MAFS5250 – Computational Methods for Pricing Structured Products

Topic 6 – Advanced numerical schemes for pricing path dependent options

6.1 Discretely sampled fixed strike Asian option

- Change of numeraire approach
- Choice of Markovian state variable
- Construction of the Crank-Nicolson scheme

6.2 Discretely sampled lookback options

- Fixed strike lookback call option
- Floating strike lookback option
6.1 Discretely sampled fixed strike Asian options

Change of numeraire approach

Reference:

- We make use of change of numeraire techniques to obtain the option price as a function of time and a one-dimensional Markovian state variable. The state variable exhibits jumps at the observation points with probability one.

- Between two observation points, the state variable evolves continuously, so the price function is governed by a PDE (consequence of the Feynman-Kac representation Theorem).
• A coupled sequence of PDEs are resulted, the first PDE generates the terminal condition of the second one.

• The Asian option model is converted into a barrier option pricing problem. Why do we have the “barrier type” behavior? This is because when the state variable goes beyond a certain level, typically in the status of being currently in-the-money, the optionality in the payoff function disappears. It is then possible to derive the risk adjusted expectation of the terminal payoff in closed form.

• The Crank-Nicolson scheme is adopted for the numerical solution of the resulting PDE. This is to avoid the odd-even problems in pricing barrier options under tree calculations.

• Highly accurate: within 1% accuracy even with 50 time steps between successive observation time points.
**Model formulation: change of numeraire and Markov representation**

Under a risk neutral measure $Q$, the stock price dynamics is given by

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t.$$ 

Based on the risk neutral valuation principle where the money market account is used as the numeraire, the time-$t$ price of the contingent claim $H$ at time $T$ is given by

$$F_t = E^Q_t[e^{-r(T-t)}H].$$

We apply the change of measure with the stock price as the numeraire, which is defined by

$$\frac{dQ'}{dQ}_{\mathcal{F}_0} = \frac{S_te^{qt}}{S_0} / e^{rt} = \frac{S_t}{S_0e^{(r-q)t}}, \quad t > 0.$$ 

- One unit of asset at $t = 0$ will grow to $e^{qt}$ units at time $t$, if the dividends are all used to buy additional units of asset.
The solution to $S_t$ is known to be

$$S_t = S_0 e^{(r-q-\frac{\sigma^2}{2})t+\sigma W_t},$$

so we obtain the share measure $Q'$ defined by the Radon-Nikodym derivative

$$\frac{dQ'}{dQ} \bigg|_{\mathcal{F}_0} = e^{-\frac{\sigma^2}{2}t+\sigma W_t}.$$

Recall the Girsanov theorem: Suppose $W_t^P$ is $P$-Brownian, and

$$\frac{dP'}{dP} = e^{\left(-\mu W_t^P - \frac{\mu^2}{2}t\right)},$$

then $W_t^{P'} = W_t^P + \mu t$ is $P'$-Brownian. By choosing $\mu$ to be $-\sigma$ in the above formula, we observe that

$$W_t' = W_t - \sigma t$$

is $Q'$-Brownian when $W_t$ is $Q$-Brownian.
Accordingly, the dynamics of $S_t$ under $Q'$ is governed by

$$\frac{dS_t}{S_t} = (r - q + \sigma^2) t + \sigma dW'_t, \quad t > 0.$$  

By applying the change of measure from $Q$ to $Q'$ conditional on $\mathcal{F}_t$, we obtain

$$F_t = E^Q_t \left[ e^{-r(T-t)} H \right] = E^Q_t \left[ \frac{e^{-r(T-t)} H}{S_t e^{q t}} \right] = S_t E^Q_t \left[ \frac{e^{-q(T-t)} H}{S_T} \right].$$

**Remark**

Here, we use the numeraire change formula:

$$\left. \frac{dQ^M}{dQ^N} \right|_{\mathcal{F}_t} = \frac{M(T)}{M(t)} \bigg/ \frac{N(T)}{N(t)},$$

where $M(t)$ and $N(t)$ are numeraires. When the money market account is used as the numeraire, the discount factor is $e^{-r(T-t)}$. With the stock price as the numeraire, the “pseudo” discount factor is $e^{-q(T-t)}$. 
Dimension reduction of a path dependent option model

When the contingent payoff $H$ depends on the path history of $S_t$ up to $T$, then it is necessary to keep track of the path realization of $S_t$ when we compute $F_t$ using the risk neutral valuation principle.

It may be possible to reduce the dimension of the path dependent option model if we can find a Markov process $x_t$ such that under $Q'$

$$dx_t = \mu_x(x_t, t)\,dt + \sigma_x(x_t, t)\,dW'_t,$$

and together with the satisfaction of the property:

$$\frac{H}{S_T} = \xi(x_T),$$

for some function $\xi$. 
• If the above properties hold true, then the deflated option price

\[ f = \frac{F}{S} \]

will be a function of \((x_t, t)\) only. By the Feynman-Kac representation theorem, \(f\) satisfies the following pde

\[
\frac{\partial f}{\partial t} + \mu x \frac{\partial f}{\partial x} + \frac{\sigma^2 x}{2} \frac{\partial^2 f}{\partial x^2} - qf = 0,
\]

subject to the terminal condition: \(f(x, T) = \xi(x)\).

