<td>625.000</td>
</tr>
<tr>
<td>Weekly sampling</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>627.061</td>
<td>625.000</td>
</tr>
<tr>
<td>1.00</td>
<td>627.061</td>
<td>625.000</td>
</tr>
<tr>
<td>2.00</td>
<td>627.061</td>
<td>625.000</td>
</tr>
</tbody>
</table>
Parameter values: $S_0 = 1100, r = 0.05, q = 0.00$ and $\sigma = 0.25$.

The values computed using our approach are obtained using a Crank-Nicholson scheme.

The columns labelled Grid 1, Grid 2, and Grid 3 refer to grids with (i) $\Delta t = 1/1040, \Delta z = 0.016$; (ii) $\Delta t = 1/2080, \Delta z = 0.008$; and (iii) $\Delta t = 1/4160, \Delta z = 0.004$, respectively.

- The numerical results obtained using both approximations are reasonably accurate for daily sampling but not very accurate when we consider weekly sampling.
Presence of volatility skew

The local volatility function is chosen to be

$$\sigma(S, t) = 0.281 + 0.002538(t + t^2) + (0.207 + 0.033t + 0.218t^2) \times \tanh\left(\frac{-27.42 - 4.71t}{1 + 28.27t} \ln \frac{S}{S_0} + \frac{0.025 + 0.29t}{1 + 1.85t}\right).$$

To perform the Monte Carlo simulation, they use the Milstein scheme in order to minimize the discretization error:

$$S_{T+\Delta t} = S_T + (r - q)S_T\Delta t + \sigma(T, S_T)S_T\epsilon\sqrt{\Delta t} + \frac{1}{2}\sigma(T, S_T)\left\{\sigma(T, S_T) + S_T\frac{\partial\sigma}{\partial S}(T, S_T)\right\} (\epsilon^2 - 1)\Delta t,$$

where $\epsilon$ is drawn from a standard normal distribution. Also they use antithetic variates for the purposes of variance reduction. The Monte Carlo values reported are based on simulating 1,000,000 paths with a time step of $\Delta t = 1/1300$. 
Table 2 Comparison of expected values of the unrealized variance with
daily sampling in the presence of a volatility skew

<table>
<thead>
<tr>
<th>Tenor</th>
<th>MC</th>
<th>App 1</th>
<th>App 2</th>
<th>Little-Pant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 0.5$</td>
<td>910.06</td>
<td>910.55</td>
<td>910.55</td>
<td>909.49</td>
</tr>
<tr>
<td></td>
<td>(1.29)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T = 1.0$</td>
<td>1004.53</td>
<td>1005.33</td>
<td>1005.43</td>
<td>1004.79</td>
</tr>
<tr>
<td></td>
<td>(1.42)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T = 2.0$</td>
<td>1470.73</td>
<td>1471.97</td>
<td>1472.06</td>
<td>1471.55</td>
</tr>
<tr>
<td></td>
<td>(2.08)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here, $S_0 = 1100$, $r = 0.05$, $q = 0.00$. They have taken $\Delta t = 1/2080$ and $\Delta z = 0.004$ in the Crank-Nicholson scheme.
Table 3 Comparison of expected values of the unrealized variance with weekly sampling in the presence of a volatility skew

<table>
<thead>
<tr>
<th>Tenor</th>
<th>MC</th>
<th>App 1</th>
<th>App 2</th>
<th>Little-Pant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 0.5$</td>
<td>904.95</td>
<td>910.55</td>
<td>910.94</td>
<td>905.16</td>
</tr>
<tr>
<td></td>
<td>(1.28)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T = 1.0$</td>
<td>1001.69</td>
<td>1005.33</td>
<td>1005.81</td>
<td>1001.96</td>
</tr>
<tr>
<td></td>
<td>(1.42)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T = 2.0$</td>
<td>1468.52</td>
<td>1471.97</td>
<td>1472.45</td>
<td>1468.32</td>
</tr>
<tr>
<td></td>
<td>(2.08)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here, $S_0 = 1100, r = 0.05, q = 0.00$. They have taken $\Delta t = 1/2080$ and $\Delta z = 0.004$ in the Crank-Nicholson scheme.

- Using a local volatility surface leads to larger differences between the present approach and the two approximations.
Table 4 Expected values of the unrealized variance with weekly sampling over one year computed using our approach and computational times for each value

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$\Delta z$</th>
<th>0.064</th>
<th>0.032</th>
<th>0.016</th>
<th>0.008</th>
<th>0.004</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/260</td>
<td>1007.07</td>
<td>1003.20</td>
<td>1002.20</td>
<td>1001.96</td>
<td>1001.90</td>
<td></td>
</tr>
<tr>
<td>2s</td>
<td>4s</td>
<td>13s</td>
<td>42s</td>
<td>157s</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/520</td>
<td>1007.10</td>
<td>1003.23</td>
<td>1002.24</td>
<td>1001.99</td>
<td>1001.93</td>
<td></td>
</tr>
<tr>
<td>3s</td>
<td>7s</td>
<td>23s</td>
<td>75s</td>
<td>281s</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/1040</td>
<td>1007.12</td>
<td>1003.25</td>
<td>1002.25</td>
<td>1002.01</td>
<td>1001.95</td>
<td></td>
</tr>
<tr>
<td>5s</td>
<td>15s</td>
<td>43s</td>
<td>143s</td>
<td>528s</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2080</td>
<td>1007.13</td>
<td>1003.26</td>
<td>1002.26</td>
<td>1002.02</td>
<td>1001.96</td>
<td></td>
</tr>
<tr>
<td>11s</td>
<td>28s</td>
<td>85s</td>
<td>276s</td>
<td>1023s</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/4160</td>
<td>1007.13</td>
<td>1003.26</td>
<td>1002.26</td>
<td>1002.02</td>
<td>1001.96</td>
<td></td>
</tr>
<tr>
<td>22s</td>
<td>56s</td>
<td>56s</td>
<td>544s</td>
<td>2013s</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- The scheme is much more sensitive to changes in $\Delta z$ than in $\Delta t$. It is advisable to work with larger time steps and refine the spatial mesh to improve accuracy.
Volatility swaps

Define $Z_i = \frac{1}{i} \sum_{j=1}^{n} R_{ij}^2$ and let $L$ be the notional.

Terminal payoff of volatility swap $= L(100\sqrt{\overline{F}_A Z_N} - K_{\text{vol}})$, where $K_{\text{vol}}$ is the strike.

Terminal payoff of a capped volatility swap $= L[\min(100\sqrt{\overline{F}_A Z_N}, \sigma_{R,\text{max}}) - K_{\text{vol}}]$, where $\sigma_{R,\text{max}}$ is the cap value on $\sigma_R$.