• The solution of the one-dimensional Black-Scholes type pde can be solved by finite difference calculations.
Andreasen manages to find such a Markov representation for discretely sampled Asian and lookback options with either fixed strike or floating strike. The payoff structures of many structured products and variable annuities resemble the Asian style payoff. The same approach can be adopted provided that the corresponding Markov representation can be found.

The numerical pricing of discretely sampled path dependent option is considered more difficult compared to the continuously sampled counterparts. We have jump of the path dependent state variable across a fixing date in discretely sampled options.
Analogy with the continuously monitored fixed strike Asian call option

Define $I_t = \int_0^t S_u \, du$ so that the arithmetic average is given by

$$A_T = \frac{1}{T} \int_0^T S_u \, du = \frac{I_T}{T}.$$  

The time-0 price of the continuously monitored fixed strike Asian call with strike price $K$ is given by

$$c_{fix}(S_0; T, K) = e^{-rT} \mathbb{E}_Q \left[ \max \left( \frac{1}{T} \int_0^T S_u \, du - K, 0 \right) \right].$$

By defining $x_T = \frac{1}{S_T} \left( \frac{I_T}{T} - K \right)$ and $\frac{dQ'}{dQ} = \frac{S_T}{S_0} e^{-(r-q)T}$, we have

$$\frac{c_{fix}(S_0; T, K)}{S_0} = e^{-rT} \mathbb{E}_{Q'} \left[ \frac{S_T}{S_0} \max (x_T, 0) \right]$$

$$= e^{-qT} \mathbb{E}_{Q'} \left[ \max (x_T, 0) \right].$$

The deflated call price $c_{fix}/S_0$ is given by the expectation of the terminal payoff $(x_T)^+$ under $Q'$ discounted by $e^{-qT}$. 
Asian call option with fixed strike (discretely sampled)

Let the fixing dates be \( t_1, t_2, \ldots, t_n \), and let the initial time be \( t_0 = 0 \) and terminal date be \( t_{n+1} = T \). We allow the flexibility that \( t_1 = t_0 \) and \( t_n = T \). Hence, we have

\[
0 = t_0 \leq t_1 < \cdots < t_n \leq t_{n+1} = T.
\]

Denote the index of the last fixing date associated with the current time \( t \) by

\[
m(t) = \sup \{1 \leq i \leq n : t_i \leq t\}.
\]
Choice of the Markovian state variable

Define the running sum of discretely sampled stock prices up to time $t$ by

$$I(t) = \sum_{1 \leq i \leq n, t_i \leq t} S(t_i) = \sum_{i=1}^{m(t)} S(t_i).$$

The time-$t$ price of the Asian call option with fixed strike $K$ is

$$F_t = S_tE_t^{Q'} \left[ e^{-q(T-t)} \left( \frac{I(T)}{n} - K \right)^+ \right].$$

Define the stochastic process

$$x_t = \frac{1}{n} I(t) - K,$$

which jumps by a deterministic amount $\frac{1}{n}$ when the calendar time moves across an observation point $t_i$. 
• In fact, for $1 \leq i \leq n$, we have

$$x(t_i) = \frac{\frac{1}{n} \sum_{j=1}^{i} S(t_j) - K}{S(t_i)} $$

$$= \frac{\frac{1}{n} \sum_{j=1}^{i-1} S(t_j) - K}{S(t_i)} + \frac{1}{n} = x(t_i^-) + \frac{1}{n}. $$

• At times between successive observation dates, the process $x_t$ evolves continuously since $I(t)$ is held fixed and only $S_t$ changes continuously as time evolves. The dynamics of $x_t$ under $Q'$ with jump is

$$dx(t) = -(r - q)x(t)dt - \sigma x(t)dW'(t) + \frac{1}{n} dm(t).$$

To show the above dynamics equation of $x_t$, recall that under $Q'$

$$\frac{dS_t}{S_t} = (r - q + \sigma^2) dt + \sigma dW'_t$$

so

$$\frac{d\left(\frac{1}{S_t}\right)}{\frac{1}{S_t}} = \left[-(r - q + \sigma^2) + \sigma^2\right] dt - \sigma dW'_t$$

$$= -(r - q) dt - \sigma dW'_t.$$
Under $Q'$, the evolution of $x_t$ depends only on $x_t$ itself and known jumps at $t_i$, $i = 1, \ldots, n$; so $x_t$ is Markovian. Therefore, $F_t/S_t$ depends on $x_t$ and $t$ only, so we write

$$
\frac{F_t}{S_t} = f(x_t, t) = E^Q_t \left[ e^{-q(T-t)x_T^+} \bigg| x_t \right], \quad x_T^+ = \max(x_T, 0).
$$

Furthermore, once $x_t \geq 0$, then $x_u \geq 0$ for all $u \geq t$ since $I(t)$ is non-decreasing in $t$.