The cap feature is used to limit the liability of the short volatility position. In market practice, $\sigma_{R,\text{max}}$ is typically set to be 2.5 times $K_{\text{vol}}$. 
Challenges in pricing volatility swaps

Due to the occurrence of $\sqrt{F_A Z_N}$ in the terminal payoff of a volatility swap, we cannot decompose the expectation calculation of $\sqrt{F_A Z_N}$ into $N$ expectations, like that in variance swap.

Instead, it is necessary to augment the price function by two path-dependent state variables whose updating rules are specified by

$$I_{t_i}^+ = S_{t_i} \quad \text{and} \quad Z_{t_i}^+ = Z_{t_i}^- + \frac{R_i^2 - Z_{t_i}^-}{i},$$

where

$$R_i = \ln \frac{S_{t_i}}{S_{t_{i-1}}} = \ln \frac{S_{t_i}}{I_{t_i}^-}.$$ 

Note that $I$ keeps the information of the stock price at the previous monitoring time.
Outline of numerical solution procedure

The value function is a function of the underlying stock price $S$ and time $t$, parameterized by $I$ and $Z$, that is $V = V(S, t; I, Z)$.

- Between two monitoring time instants, the governing equation is the usual Black-Scholes type equation with local volatility function $\sigma_t = \sigma(S_t, t)$.

- Across a monitoring instant $t_i$, both state variables $I$ and $Z$ are updated according to the updating rules. However, there is no jump in the value function across $t_i$.

In other words, the pricing procedure can be mimicked from that of participating policies except that there are two path dependent state variables, namely, $I$ and $Z$. 
5.3 Convertible bonds

Embedded features

*Combination of bonds and equities – bond plus conversion option*

- Bondholder has the right to convert the bond into common shares of issuer’s company at some contractual price (conversion number may change over time).

- Issuer’s call and holder’s put
  - Hard call and Parisian soft call provision; notice period requirement
Holder’s perspective: take advantage of the future potential growth of issuer’s company

Issuer’s perspective: raise capital at a lower cost by the provision of conversion privilege to the bondholders

- High default risk when the stock price level is very low.
- Conversion premium = value above the bond floor.
The conversion option is similar to a call option on the underlying stock, where the call’s strike price equals the bond floor value.

**Equity perspective on convertibles**

- To take advantage of the upside potential growth of the underlying stock (participation into equity).
- Swapping the variable stock dividends in return for fixed coupon payments until the earlier of the maturity date and conversion date.

**Fixed income perspective on convertibles**

- Provides the “bond floor” value.
- Conversion option that allows the investor to exchange the straight bond for a fixed number of shares.
Call terms

Issuer has the right to call back the bond at a pre-specified call price prior to final maturity, usually with a notice period requirement. Upon call, the holder can either convert the bond or redeem at the call price.

Issuer’s perspective on the call right

- To have the flexibility to call if they think they can refinance the debt more cheaply at a lower interest rate.

- To force bondholders to convert debt into equity, which can reduce the company’s debt level and result in a beneficial effect on the balance sheet. The issuer has the flexibility to shift debt into equity to reduce the leverage of the firm. It is used as a tool by the issuer for possible future equity financing – managing the debt / equity balance.
Call protection

Hard (or absolute):

To protect the bond from being called for a certain period of time.

Soft (or provisional):

The issuer is allowed to call only when certain conditions are satisfied. Say the closing price of the stock has been in excess of 150% of the conversion price on any 20 trading days within 30 consecutive days.

Role of the call protection

To preserve the value of the equity option for the bondholders. The premium of the conversion right has been paid upfront at the time of purchase. While waiting for the stock price to increase, convertibles typically provide more income than the stock. Without the call protection, this income stream could be called away too soon. Hard call protection of a longer time period is more desirable for the investors.
Put feature

Allows the holder to sell back the bond to the issuer in return for a fixed sum. Usually, the put right lasts for a much shorter time period than the life of the bond.

- The holder is compensated for the lesser amount of coupons received in case the equity portion of the convertible has a low value.

- It helps immunize the holder against the risk of rising interest rates by effectively reducing the year to maturity. With a smaller value of duration*, the convertible price becomes less sensitive to interest rates.

* Duration $D$ is the weighted average of the times of cash flow stream, weighted according to the present value of the cash flow amount. The percentage change in bond price $P$ is proportional to negative yield change, where the proportional constant is the duration: $\frac{\Delta P}{P} \approx -D \times$ yield change.
Convertible bond issued by the Bank of East Asia

US$250,000,000

2.00 percent Convertible Bonds due 2003

<table>
<thead>
<tr>
<th>Issue date</th>
<th>July 19, 1996</th>
</tr>
</thead>
<tbody>
<tr>
<td>Issue price</td>
<td>100 percent of the principal amount of the Bonds, plus accrued interest, if any, from July 19, 1996 (in denominations of US$1,000 each)</td>
</tr>
<tr>
<td>Conversion period</td>
<td>From and including September 19, 1996 up to and including July 7, 2003</td>
</tr>
</tbody>
</table>
**Conversion feature**

| Conversion price | HK$31.40 per Share and with a fixed rate of exchange on conversion of HK$7.7405 = US$1.00. |
| Dilution protection clause | The Conversion Price will be subject to adjustment for, among other things, subdivision or consolidation of the Shares, bonus issues, right issues and other dilutive events. |
Put feature

Redemption at the option of the bondholders

On July 19, 2001, the Bonds may be redeemed at the option of the Bondholders in US dollars at the redemption price equal to 127.25 percent of the principal amount of the Bonds, together with accrued interest.

The investors are protected to have 27.25% returns on the bond investment upon early redemption by the holder.
Call feature

Redemption at the option of the issuer

On or after July 19, 1998, the Issuer may redeem the Bonds at any time in whole or in part at the principal amount of each Bond, together with accrued interest, if for each of 30 consecutive Trading Days, the last of which Trading Days is not less than five nor more than 30 days prior to the day upon which the notice of redemption is first published, the closing price of the Shares as quoted on the Hong Kong Stock Exchange shall have at least 130 percent of the Conversion Price in effect on such Trading Day.
Soft call protection – Parisian feature

The closing price has to be above 130 percent of the conversion price on consecutive 30 trading days.

- On the date of issuance of the notice of redemption (taken as day 0), the Issuer looks back 5 to 30 days (corresponds to $[-30, -5]$ time interval) to check whether the history of the stock price path satisfies the Parisian constraint. That is, the last of the 30 trading days (with closing price above 130% of the conversion price) falls in $[-30, -5]$ time interval.

- From Issuer’s perspective, when the Parisian constraint has been satisfied, the Issuer has 5 to 30 days to decide on redemption or not.
Modeling considerations in convertible bond pricing models


Choices of the underlying state variables

- Firm value versus stock price

  Earliest works use the value of the issuing firm as the underlying state variable. From corporate finance perspective, the firm value model can incorporate the balance sheet information on the firm’s liabilities.