For $x_t \geq 0$, we have

$$
f(x_t, t) = E^Q_t \left[ e^{-q(T-t)x_T} \bigg| x_t \right].
$$

One can obtain the following closed form formula:

$$
e^{-q(T-t)}E^Q_t [x_T | x_t] = e^{-r(T-t)}x_t + \frac{1}{n \sum_{i : t < t_i \leq t_n}} e^{-r(T-t_i) - q(t_i - t)}.
$$

For convenience, we write the above known solution as $g(x_t, t)$. 
Proof

We multiply the SDE for $x(t)$ throughout by the integrating factor $e^{(r-q)t}$:

$$d[e^{(r-q)t}x(t)] = -e^{(r-q)t} \sigma x(t) \, dW'(t) + \frac{1}{n} e^{(r-q)t} \, dm(t).$$

Integrating both sides with respect to $u$ from $t$ to $T$, we obtain

$$e^{(r-q)T} x_T - e^{(r-q)t} x_t = -\int_t^T e^{(r-q)u} \sigma x(u) \, dW'(u) + \frac{1}{n} \int_t^T e^{(r-q)u} \, dm(u).$$

Since $m(t)$ has unit jump across the fixing dates $t_i$, $i = 1, 2, \ldots, n$, we may write

$$dm(u) = \sum_{i: t < t_i \leq t_n} \delta(u - t_i), \quad t \leq u \leq T,$$

so that

$$\int_t^T e^{(r-q)u} \, dm(u) = \int_t^T e^{(r-q)u} \sum_{i: t < t_i \leq t_n} \delta(u - t_i) \, du = \sum_{i: t < t_i \leq t_n} e^{(r-q)t_i}$$
Next, we take the expectation under $Q'$ conditional on $\mathcal{F}_t$. Note that the expectation of the stochastic integral equals zero due to the differential Brownian (whose expectation is zero). We then have

$$E^Q_{t} [x_T] = e^{-(r-q)(T-t)} x_t + \frac{1}{n} e^{-(r-q)T} \sum_{i:t < t_i \leq t_n} e^{(r-q)t_i}.$$ 

Note that $e^{(r-q)(t_i-t)}$ may be treated as the discount factor over $(t, t_i)$ with discount rate $-(r - q)$. Note the negative sign in the discount rate, corresponding to the negative drift rate of $x_t$ under $Q'$. The growth factor from $t$ to $T$ is given by $e^{-(r-q)(T-t)}$. The time-$T$ future value of $\frac{1}{n}$ added to $x_t$ at fixing date $t_i$ is given by

$$\frac{1}{n} e^{-(r-q)(T-t)} e^{(r-q)(t_i-t)} = \frac{1}{n} e^{-(r-q)T} e^{(r-q)t_i}.$$
Lastly, we obtain

\[ g(x_t, t) = e^{-q(T-t)} E^Q_t [x_T] \]
\[ = e^{-r(T-t)} x_t + \frac{1}{n} \sum_{i : t < t_i \leq t_n} e^{-r(T-t_i)-q(t_i-t)}. \]

When the current level of \( x_t \) is below \( x = 0 \), the process \( x_t \) may pass through the level \( x = 0 \) at some future sample point \( t_i \), where \( m(t) < i \leq n \).

- If crossing does occur at time \( t_i, i \leq n \), then
  \[ f(x_{t_i}, t_i) = g(x_{t_i}, t_i) = g \left( x_{t_i} + \frac{1}{n}, t_i \right). \]

- On the other hand, if crossing never occurs, then the terminal payoff \( \max(x_T, 0) \) is zero.
Analogy as an up-and-in barrier option

(first passage time problem for a Markov process)

The Markov process $x_t$ has a finite known jump at $\frac{1}{n}$ at each sample point. Here, $\tau$ is the first sample time that $x_\tau \geq 0$. 
The crossing of $x_t$ across $x = 0$ can only occur at one of the sample points. If crossing has not occurred prior to a sample point, the sign of $\frac{1}{n}I(t) - K$ remains negative at times between 2 observation points. Hence, $x_t$ remains to be negative at least until reaching the next sample point.

To get rid of the discontinuous dynamics of $x(t)$, we define

$$y(t) = x(t) - \frac{m(t)}{n}.$$  

We then have

$$dy(t) = -(r - q) \left[ y(t) + \frac{m(t)}{n} \right] dt - \sigma \left[ y(t) + \frac{m(t)}{n} \right] dW_t'.$$
Since $e^{-qt}f(t)$ is a $Q'$-martingale, we deduce that the governing equation for $f$ is given by

$$qf = \frac{\partial f}{\partial t} - (r - q) \left[ y(t) + \frac{m(t)}{n} \right] \frac{\partial f}{\partial y} + \frac{\sigma^2}{2} \left[ y(t) + \frac{m(t)}{n} \right]^2 \frac{\partial^2 f}{\partial y^2}.$$ 

What is the domain of definition in the $(y, t)$-plane for $t_{i-1} < t < t_i$?

Note that $\{ (x, t) : x < 0, 0 < t < T \}$ is the domain of definition in the $(x, t)$-plane since the Asian option can be visualized as an up-and-out barrier option with an upper barrier at $x = 0$.

At $t_0 < t < t_1$, $y(t) = x(t)$; when $t_{n-1} < t < t_n$, $y(t) = x(t) - \frac{n-1}{n}$. In general, when $t_{i-1} < t < t_i$, $y(t) = x(t) - \frac{i-1}{n}$, $i = 1, 2, ..., n$. 
We perform the numerical calculations backward in the calendar time, starting from the last period: \( t_n < t \leq t_{n+1} = T \).

In the last time period: \( t_n < t \leq t_{n+1} \) (between the last monitoring instant and maturity date), we have \( y = x - 1 \). Recall that

\[
f(x_t, t) = g(x_t, t) \quad \text{for} \quad x_t \geq 0 \Leftrightarrow y_t \geq -1.
\]

Suppose \( x_t < 0 \) in the last time period, then the option is sure to expire out-of-the-money. We then have

\[
f(y, t) = 0 \quad \text{for} \quad y < -1 \text{ and } t \geq t_n.
\]

In general, for \( t_{i-1} < t < t_i \), the terminal condition is given by

\[
f(y, t_i^-) = \begin{cases} 
  f(y, t_i^+) & \text{if } y < -\frac{i}{n} \\
  g(y + \frac{i}{n}, t_i) & \text{if } -\frac{i}{n} \leq y < -\frac{i-1}{n}.
\end{cases}
\]

We observe continuity of \( y \) across \( t_i \) while \( x \) has a jump of \( \frac{1}{n} \) across \( t_i \).
For $t_{i-1} < t < t_i$, $x = y + \frac{i-1}{n}$; and $x < 0 \iff y < -\frac{i-1}{n}$.
• Note that the state space of $y$ within which the numerical solution of $f$ is sought changes as time progresses.

• The state space is kept constant for all $t$ between 2 observation points. Running backward in time, the newly added region is subject to the terminal condition as specified by the known function $g(\cdot, \cdot)$.

• Between 2 observation points, say within the time interval $[t_{i-1}, t_i]$, the pde is solved numerically using the Crank-Nicolson scheme.

When $y$ lies within $\left[-\frac{i}{n}, -\frac{i-1}{n}\right]$ at $t_i^-$, the Asian option becomes in-the-money when $t$ advances across $t_i$ (due to the jump of $\frac{1}{n}$ in $x_t$ at $t = t_i$). By virtue of continuity of option value, we have $f(y, t_i^-) = g\left(y + \frac{i}{n}, t_i\right)$. Otherwise, when $y < -\frac{i}{n}$, we have continuity of $f(y, t)$ across $t = t_i$. 

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Construction of the Crank-Nicolson scheme

Rewrite the governing partial differential equation between two successive observation dates as

\[
\left[-q + \frac{\partial}{\partial t} + \mu(y) \frac{\partial}{\partial y} + v(y) \frac{\partial^2}{\partial y^2}\right] f = 0,
\]

where

\[
\mu(y) = -(r - q) \left[ y + \frac{m(t)}{n} \right] \quad \text{and} \quad v(y) = \frac{\sigma^2}{2} \left[ y + \frac{m(t)}{n} \right]^2.
\]

We suppress the dependence of \( \mu \) and \( v \) on \( t \) since \( m(t) \) is constant in each interval \([t_{i-1}, t_i]\).
Approximate the differential operators in the partial differential equation by centered differences at \((y, t + \Delta t/2)\). We obtain the Crank-Nicolson scheme:

\[
\begin{align*}
-\frac{q}{2} + \frac{1}{\Delta t} + \frac{\mu(y)}{2} \delta_y + \frac{v(y)}{2} \delta_{yy} \left[ f(y, t + \Delta t) - f(y, t) \right]
\end{align*}
\]

where the difference operators are

\[
\delta_y h(y) = \frac{h(y + \Delta y) - h(y - \Delta y)}{2\Delta y},
\]

\[
\delta_{yy} h(y) = \frac{h(y + \Delta y) - 2h(y) + h(y - \Delta y)}{\Delta y^2}.
\]

The artificial boundary conditions: \(\delta_{yy} f = 0\) is applied at the lower boundary \(y_{\text{min}}\) and \(\frac{\partial f}{\partial t} = qf\) at the upper boundary [known solution \(g(x, t)\) at \(y = y_{\text{max}}\) is subject to the discount rate \(q\) when \(x_t\) is held fixed and time dependence is only exhibited only in those terms that involve \(e^{-q(t_i-t)}\)].
For the interval \([t_{i-1}, t_i]\), we limit our state space to the discrete grid:
\[
\{(y_l, s_k)\}_{k=0,1,...,K; l=0,1,...,L}
\]
with
\[
\begin{align*}
    s_k &= t_{i-1} \frac{K - k}{K} + t_i \frac{k}{K} \\
    y_l &= y_{\text{min}} \frac{L - l}{L} + y_{\text{max}} \frac{l}{L}.
\end{align*}
\]
Here, \(\Delta t = \frac{t_i - t_{i-1}}{K}\) and \(\Delta y = \frac{y_{\text{max}} - y_{\text{min}}}{L}\). We have
\[
y_{\text{max}} = -\frac{i - 1}{n}
\]
and we set \(y_{\text{min}} = -2\) (for maturity less than one year).

It is convenient to choose the same stepwidth \(\Delta y\) at all time intervals. Since the state space of \(y\) increases in size in the backward induction procedure, so \(L\) increases accordingly with an increase in \(y_{\text{max}}\).