  The firm’s debt and equity are claims contingent on the firm value, and options on its debt and equity become compound options on this variable.
Advantage: Dilution effect on equity upon conversion of the bond into shares can be modeled directly

Disadvantage: Since the value of the firm is not a traded asset, parameter estimation is difficult. Any other liabilities of the firm that are more senior than the convertible must be simultaneously valued.

• Most pricing models use the issuing firm’s stock price.
  – Since stock is a traded asset, so parameter estimation is easy. Also, the use of risk neutral valuation principle is more convincing. Hedging ratios can be computed easily.
  – There is no need to estimate the values of other more senior claims.
How about stochastic riskfree interest rate?

• Addition of the stochastic interest rate as an additional state variable increases the dimensionality of the pricing model.

• Practitioners often regard a convertible bond primarily as an equity instrument, where the main factor is the stock price. The random nature of the riskfree interest rate is of second order importance as the role of interest rate serves as the discount rate but not in the payoff structure.
Brennan and Schwartz (1980) conclude that “for a reasonable range of interest rates the errors from the non-stochastic interest rate model are likely to be slight.”

Why does interest rate fluctuation have lower impact on convertible bond value? When interest rate increases, the conversion option increases in value due to the drop in the bond floor value. Here, the bond floor can be visualized as the strike price of the conversion option (the underlying call option increases in value when the strike price decreases). The drop in value of the bond component is compensated by the increase in value of the conversion option.

- Quite often, the interest rate $r$ and stock price $S$ are negatively correlated. When the correlation coefficient $\rho_{Sr}$ is negative, an increase in $r$ leads to a drop in the bond component and a drop in $S$ (lowering of the equity component). Hence, high negative value of correlation may lead to higher sensitivity of the convertible bond value to interest rate fluctuation.
Other considerations in modeling

- Modeling of the default risk
  - arrival of the default event
    Structural approach versus reduced form approach (hazard rate)
  - loss upon default
    Under the stock price model, what would be the drop of the stock price upon default?

- Issuer’s call provision
  - soft call requirement, trigger prices
  - call notice period

- Dilution upon holders’ conversion – more shares are issued

- Holder’s put right
Formulation of the continuous time pricing model (zero default risk)

Under a risk neutral measure $Q$, the dynamics of $S_t$ is governed by

$$\frac{dS_t}{S_t} = [(r(t) - q(t))] dt + \sigma dZ_t.$$ 

Let $V$ be the price function of the convertible bond, and we define

$$\mathcal{L}V = -\frac{\partial V}{\partial t} - \left[\frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} - rV\right].$$ 

A convertible has the following contractual features:

- A continuous put provision with an exercise price $B_p$.
- A continuous conversion provision – conversion into $k$ shares.
- A continuous (time-dependent) call provision with call price $B_c$, where $B_c > B_p$ at all times. Holders can convert the bond if called.
Separate into two cases: \( B_c > kS \) or \( B_c \leq kS \)

- \( B_c \leq kS \) (the conversion option is sufficiently deep-in-the-money)

Recall that \( V \geq kS \) (with regard to the conversion right) while the issuer can cap the convertible bond value by \( B_c \) if the bond is not converted upon recall. Without solving any pricing models, the convertible value is easily seen to be simply \( V = kS \) since the holder would choose to convert immediately upon recall.

- \( B_c > kS \) [linear complementarity formulation]

\[
\begin{align*}
&\left( \begin{array}{c}
\mathcal{L}V = 0 \\
V - \max(B_p, \kappa S) > 0 \\
V - B_c < 0
\end{array} \right) \lor \\
&\left( \begin{array}{c}
\mathcal{L}V > 0 \\
V - \max(B_p, \kappa S) = 0 \\
V - B_c < 0
\end{array} \right) \lor \\
&\left( \begin{array}{c}
\mathcal{L}V < 0 \\
V - \max(B_p, \kappa S) > 0 \\
V - B_c = 0
\end{array} \right)
\end{align*}
\]

\((C_1) \lor (C_2) \lor (C_3)\) is interpreted as one and only one of the three conditions: \( C_1, C_2, C_3 \), holds at each point in the solution domain.
• When the convertible bond remains alive, the value of the convertible bond is given by $\mathcal{L}V = 0$ subject to the constraints:

$$V > \max(B_p, kS), \quad V < B_c.$$  

• When it is optimal for the holder to put or convert, the rate of returns from holding the $\Delta$-hedged portfolio is less than $r$ and we observe $\mathcal{L}V > 0 \Leftrightarrow d\Pi < r\Pi dt$. In this case,

$$V = \max(B_p, kS) \quad \text{where} \quad V < B_c. \quad \text{[Recall} \quad B_c > B_p]$$

• When it is optimal for the issuer to call, we have $V = B_c \quad [B_c > kS]$ and $V > \max(B_p, kS)$. In this case, if the convertible bond is not called, then the rate of returns from holding the $\Delta$-hedged portfolio is higher than $r$. That is, $d\Pi > r\Pi dt \Leftrightarrow \mathcal{L}V < 0$.

The price function is known to be $V = kS$ when $B_c \leq kS$, so the computational domain can be confined to $0 \leq S \leq \frac{B_c}{k}$.
When $\tau = 0.25$ and $\tau = 10$, the bond price curves intersect the conversion value line (shown as dotted-dashed line) and the cap value curve $c_n(S, \tau_n)$ (shown as dashed line), respectively. When $\tau = 1.5$, the price curve ends at the intersection point of the conversion value line and the cap value curve.
Auxiliary conditions

- Suppose we assume the coupon rate to be higher than the riskfree rate so that the put provision becomes non-effective. At $S = 0$, the convertible remains alive and $LV = 0$ becomes $-\left[\frac{\partial V}{\partial t} - r(t)V\right] = 0$ [equity component disappears completely]. So far we have neglected default risk in the modeling process. The convertible is essentially at its bond floor value, which lies below the call price and above the corresponding conversion value.

- As $S$ is sufficiently close to the upper bound, the bond value is almost linear in $S$ so that $\frac{\partial^2 V}{\partial S^2} \rightarrow 0$. This is because $V$ tends to $kS$ when $S \uparrow \infty$.

The terminal condition is given by

$$V(S, T) = \max(P, kS)$$

where $P$ is the par value of the bond.
Earlier attempt in modeling default risk

- Ad-hoc approach – use of the risky discount rate
  
  However, different components of the convertible bond are subject to different default risk
  
  - cash-only part (bond part); equity part (conversion option)

- Stock price instantly jumps to zero upon default
  
  - Shares in firms filing for bankruptcy in the US had abnormal returns of about $-65\%$ during the three years prior to a bankruptcy announcement, and had abnormal returns of about $-30\%$ around the announcement.
  