- If we set \(y_{\text{min}} = -2\) for all time intervals, we observe the advantage of having a smaller computational domain in later time intervals (saving of computational efforts).
For the interior nodes, \( l = 1, 2, \ldots, L \), we have

\[
\left(\frac{q}{2} + \frac{1}{\Delta t}\right) f(s_k, y_l) - \frac{\mu(y_l)}{2} \frac{f(s_k, y_{l+1}) - f(s_k, y_{l-1})}{2\Delta y} \\
- \frac{v(y_l)}{2} \frac{f(s_k, y_{l+1}) - 2f(s_k, y_l) + f(s_k, y_{l-1})}{\Delta y^2} \\
= \left(\frac{-q}{2} + \frac{1}{\Delta t}\right) f(s_{k+1}, y_l) + \frac{\mu(y_l)}{2} \frac{f(s_{k+1}, y_{l+1}) - f(s_{k+1}, y_{l-1})}{2\Delta y} \\
+ \frac{v(y_l)}{2} \frac{f(s_{k+1}, y_{l+1}) - 2f(s_{k+1}, y_l) + f(s_{k+1}, y_{l-1})}{\Delta y^2}.
\]

Applying \( \delta_{yy} = 0 \) at the \( (k, L)^{th} \) node, we obtain

\[
f(s_k, y_{L+1}) = 2f(s_k, y_L) - f(s_k, y_{L-1}),
\]

where \( f(s_k, y_{L+1}) \) is a fictitious point outside the computational domain.
We are marching backward in time, starting from \( k = K - 1 \) down to \( k = 0 \). The values at \( k = K \) are inferred from the last time interval.

The matrix system of equations of the Crank-Nicolson scheme:

\[
\begin{pmatrix}
- A_0 & - \\
- A_1 & - \\
  & \\
  & \\
- A_{L-1} & - \\
- A_L & -
\end{pmatrix}
\begin{pmatrix}
f(s_k, y_0) \\
f(s_k, y_1) \\
  \\
  \\
f(s_k, y_{L-1}) \\
f(s_k, y_L)
\end{pmatrix}
= 
\begin{pmatrix}
- B_0 & - \\
- B_1 & - \\
  & \\
  & \\
- B_{L-1} & - \\
- B_L & -
\end{pmatrix}
\begin{pmatrix}
f(s_{k+1}, y_0) \\
f(s_{k+1}, y_1) \\
  \\
  \\
f(s_{k+1}, y_{L-1}) \\
f(s_{k+1}, y_L)
\end{pmatrix}.
\]

Here, both coefficient matrices are essentially tridiagonal. We would like to find the explicit representation of the rows \( A_\ell \) and \( B_\ell \); and special care is taken at \( \ell = 0 \) and \( \ell = L \) so as to incorporate the numerical boundary conditions.
At $\ell = L$, the numerical scheme becomes
\[
\left( \frac{q}{2} + \frac{1}{\Delta t} \right) f(s_k, y_L) - \frac{\mu(y_L)}{2\Delta y} \left[ f(s_k, y_L) - f(s_k, y_{L-1}) \right] = \left( -\frac{q}{2} + \frac{1}{\Delta t} \right) f(s_{k+1}, y_L) + \frac{\mu(y_L)}{2\Delta y} [f(s_{k+1}, y_L) - f(s_{k+1}, y_{L-1})].
\]

The last two components in the row vector $B_L$ are
\[
B_{L,L-1} = -\frac{\mu(y_L)}{2\Delta y}, \quad B_{L,L} = -\frac{q}{2} + \frac{1}{\Delta t} + \frac{\mu(y_L)}{2\Delta y};
\]
and the last two components in the row vector $A_L$ are
\[
A_{L,L-1} = \frac{\mu(y_L)}{2\Delta y}, \quad A_{L,L} = \frac{q}{2} + \frac{1}{\Delta t} - \frac{\mu(y_L)}{2\Delta y}.
\]
At \( y = y_{\text{max}} \) (node \( \ell = 0 \)), the differential equation becomes \( \frac{\partial f}{\partial t} = q f \) so that the discretized scheme at \( y_0 \) is

\[
\frac{f(s_{k+1}, y_0) - f(s_k, y_0)}{\Delta t} = q \frac{f(s_k, y_0) + f(s_{k+1}, y_0)}{2}
\]

giving

\[
\left( \frac{q}{2} + \Delta t \right) f(s_k, y_0) = \left( -\frac{q}{2} + \Delta t \right) f(s_{k+1}, y_0).
\]

Hence, the two row vectors \( A_0 \) and \( B_0 \) each contains only one component, namely,

\[
A_{0,0} = \frac{q}{2} + \frac{1}{\Delta t} \quad \text{and} \quad B_{0,0} = -\frac{q}{2} + \frac{1}{\Delta t}.
\]

Since we have set \( \delta_{yy} = \delta_y = 0 \) at \( y_0 \), it is not surprising that \( \mu(y_0) \) and \( v(y_0) \) do not appear in \( A_0 \) and \( B_0 \).
We solve a sequence of matrix equations from $k = K - 1, K - 2, \ldots, 0$:

$$A f(s_k) = B f(s_{k+1}),$$

where $f$ is the vector

$$f(s_k) = (f(s_k, y_0), \cdots, f(s_k, y_L))^T$$

and $A$ and $B$ are $(L + 1)$-dimensional tridiagonal matrices with the $l^{th}$ rows given by

$$A_l = \left(0, \cdots, 0, \frac{\mu(y_l)}{4\Delta y} - \frac{v(y_l)}{2(\Delta y)^2}, \frac{q}{2} + \frac{1}{\Delta t} + \frac{v(y_l)}{(\Delta y)^2}, -\frac{\mu(y_l)}{4\Delta y} - \frac{v(y_l)}{2(\Delta y)^2}, 0, \cdots, 0\right)$$

$$B_l = \left(0, \cdots, 0, -\frac{\mu(y_l)}{4\Delta y} + \frac{v(y_l)}{2(\Delta y)^2}, -\frac{q}{2} + \frac{1}{\Delta t} - \frac{v(y_l)}{(\Delta y)^2}, \frac{\mu(y_l)}{4\Delta y} + \frac{v(y_l)}{2(\Delta y)^2}, 0, \cdots, 0\right)$$