  - characterized by a gradual erosion of the stock price prior to the event, followed by a significant decline upon announcement.
Lattice tree algorithm (ad-hoc approach)

Incorporation of default risk, call and conversion features

- stock price process follows the binomial random walk
- interest rates are deterministic

Two discount rates

1. If the convertible is certain to remain a bond, it is appropriate to use a discount rate corresponding to the creditworthiness of the issuer, namely, the risky rate.

2. Suppose the bond is certain to be converted, it is then appropriate to use the riskfree rate since the fair value of the equity part is priced under the risk neutral valuation principle.
At maturity, the holder chooses the maximum between the par value and the value of stocks received upon conversion.

*How to account for the creditworthiness of the issuer?*

The discount rate to be used when we roll back is given by

\[ pw_u + (1 - p)w_d. \]

Here, \( p \) is the probability to an upward node where the discount rate is \( w_u \) and \( (1 - p) \) is probability to a downward node with \( w_d \). The appropriate discount rate is the weighted average of the discount rates at the nodes in the next time step.
Deficiencies of this ad-hoc approach

The credit spread of the bond component reflects the combination of default probability and loss upon default. The proper modeling of default must include

(i) probability of arrival of default

(ii) drop in the stock price upon default

(iii) recovery rate upon default.

- The tree methodology remains to be a feasible tool once we have found the “modified” drift rate and risky discount rate.
Interaction between conversion and calling

conv = value of stocks received if conversion takes place
call = call price
roll = value given by the rollback (neither converted nor recalled)

Six possible permutations on their relative values

(i) conv < call

\[
\begin{align*}
\text{conv} & < \text{call} < \text{roll} \\
\text{conv} & < \text{roll} < \text{call} \\
\text{roll} & < \text{conv} < \text{call}
\end{align*}
\]

(ii) call < conv

\[
\begin{align*}
\text{call} & < \text{conv} < \text{roll} \\
\text{call} & < \text{roll} < \text{conv} \\
\text{roll} & < \text{call} < \text{conv}
\end{align*}
\]
Premature conversion into shares is optimal only when the stock pays dividends at a yield higher than the coupon rate.

**Dynamic programming procedure:**

\[
\max(\min(\text{roll}, \text{call}), \text{conv}).
\]

At each node, the optimal strategy of the holder is exemplified by taking the maximum of \(\min(\text{roll}, \text{call})\) and \(\text{conv}\).

- The maximum reflects the conversion right, which persists with or without recall by the issuer.

- The bond value before potential conversion is seen to be \(\min(\text{roll}, \text{call})\) since the issuer would initiate calling when the roll value shoots beyond the call price.
Alternative dynamic programming procedure:

\[
\min(\max(roll, conv), \max(call, conv))
\]

- \(\max(roll, conv)\) represents the optimal strategy of the holder.

- Upon recall, the holder chooses to accept the call price or convert into shares. This can be represented by \(\max(call, conv)\).

The issuer chooses to recall or to abstain from recalling in order to minimize the option value.

These two procedures can be shown to be mathematically equivalent if we apply the distributive rule of sequencing the order on the “max” and “min” operations.
What would happen when \( \text{call} < \text{conv} \)?

- This occurs when the stock price level is sufficiently high, that is, the conversion option is sufficiently deep-in-the-money. Since the convertible bond value is always equal or above \( \text{conv} \), so the issuer initiates calling immediately.

- Upon calling, the holder chooses to convert into stocks since \( \text{conv} > \text{call} \).

- This represents a straightforward case since convertible value = \( \text{conv} \) for sure, and there is no need to perform any numerical calculations to find the convertible bond value.
Provided that we rule out the scenario where $\text{call} < \text{conv}$ in our calculations, the third alternative for the third dynamic programmic procedure is given by

$$\min(\text{call}, \max(\text{conv}, \text{roll})).$$

Under the scenario $\text{conv} < \text{call}$, the holder has the optionality to convert into shares but the convertible bond value is always capped by $\text{call}$.

The 3 choices of dynamic programming procedures give the same set of outcomes

<table>
<thead>
<tr>
<th>outcome</th>
<th>call</th>
<th>roll</th>
<th>conv</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{conv} &lt; \text{call} &lt; \text{roll}$</td>
<td>$\text{call}$</td>
<td>$\text{roll}$</td>
<td>$\text{conv}$</td>
</tr>
</tbody>
</table>
Summary

If call price < conversion value, then the convertible bond value is simply given by the conversion value (straightforward case). We are interested to find the price function within the range of stock price such that

\[
\text{call price} > \text{conversion value.}
\]

That is, we are interested to compute the price function whose value is capped by “call price”. Hence, the dynamic programming procedure

\[
\min(\text{call}, \max(\text{conv}, \text{roll}))
\]

makes good sense.

- The convertible bond value lies between the “call price” (upper obstacle function) and conversion value (lower obstacle function).
- The call price has “stock price” dependence if we include the consideration of the notice period requirement.
Delayed call phenomenon

Convertible bonds are recalled by issuers only when the stock price is sufficiently well above the “theoretical” critical recall price. In this case, the holders are almost sure to make the “so called” forced conversion into shares. This is consistent with one of the corporate finance considerations – delayed equity financing.

- How to incorporate this behavior into the pricing model?
- What other factors that affect the determination of the optimal “recall price”? 
Example

A 9-month discount bond issued by XYZ company with a face value of $100. Assume that it can be exchanged for 2 shares of company’s stock at any time during the 9 months.

- It is callable for $115 at any time.

- Initial stock price = $50, \( \sigma = 30\% \) per annum and no dividend; risk-free yield curve to be flat at 10\% per annum.

- Yield curve corresponding to bonds issued by the company to be flat at 15\%.

- Tree parameters are: \( u = 1.1618, d = 0.8607, p = 0.5467 \),

\[
R = e^{0.1\Delta t} = 1.0253.
\]

- At maturity, the convertible is worth \( \max(100, 2S_T) \).
Binomial tree for pricing a risky convertible bond

upper figure: stock price
middle figure: discount rate
lower figure: value of convertible
At node $D$

Roll back gives the convertible bond value

$$(0.5467 \times 156.84 + 0.4533 \times 116.18)e^{-0.1\times0.25} = 134.98.$$  

The bondholder is indifferent to conversion or hold. Upon call, the holder will choose to convert, so the issuer is also indifferent as to whether the bond is called. The correct discount rate at node D is 10% since the convertible is considered 100% equity at this node.