for $l = 1, \cdots, L - 1$. 
and

\[
A_0 = \left( \frac{q}{2} + \frac{1}{\Delta t}, 0, 0, \ldots, 0 \right),
\]

\[
A_L = \left( 0, \ldots, 0, \frac{\mu(y_L)}{2\Delta y}, \frac{q}{2} + \frac{1}{\Delta t} - \frac{\mu(y_L)}{2\Delta y} \right),
\]

\[
B_0 = \left( -\frac{q}{2} + \frac{1}{\Delta t}, 0, 0, \ldots, 0 \right),
\]

\[
B_L = \left( 0, \ldots, 0, -\frac{\mu(y_L)}{2\Delta y}, -\frac{q}{2} + \frac{1}{\Delta t} + \frac{\mu(y_L)}{2\Delta y} \right).
\]
**Backward induction procedure**

- We start at time \( t_n^- \). By applying the known analytic solution at \( t_n \), we obtain \( f(t_n^-) \). We then numerically solve backward in time to \( t_{n-1}^+ \) by solving the matrix system: \( A f(s_k) = B f(s_{k+1}) \).

- At time \( t_{n-1}^+ \), the numerical solution together with the function \( g(\cdot, t_{n-1}^-) \) serves as the terminal condition for the numerical solution over the next time interval \( (t_{n-2}, t_{n-1}) \).

The finite difference algorithm is surprisingly accurate. The maximum relative error compared with the Monte Carlo simulation results is approximately 0.4%.
The parameters are: \( r = 0.05, \ q = 0.0, \ \sigma = 0.2, \ T = 1.0, \ t = 0.0, \ n = 10, \ S(0) = 100.0, \ t_i = 0.1i. \) MC refers to Monte Carlo solution, and FD refers to finite difference solution. The different I’s refer to the number of time steps. We used \( I/10 \) number of steps per jump size \( 1/n \) in the \( y \) direction. The Monte Carlo prices are based on \( 10^5 \) simulations with a control variate technique. The standard deviations of the Monte Carlo option prices are approximately \( 3.0 \times 10^{-3} \). Reported CPU times are for all 9 strikes.

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<th>FD ((I = 100))</th>
<th>FD ((I = 50))</th>
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</table>
6.2 Discretely sampled lookback options

Fixed strike lookback call option

For \( t \geq 0 \), we define the discretely monitored realized maximum of the asset price \( S(t) \) by

\[
\bar{S}(t) = \sup_{1 \leq i \leq m(t)} S(t_i)
\]

with \( \bar{S}(t) = 0 \) for \( t < t_1 \) for notational convenience.

Terminal payoff = \((\bar{S}(T) - K)^+\).

First, we solve for the time-\( t \) option value when \( \bar{S}(t) \geq K \).

Next, we solve for the case \( \bar{S}(t) < K \). This is done by treating the option pricing problem as a first passage problem of \( S \) to the level to \( K \), where the payoff at the first passage time is equal to the option value corresponding to \( \bar{S}(t) \geq K \).
1. $\bar{S}(t) = 0$ for $0 \leq t < t_1$ and $\bar{S}(t) = S(t_1)$ for $t_1 \leq t < t_2$. Here, $t_n$ is the last observation date and $T = t_{n+1}$ is the maturity date.

2. $\bar{S}(t)$ is updated to a higher value on an observation date if a new maximum asset price is realized.
Once $\bar{S}(t) \geq K$, $t < T$, the fixed strike lookback call is guaranteed to be in-the-money. The time-$t$ option price is given by

$$F(S(t), t) = E_t^Q \left[ e^{-r(T-t)} (\bar{S}(T) - K) \right]$$

$$= S_t E_t^{Q'} \left[ e^{-q(T-t)} \frac{\bar{S}(T)}{S(T)} \right] - e^{-r(T-t)} K.$$

This is consistent with the observation that $F_t/e^{qt}S_t$ is $Q'$-martingale. Define $x(t) = \frac{\bar{S}(t)}{S(t)}$ for $t \geq t_1$. Note that we have adopted the change of measure from $Q$ to $Q'$ since it is more convenient to work with $x(t)$.

- Unlike the Asian option model, the numerical calculation of the closed form analytic solution to $E_t^{Q'} \left[ e^{-q(T-t)} \frac{\bar{S}(T)}{S(T)} \right]$ is more cumbersome than its numerical solution via finite difference scheme since the closed form formula involves $n$-dimensional cumulative normal distribution functions.
Potential updating of recorded realized maximum value across a sampling date

It may occur that \( S(t_i^-) \geq \bar{S}(t_i^-) = \bar{S}(t_{i-1}) \). That is, the stock price at \( t_i^- \) is equal to or higher than the recorded realized maximum at the last fixing date. In this case, we have \( x(t_i) = 1 \) since the updated realized maximum \( \bar{S}(t_i) \) becomes \( S(t_i) \) [by virtue of continuity of \( S(t) \) across \( t_i \)].

If otherwise, suppose we have \( S(t_i^-) < \bar{S}(t_i^-) \) and so \( x(t_i^-) > 1 \). There will be no updated realized maximum. By continuity, we have \( x(t_i) = x(t_i^-) \).

The jump amount on \( x(t) \) across \( t_i^- \) is equal to (i) zero if \( S(t_i^-) < \bar{S}(t_i^-) \). If otherwise, the jump amount is \( 1 - x(t_i^-) \) when \( x(t_i^-) \leq 0 \iff S(t_i^-) \geq \bar{S}(t_i^-) \).
In summary, for $1 \leq i \leq n$,

$$x(t_i) = \begin{cases} 
1 & \text{if } x(t_i^-) \leq 1 \\
x(t_i^-) & \text{if } x(t_i^-) > 1
\end{cases}.$$  \hspace{1cm} (i)

On the observation date $t_i$, we always have $x(t_i) \geq 1$. After then, $x(t)$ may drop below 1 when $S(t)$ increases beyond $\overline{S}(t_i)$. 

$\overline{S}(t_i)$
The process $x_t$ may jump by an amount $(1-x(t^-))^+$ when $t$ falls on an observation date. Such jump occurs when a new realized maximum of the stock price is recorded on that observation date. In between two successive observation dates, the evolution of $x_t$ is continuous. Under $Q'$, the dynamics of $x_t$ is governed by

$$dx(t) = -(r - q)x(t^-)dt - \sigma x(t^-)dW'(t) + (1 - x(t^-))^+ dm(t)$$

$x(t_1) = 1$.

Therefore, $x_t$ is a Markov process with domain on $x > 0$.

Define

$$f(x(t), t) = E_{t}^{Q'} \left[ e^{-q(T-t)} \frac{\bar{S}(T)}{S(T)} \right] = E_{t}^{Q'} \left[ e^{-q(T-t)} x(T) | x(t) \right],$$

taking note of the Markovian property of $x_t$. 
When \( t \geq t_1 \), suppose \( \bar{S}(t) \geq K \), then

\[
F(S(t), t) = S(t)f(x(t), t) - e^{-r(T-t)}K. 
\]  
(A)

At the first time \( t_i > t \) (\( i \leq n \)) with \( S \) staying at or above \( K \), \( x(t_i) = 1 \) since a jump of \( x(t) \) to the value one must occur. We then obtain the payoff

\[
F(S(t_i), t_i) = S(t_i)f(1, t_i) - e^{-r(T-t_i)}K. 
\]  
(i)

- Note that \( \bar{S}(t) - K \) can change sign only on the observation dates at which potential updating of recorded maximum value may occur.
How about when $\bar{S}(t) < K$?

In general, for $t \geq 0$ with $\bar{S}(t) < K$, we have

$$F(S(t), t) = E_Q\left[ e^{-r(\tau^*-t)}\{S(\tau^*)f(1, \tau^*) - e^{-r(T-\tau^*)}K\}1_{\tau^* \leq t_n}S(t) \right],$$

where $\tau^*$ is the first passage time defined by

$$\tau^* = \inf_{i=1, \ldots, n} \{t_i : S(t_i) \geq K\},$$

with the convention: $\inf \phi = \infty$.

Thus, we again have the parallel formulation like that of an up-and-in barrier option.

**Numerical procedure**

1. First, solve for $f(x, u)$ for all $(x, u)$ with $u \geq \max(t, t_1)$.

2. If $\bar{S}(t) \geq K$, then the option price is given by eq. (A). Otherwise, solve the first passage time problem as depicted in eq. (B).
Numerical solution for $f(y,t)$, corresponding to the scenario $\overline{S}(t) \geq K$

Let $y = \ln x$ for the jump dynamics, it is seen that when $y < 0$ (which is equivalent to $x < 1$) at $t_i^-$, then $y$ jumps to 0 across $t_i$. When $y \geq 0$ at $t_i^-$, $y$ remains continuous across $t_i$.

Observing that $e^{-qt}f(x,t)$ is a $Q'$-martingale, for $t_{i-1} < t < t_i$, $i > 1$, and the incorporation of the jump dynamics in $y$ via the auxiliary condition on an observation date, $f$ is governed by

$$qf = \frac{\partial f}{\partial t} - \left( r - q + \frac{\sigma^2}{2} \right) \frac{\partial f}{\partial y} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial y^2},$$

subject to

$$f(y, t_i^-) = \begin{cases} f(0, t_i^+) & \text{if } y < 0 \\ f(y, t_i^+) & \text{if } y \geq 0 \end{cases}$$

$$f(y, T) = e^y.$$

We solve for $f(y,t)$ for $t_i^+ < t < T$. Note that $f(y,t)$ is not defined for $t < t_1$ since no maximum value has yet recorded.
Calculations for \( f \) \( [\bar{S}(t) \geq K] \),
time marching in \( \tau = T - t \), where \( \tau \) is the time to expiry.

\[
qf = \frac{\partial f}{\partial t} - \left( r - q + \frac{\sigma^2}{2} \right) \frac{\partial f}{\partial y} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial y^2},
\]

\(-\infty < y < \infty, \quad t_1 < t < T = t_{n+1}.\)

Jump condition at \( t_i \): \( f(y, t_i^-) = \begin{cases} f(0, t_i^+) & \text{if } y < 0, \\ f(y, t_i^+) & \text{if } y \geq 0 \end{cases} \), \( i = n - 1, n - 2, \ldots, 1; \)

Terminal payoff: \( f(y, T) = e^y. \)