At node $F$

The correct discount rate is 15% since the convertible is certain not to be converted if node $F$ is reached.
At node $E$

The discount rate is given by the weighted average:

$$0.5467 \times 10\% + 0.4533 \times 15\% = 12.27\%.$$ 

The roll back value of convertible at $E$

$$(0.5467 \times 116.18 + 0.4533 \times 100)e^{-0.1227 \times 0.25} = 105.56.$$

Since roll back value is higher than the conversion value (which is $2 \times 50 = 100$) and lower than the call price of 115, so the bond should be neither converted nor called. The discount rate is then kept at 12.27\%.
At node $B$

The discount rate is given by the weighted average of those at nodes $D$ and $E$:

$$0.5467 \times 10\% + 0.4533 \times 12.27\% = 11.03\%$$

and roll back value of convertible is

$$(0.5467 \times 134.99 + 0.4533 \times 105.56)e^{-0.1103 \times 0.25} = 118.34.$$

It is optimal to call the bond at node $B$ so that it causes immediate conversion and lead to $116.18$.

Note that continuation is assumed when the roll back value is calculated, we use the weighted averaged discount rate in the preliminary calculation. However, the discount rate at node $B$ should be taken to be 10% subsequently since optimal conversion into shares would take place at this node.
At node $A$

The discount rate is

$$0.5467 \times 10\% + 0.4533 \times 13.51\% = 11.59\%.$$  

The convertible value at node $A$ is given by the roll back value

$$(0.5467 \times 116.18 + 0.4533 \times 98.00)e^{-0.1159 \times 0.25} = 104.85,$$

since neither call or conversion would occur.

*Value of the embedded option features*

If the bond has no conversion option, its value is

$$e^{-0.75 \times 0.15} = 89.36.$$  

The value of the conversion option minus the issuer’s call right $=$ $104.85 - 89.36 = 15.49$. 
Pricing model of a convertible bond with credit risk

- We adopt the one-factor contingent claims model with stock price as the underlying state variable.

- We assume constant interest rate and model the arrival of default by a Poisson arrival process with constant hazard rate.

- The stock price $S_t$ under the risk neutral valuation framework is assumed to follow the Geometric Brownian process

$$
\frac{dS_t}{S_t} = (r - q)dt + \sigma_S dZ_t,
$$

where $r$ is the riskless interest rate, $q$ and $\sigma_S$ are the constant dividend yield and volatility of the stock price, respectively, and $Z_t$ is the standard Wiener process.
• Conditional on no prior default up to time $t$, the probability of default within the time period $(t, t + dt)$ is $h \, dt$, where $h$ is the constant hazard rate of arrival of default.

• Assume that upon default the bondholder receives the fraction $R$ (recovery rate) of the bond value and the stock price drops to zero instantaneously, the corresponding governing equation for the convertible bond price function $V(S, t)$ is given by

$$
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q + h) S \frac{\partial V}{\partial S} - [r + (1 - R)h] V + c(t) = 0,
$$

$$
0 < S < S^*(\tau), \quad 0 < t < T.
$$
Intuitive observations on the governing equation

1. With probability \( h dt \) over \((t, t+dt)\), the stock price drops to zero due to default. Under expectation calculations, this can be visualized to a negative dividend yield \( h \) of the stock so that the drift rate is modified to \( r - q + h \).

2. The convertible bond value is discounted at the risky rate \( r + (1 - R)h \). Here, \((1 - R)h\) is commonly recognized as the credit spread. Investors demand the rate of return to be higher than the riskfree interest rate by a certain spread due to potential loss upon default. Both \( r \) and \((1 - R)h\) represent the rate of decrease in the bond value over time.
3. The bond price function satisfies the governing equation only in the continuation region \( \{(S, t) : 0 < S < S^*(t), 0 < t < T\} \), where the bond remains alive. Here, \( S^*(t) \) denotes the critical stock price at which the bond ceases to exist either due to either early conversion or calling, and \( T \) is the bond maturity date.

4. The source term \( c(t) \) arises from the coupon payment stream. The external cash payout rate may be represented by \( c(t) = \sum_{i=1}^{N} c_i \delta(t-t_i) \), where \( c_i \) is the coupon payment paid on the discrete coupon payment dates \( t_i, i = 1, 2, \ldots, N \). The Dirac function \( c_i \delta(t-t_i) \) indicates the discrete nature of a coupon payment. The coupon rate becomes infinite at \( t_i \) while the coupon amount collected across \( t_i \) is \( c_i \).
Derivation of the governing equation

As usual, consider the portfolio \( \Pi = -V + \Delta S \), where \( \Delta \) units of the underlying stock are held to hedge against the short position of one unit of the convertible bond. Using Ito’s lemma and considering the portfolio value under no-default and default, the expectation of \( d\Pi \) is given by

\[
d\Pi = (1 - h dt) \left[ -\left( \partial V \partial t + \sigma^2 S^2 \partial^2 V \right) dt - \frac{\partial V}{\partial S} dS \right.
\]

\[
+ \Delta dS + \Delta qS dt - c dt \left. \right] - h dt \left[ -(1 - R)V + \Delta S \right].
\]

As revealed by the negative sign in front of \( h dt \), the last term \( -(1 - R)V + \Delta S \) represents the loss in the portfolio value. Now, we take \( \Delta = \frac{\partial V}{\partial S} \) so that the stochastic terms involving \( dS \) vanish.

Assuming that the default risk is firm-specific, based on the Capital Asset Pricing Model, we then have

\[
E[d\Pi] = r \Pi dt.
\]
This is because an investor will not be compensated with excess expected rate of return above the riskfree interest rate when the underlying risks in the portfolio are firm-specific. Collecting the terms, we obtain

\[- \left( \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) + qS \frac{\partial V}{\partial S} - hS \frac{\partial V}{\partial S} + h(1 - R)V - c(t) \]

\[= r \left( -V + S \frac{\partial V}{\partial S} \right) \]

so that

\[\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q + h)S \frac{\partial V}{\partial S} - [r + (1 - R)h]V + c(t) = 0.\]

For the source term arising from the discrete coupons, we observe

\[\int_0^u c(t) \, dt = \sum_{i=1}^N c_i H(u - t_i), \quad \text{where } H(u - t_i) = \begin{cases} 1 & u \geq t_i, \\ 0 & u < t_i. \end{cases}\]

so that

\[c(t) = \sum_{i=1}^N c_i \delta(t - t_i).\]
Auxiliary conditions

(i) Terminal payoff on the maturity date $T$

The terminal value of $V$ is given by

$$V(S,T) = (c_N + P)1_{\{P \geq nS\}} + nS1_{\{P < nS\}},$$

where $1_A$ is the indicator function for the event $A$. Here, $P$ denotes the par value of the bond, $c_N$ is the last coupon payment and $n$ is the number of units of stock to be exchanged for the bond upon conversion. The last coupon is not paid if the bondholder chooses to convert at maturity.
(ii) Conversion policy

Since the bondholders have the right to convert the bond into $n$ units of stock at any time, the convertible bond always stays equal or above the conversion value. Upon voluntary conversion, the value of the bond equals the conversion value identically. We then have

$$V(S,t) > nS \quad \text{when the convertible bond remains alive},$$
$$V(S,\bar{t}) = nS \quad \text{when the convertible bond is converted},$$

where $\bar{t}$ is the optimal time of conversion chosen by the bondholders.

- The screw clause stipulates that the accrued interest will not be paid upon voluntary conversion. This clause may inhibit bondholders to convert optimally when a coupon date is approaching.
(iii) Calling policy

- The convertible bond indenture usually contains the hard call provision where the bond cannot be called for redemption or conversion by the bond issuer in the early life of the bond. This serves as a protection for the bondholders so that the privilege of awaiting growth of the equity component will not be called away too soon.

- Let $[T_c, T]$ denote the callable period, that is, the bond cannot be called during the earlier part of the bond life $[0, T_c)$.

- Upon calling, the bondholders can decide whether to redeem the bond for cash or convert into shares at the end of the notice period of $t_n$ days.
Notice period requirement

- Let $\hat{t}$ denote the date of call so that $\hat{t} + t_n$ is the conversion decision date for the bondholders.

- The bondholders essentially replace the original bond at time $\hat{t}$ by a new derivative that expires at the future time $\hat{t} + t_n$ and with terminal payoff $\max(nS, K + \hat{c})$, where $\hat{c}$ is the accrued interest from the last coupon date to the time instant $\hat{t} + t_n$, and $K$ is the pre-specified call price of the convertible bond.
• We write $V_{new}(S, t; K, t_n)$ as the value of this new derivative. When there is no soft call requirement (a constraint that is related to stock price movement over a short period prior to calling), the convertible bond value should be capped by $V_{new}$. The convertible bond should be called once its value reaches $V_{new}(S, t; K, t_n)$.

\[
V(S, t) \leq V_{new}(S, t; K, t_n) \quad \text{within the callable period},
\]
\[
V(S, \hat{t}) = V_{new}(S, t; K, t_n) \quad \text{at the calling moment}.
\]

• When there is a soft call requirement, it is possible that $V(S, t)$ stays above $V_{new}(S, t; K, t_n)$. 
(iv) Coupon payments

By no arbitrage argument, there is a drop in bond value of amount that equals the coupon payment \( c_i \) across a coupon payment date \( t_i, i = 1, 2, \cdots , N \). We have

\[
V(S, t_i^+) = V(S, t_i^-) - c_i, \quad i = 1, 2, \cdots , N.
\]

Remark

The interaction of the optimal conversion and calling policies determines the early termination of the convertible bond. The synergy of these two features can be treated effectively via dynamic programming procedure in the numerical schemes.
Finite difference algorithms

• We adopt the log-transformed variable $x = \ln S$, and define time to expiry $\tau = T - t$. Let $V^m_j$ denote the numerical approximation of $V(x, \tau)$ at the grid point $x = j\Delta x$ and $\tau = m\Delta t$, where $\Delta x$ and $\Delta t$ are the respective stepwidth and time step.

• The explicit finite difference scheme takes the following basic form

$$V^m_{j+1} = p_u V^m_{j+1} + p_m V^m_j + p_d V^m_{j-1} - \left[ r + (1 - R)h \right] V^m_j + c_i \mathbf{1}_{\{E_i\}}.$$  

The probabilities of upward jump, zero jump and downward jump of the logarithm of the stock price, $x = \ln S$ are given by

$$p_u = \frac{1}{2\lambda^2} + \frac{\left( r - q + h - \frac{\sigma^2_S}{2} \right) \sqrt{\Delta t}}{2\lambda \sigma_S},$$  

$$p_m = 1 - \frac{1}{\lambda^2}, \quad p_d = \frac{1}{2\lambda^2} - \frac{\left( r - q + h - \frac{\sigma^2_S}{2} \right) \sqrt{\Delta t}}{2\lambda \sigma_S},$$

respectively, and $\Delta x = \lambda \sigma_S \sqrt{\Delta t}$ for some parameter $\lambda$. 

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• Here, $E_i$ denotes the event that the coupon payment $c_i$ is paid at $t_i$. When the payment date $t_i$ is bracketed between two successive time levels $m\Delta t$ and $(m+1)\Delta t$, the bond values $V_{j}^{m+1}$ are increased by an extra amount $c_i$ due to the discrete coupon payment.

• The “initial” values $V_{j}^{0}$ at time level $n = 0$, which correspond to the terminal payoff values of the bond, are given by

$$V_{j}^{0} = c_N + \begin{cases} 
  P & \text{if } x_j \leq \ln \frac{P}{n} \\
  ne^{x_j} & \text{otherwise}
\end{cases}$$
Interaction of the callable and convertible features

- Apply the full dynamic programming procedure

\[
\min(\max(V_{cont}, V_{conv}), \max(V_{call}, V_{conv}))
\]

at those nodes where conversion and calling are allowed.

When the calling right is non-operative (say, during the period under the hard call constraint) where the conversion right exists only, the full dynamic programming procedure reduces to the partial dynamic programming procedure

\[
V_j^n = \max(V_{cont}, V_{conv}).
\]
Soft call requirement

- To incorporate the soft call requirement, we model the associated Parisian feature using the forward shooting grid approach, where an extra dimension is added to capture the excursion of the stock price beyond some predetermined trigger level $B$. With the inclusion of the path dependence of the stock price associated with the soft call requirement, the finite difference scheme is modified as follows:

$$
V_{j,k}^{m+1} = p_u V_{j+1,g(k,j+1)}^m + p_m V_{j,g(k,j)}^m + p_d V_{j-1,g(k,j-1)}^m \\
- [r + (1 - R)h] V_{j,g(k,j)}^m + c_i 1\{E_i\}.
$$
Cumulative Parisian feature to activate calling

- The grid evolution function assumes the form

\[ g_{\text{cum}}(k, j) = k + 1_{\{x_j > \ln B\}} \]

for cumulative counting of number of days that the stock price has been staying above the level \( B \).

- Suppose the condition of \( M \) cumulative days of breaching is required in order to activate the calling right, the full dynamic programming procedure is applied only when the condition \( g_{\text{cum}} \geq M \) has been satisfied.
Anatomy of the embedded features

- Using the one-factor defaultable convertible bond pricing model, we explore the dependence of the convertible bond value on the coupon payment streams, conversion ratio and soft call constraint.

- We examine the interaction of the callable and conversion features and show how the notice period requirement affects the critical stock price at which the convertible bond is terminated prematurely either by optimal calling or voluntary conversion.
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>par value</td>
<td>100</td>
</tr>
<tr>
<td>annualized volatility</td>
<td>20%</td>
</tr>
<tr>
<td>dividend yield per year</td>
<td>1%</td>
</tr>
<tr>
<td>maturity date</td>
<td>5 years</td>
</tr>
<tr>
<td>coupon rate</td>
<td>2% per annum, paid semi-annually</td>
</tr>
<tr>
<td>conversion ratio</td>
<td>1</td>
</tr>
<tr>
<td>call period</td>
<td>starting 1.0 years from now till maturity</td>
</tr>
<tr>
<td>conversion period</td>
<td>throughout the life</td>
</tr>
<tr>
<td>call price</td>
<td>140</td>
</tr>
<tr>
<td>riskless interest rate</td>
<td>flat at 5% per annum</td>
</tr>
<tr>
<td>hazard rate</td>
<td>0.02</td>
</tr>
<tr>
<td>recovery rate</td>
<td>0.8</td>
</tr>
</tbody>
</table>
Plot of convertible bond value against time at different fixed values of stock price (dotted curve corresponds to $S = 70$, solid curve corresponds to $S = 100$ and dashed curve corresponds to $S = 120$).
At low stock price level \((S = 70)\)

- The convertible bond behaves like a simple coupon bond, and its value increases with time since the riskless interest rate is higher than the coupon rate (see the lower dotted curve).

- At maturity, the bond value matches the total value of par plus the last coupon.

At intermediate stock price level \((S = 100)\)

- At the stock price level \(S = 100\) (same as conversion price), the convertible bond drops in value within the last coupon period (see the middle solid curve).

- The drop in value is attributed primarily to the higher rate of decrease in the value of the conversion option at times close to maturity.
At high stock price level ($S = 120$)

- At a higher stock price level $S = 120$ (20% above the conversion price), the bond value shows a trend of slight decrease with increasing time (see the upper dashed curve). The conversion option decreases in value with increasing time at a rate faster than the rate of increase in value of the bond component.

- The bond value stays almost at constant value within the last coupon period. This is because the value of a deep-in-the-money convertible bond is dominated by its equity component since the bond is almost sure to be converted into shares at maturity, so the time dependent effect of accrued interest of the bond component is negligible.
The entries in the table are convertible bond values corresponding to different conversion numbers and stock price levels.

<table>
<thead>
<tr>
<th>stock price</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
<th>1.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>85.30</td>
<td>85.67</td>
<td>86.29</td>
<td>87.19</td>
<td>88.41</td>
<td>89.97</td>
<td>91.87</td>
</tr>
<tr>
<td>100</td>
<td>94.10</td>
<td>99.47</td>
<td>105.90</td>
<td>113.18</td>
<td>121.12</td>
<td>129.56</td>
<td>138.37</td>
</tr>
<tr>
<td>120</td>
<td>101.93</td>
<td>110.18</td>
<td>119.49</td>
<td>129.56</td>
<td>140.16</td>
<td>151.14</td>
<td>162.37</td>
</tr>
<tr>
<td>130</td>
<td>106.59</td>
<td>116.29</td>
<td>126.98</td>
<td>138.37</td>
<td>150.21</td>
<td>162.37</td>
<td>174.73</td>
</tr>
<tr>
<td>140</td>
<td>111.67</td>
<td>122.77</td>
<td>134.81</td>
<td>147.45</td>
<td>160.48</td>
<td>173.77</td>
<td>187.23</td>
</tr>
<tr>
<td>150</td>
<td>117.08</td>
<td>129.56</td>
<td>142.88</td>
<td>156.73</td>
<td>170.91</td>
<td>185.30</td>
<td>199.81</td>
</tr>
</tbody>
</table>
Price sensitivity with respect to conversion number and stock price level

- At a low stock price level, the bond value is not quite sensitive to an increase in conversion number.

- The bond value is insensitive to an increase in stock price when the conversion number is low.

- Both phenomena are due to the low value of the equity component of the convertible bond. The data also reveal that the delta of the bond value increases with higher conversion number, due to an increased weight in the equity component.
Parisian soft call provision

• The current stock price is taken to be 130 and the annualized dividend yield to be 1%. We specify that the issuer can initiate the call only if the stock price stays above the trigger price consecutively or cumulatively for 30 days.

• For the purpose of comparison, the convertible bond value is found to be equal to 144.17 if there is no call feature and equal to 135.71 if there is no soft protection requirement. These two values serve as the respective upper and lower bound for the value of the bond subject to the soft call requirement.
The entries in the right two columns are values of convertible bond subject to varying levels of trigger price and under the rules of consecutive counting and cumulative counting of the number of days of breaching the trigger price.

<table>
<thead>
<tr>
<th>trigger price</th>
<th>consecutive counting</th>
<th>cumulative counting</th>
</tr>
</thead>
<tbody>
<tr>
<td>130</td>
<td>136.01</td>
<td>135.83</td>
</tr>
<tr>
<td>140</td>
<td>136.64</td>
<td>136.08</td>
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<tr>
<td>150</td>
<td>137.89</td>
<td>137.13</td>
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<tr>
<td>160</td>
<td>138.93</td>
<td>138.32</td>
</tr>
<tr>
<td>180</td>
<td>140.65</td>
<td>140.30</td>
</tr>
<tr>
<td>200</td>
<td>141.81</td>
<td>141.60</td>
</tr>
</tbody>
</table>
Price sensitivity with respect to the Parisian soft call provision

1. The bond value increases with an increasing trigger price. This is obvious since it becomes harder for the issuer to initiate the call when calling is constrained by a higher trigger price.

2. The bond value becomes higher when the soft call requirement is more stringent. This is because bondholders have better protection against calling by issuer. Also, this explains why the convertible bond has higher value under the rule of consecutive counting compared to cumulative counting.
Two-year convertible, coupons paid semi-annually

Assuming that the issuer cannot call, the curves show the plot of the critical conversion price $S_{conv}^*$ against time. Within the last coupon payment period, $S_{conv}^*$ decreases with time. At times right before a coupon date, $S_{conv}^*$ tends to infinite value.
During the hard call protection period \((0, 1)\), the bond is terminated prematurely by early conversion only. The critical conversion price \(S^*_\text{conv}\) decreases over time, and \(S^*_\text{conv}(1^-) = 122\). Over time period \((1, 2]\), \(S^*_\text{call}(t)\) increases slowly over time and exhibits a drop across a coupon date. At times close to maturity, the bond is terminated due to early conversion.
Delayed call phenomenon

• In the earlier theoretical works on optimal calling policies, Ingersoll (1977) and Brennan and Schwartz (1977) claimed that the bond issuer should call the bond whenever the convertible bond value reaches the call price.

• The notice period requirement may have profound impact on the critical call price $S_{call}^*$ since the bondholder receives upon calling a more valuable short-lived option (whose maturity date coincides with the ending of the notice period), rather than the cash amount that equals the sum of call price plus accrued interest.
We examine the impact of the notice period requirement on the theoretical critical call price, $S^\text{call}_t$. The time-averaged values of the ratio $S^\text{call}_t/X$ are obtained under varying length of the notice period and different set of parameter values.

<table>
<thead>
<tr>
<th>average value of $S^\text{call}_t/X(t)$</th>
<th>notice periods (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>15</td>
</tr>
<tr>
<td>volatility</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>1.049</td>
</tr>
<tr>
<td>30%</td>
<td>1.061</td>
</tr>
<tr>
<td>40%</td>
<td>1.057</td>
</tr>
<tr>
<td>interest rate</td>
<td></td>
</tr>
<tr>
<td>2%</td>
<td>1.043</td>
</tr>
<tr>
<td>5%</td>
<td>1.061</td>
</tr>
<tr>
<td>8%</td>
<td>1.077</td>
</tr>
<tr>
<td>coupon rate</td>
<td></td>
</tr>
<tr>
<td>1%</td>
<td>1.106</td>
</tr>
<tr>
<td>3%</td>
<td>1.073</td>
</tr>
<tr>
<td>5%</td>
<td>1.045</td>
</tr>
<tr>
<td>price rate</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>1.061</td>
</tr>
<tr>
<td>150</td>
<td>1.090</td>
</tr>
<tr>
<td>180</td>
<td>1.108</td>
</tr>
<tr>
<td>hazard rate</td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>1.065</td>
</tr>
<tr>
<td>0.03</td>
<td>1.051</td>
</tr>
<tr>
<td>0.05</td>
<td>1.046</td>
</tr>
<tr>
<td>recovery rate</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>1.078</td>
</tr>
<tr>
<td>0.5</td>
<td>1.068</td>
</tr>
<tr>
<td>0.8</td>
<td>1.061</td>
</tr>
</tbody>
</table>
• The sample calculations reveal that the so called “delayed call phenomena” may be largely attributed to the under estimation of the critical call price at which the issuer should call the bond optimally.

• A large portion of the “amount of call delay” may be eliminated when more careful contingent claims pricing calculations are performed.

• There may be other rationales from corporate finance perspectives (say, taxes, asymmetric information) which explain why issuers choose to delay their calls.

• In future empirical studies on assessing the amount of call delay due to corporate finance considerations, the more accurate theoretical critical stock price should be computed.
Reverse Convertibles

A contingent convertible (CoCo) is a debt instrument that automatically converts into equity or suffers a write down when the issuer company gets into a state of a possible nonviability.

- It is used as a readily available source of bank capital in times of crisis. The activation of the loss absorption mechanism must be a function of the capitalization levels of the issuer. The design for trigger has to be robust to price manipulation and speculative attacks.
Structure of CoCos

Graph 1

Math design features of CoCos

- Trigger
  - Mechanical
    - Book value
    - Market value
  - Discretionary
- Loss absorption mechanism
  - Conversion to equity
  - Principal write-off
**Trigger mechanism**

Triggers can be based on a mechanical rule or supervisors’ discretion

1. *Accounting trigger*
   The Lloyds and Credit Suisse CoCos have been constructed with the Core Tier 1 ratio (CT1) as an indicator of the health of the bank. The accounting trigger level may be 5% or 7%.

   - Accounting triggers may not be activated in a timely fashion, depending crucially on the frequency at which the ratio is calculated as well as the rigor and consistency of internal risk models.

2. *Regulatory trigger (nonviability trigger)*
   It is a discretionary choice in the hands of the bank’s national regulator. This type of trigger may reduce the marketability of a CoCo bond.
3. Market trigger
This could address the shortcoming of inconsistent accounting val-
uations, and reduce the scope for balance sheet manipulation and
regulatory forbearance. Share prices or CDS spreads (forward look-
ing parameter) could be used. For example, when the share price
breaches a well-defined barrier level, this will trigger the conversion
into shares.

- Creation of incentives for stock price manipulation
- Death Spiral: CoCo holders may short-sell the underlying com-
  mon stock and drive the share price down.
Loss absorption mechanism

1. Conversion-to-equity increases CET1 by converting into equity at a pre-defined conversion price $C_p$.

(a) $C_p = S^*$, where $S^*$ is the share price observed at the trigger moment $T^*$.

(b) $C_p = S_0$, where $S_0$ is the share price of the bank on the issue date.

(c) $C_p = \max(S^*, S_F)$, where $S_F$ is a certain floor value.

2. Principal writedown
   For example, the holders may lose 75% of the face value and receive the remaining 25% in cash. One criticism: issuer would have to fund a cash payout while in distress.
Trigger of an Accounting Trigger Is Associated with a Market Trigger on the Share Price
## Contingent Convertibles Examples

<table>
<thead>
<tr>
<th>Issuer</th>
<th>Conversion in Shares</th>
<th>Write Down</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Lloyds</td>
<td>Credit Suisse</td>
</tr>
<tr>
<td>Issue size</td>
<td>GBP 7 bn (32 series)</td>
<td>USD 2 bn</td>
</tr>
<tr>
<td>Rating (Fitch)</td>
<td>BB</td>
<td>BBB</td>
</tr>
<tr>
<td>Issue date</td>
<td>December 1, 2009</td>
<td>February 17, 2011</td>
</tr>
<tr>
<td>Maturity</td>
<td>10–20 year</td>
<td>30 year—callable 5.5 years</td>
</tr>
<tr>
<td>Coupon</td>
<td>1.5–2.5% increase of the coupon of the hybrid bond exchanged for the ECN</td>
<td>7.875%</td>
</tr>
<tr>
<td>Write down</td>
<td>75%</td>
<td>100%</td>
</tr>
<tr>
<td>Conversion Price</td>
<td>59 Pence</td>
<td>max (USD20, CHF20, $)</td>
</tr>
<tr>
<td>Accounting Trigger</td>
<td>Core Tier 1 Ratio</td>
<td>Core Tier 1 Ratio</td>
</tr>
<tr>
<td>Accounting Trigger Level</td>
<td>5%</td>
<td>7%</td>
</tr>
<tr>
<td>Regulatory Trigger</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>
Market size

Banks have issued approximately $70 billion worth of CoCos since 2009 to mid-2013. By comparison, during the same period, around $550 billion worth of non-CoCo subordinated debt and roughly $4.1 trillion worth of senior unsecured debt have been issued. Retail investors and private banks in Asia and Europe are attracted by the relatively high nominal yield that CoCos offer in the current low interest rate environment.

- As banks felt more pressure from markets and regulators to boost their Tier 1 capital, they started to issue CoCo with trigger levels at or above the preset minimum for satisfying the going-concern contingent capital requirement.
More than half of all CoCos are currently unrated. The existence of discretionary triggers creates valuation uncertainty.

**Pricing model**

Let $H_t$ denote the logarithm of the accounting ratio and assume it to follow an Ornstein-Uhlenbeck process, where

$$dH_t = k(m - H_t)dt + \eta \ dW_t^2.$$  

The stock price follows a geometric Brownian motion, where

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma(\sqrt{1 - \rho^2}dW_t^1 + \rho dW_t^2).$$  

The challenge is to find the conditional distribution of the stock price when $H_t$ hits a low-barrier $B$.  

References