Lookback option value \( F(S(t), t) = S(t)f(y, t) - e^{-r(T-t)}K, \quad y_t = \ln \frac{\bar{S}(t)}{S(t)}. \)
Numerical solution for $g(y, t)$, corresponding to the scenario $\bar{S}(t) < K$

For $\bar{S}(t) < K$, we may define $y = \ln \frac{S(t)}{K}$, and $g = \frac{F}{K}$. Note that $g$ is governed by

$$rg = \frac{\partial g}{\partial t} + \left(r - q - \frac{\sigma^2}{2}\right) \frac{\partial g}{\partial y} + \frac{\sigma^2}{2} \frac{\partial^2 g}{\partial y^2}$$

on $\{(y, t) : t_{i-1} < t < t_i, i = 1, 2, \ldots, n; -\infty < y < \infty\}$, subject to the auxiliary conditions:

$$g(y, t_i^-) = \begin{cases} g(y, t_i^+) & \text{if } y < 0 \\ eyf(0, t_i) - e^{-r(T-t_i)} & \text{if } y \geq 0 \end{cases}$$

$$g(y, t_n) = 0.$$

Note that if $\bar{S}(t) < K$ for $t \geq t_n$, then $F(S(t), t) = 0$ since the lookback option is sure to be out-of-the-money at expiry.
We may treat \( f \) and \( g \) in the same grid and simultaneously solve for \( f \) and \( g \) at each time step in that respective order. Be aware that the coefficients in the finite difference schemes are different since the governing equations are not the same. Also, the definitions of the independent variable \( y \) in the two formulation are different.

**Remark**

For \( y \geq 0 \) at \( t = t_i \), this corresponds to the first time that \( S(t) \) reaches \( K \) or above. The option value normalized by \( K \) is given by [see eq.(i)]

\[
g(y, t_i) = \frac{F(y, t_i)}{K} = e^y f(0, t_i) - e^{-r(T-t_i)},
\]

where \( x(t_i) = 1 \iff y = 0 \) and \( \frac{S(t_i)}{K} = e^y \).

At current time \( t \), values of options of different strikes are generated by

\[
F(S(t), t) = Kg \left( \frac{S(t)}{K}, t \right).
\]
Calculations for $g$ \([\bar{S}(t) < K]\); \(\tau\) is the time to expiry.

\[
\begin{align*}
gr &= \frac{\partial g}{\partial t} + \left( r - q - \frac{\sigma^2}{2} \right) \frac{\partial g}{\partial y} + \frac{\sigma^2}{2} \frac{\partial^2 g}{\partial y^2}, \quad -\infty < y < \infty, \quad 0 < t < t_n. \\
g(y, t_{i-1}^-) &= \begin{cases} 
g(y, t_i^+) \quad \text{if } y < 0 \\
e^{y}f(0, t_i) - e^{-r(T-t_i)} \quad \text{if } y \geq 0 \end{cases}, \quad i = n-1, n-2, \ldots, 1; \\
g(y, t_n) &= 0.
\end{align*}
\]

Lookback option value \(F(S(t), t) = Kg(y, t)\), where \(y_t = \ln \frac{S(t)}{K}\).
The parameters are: \( r = 0.05, \) \( q = 0.0, \) \( \sigma = 0.2, \) \( T = 1.0, \) \( t = 0.0, \) \( n = 10, \) \( S(0) = 100.0, \) \( t_i = 0.1i. \) MC refers to Monte Carlo solution, and FD refers to finite-difference solution. The different \( I \)'s refer to the number of time steps and also to the number of steps in the \( y \)-direction. The Monte Carlo prices are based on \( 10^5 \) simulations with a control variate technique. The standard error on the estimated Monte Carlo option prices is approximately \( 3.0 \times 10^{-3} \). Reported CPU times are for all 9 strikes.

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Floating strike lookback option

Terminal payoff $= (\bar{S}(T) - \alpha S(T))^+$. The fair option price is given by

$$F(S(t), t) = S(t)E_t^Q \left[e^{-q(T-t)} \left(\frac{\bar{S}(T)}{S(T)} - \alpha\right)^+\right]$$

$$= S(t)E_t^Q \left[e^{-q(T-t)} (x(T) - \alpha)^+ | x(t)\right]$$

$$= S(t)f(x(t), t) \text{ if } t \geq t_1.$$

It suffices to solve for $f(x, t)$ where $t \geq t_1$. For $t < t_1$, there is no recorded maximum value yet. Observe that $F(t_1) = S(t_1)f(1, t_1)$ since $\frac{\bar{S}(t_1)}{S(t_1)} = 1$ for sure. Since $f(1, t_1)$ is independent of asset price dynamics, so

$$F(S(t), t) = e^{-r(t_1-t)} E_t^Q [S(t_1)] f(1, t_1)$$

$$= e^{-r(t_1-t)} e^{(r-q)(t_1-t)} S(t) f(1, t_1)$$

$$= e^{-q(t_1-t)} S(t) f(1, t_1), \quad t < t_1.$$

Again, this result arises from the property that $F_t/e^{qt}S_t$ is $Q'$-martingale.
With $y = \ln x$, $f$ solves

$$qf = \frac{\partial f}{\partial t} - \left( r - q + \frac{\sigma^2}{2} \right) \frac{\partial f}{\partial y} + \frac{\sigma^2 \partial^2 f}{2 \partial y^2}$$

where $t_{i-1} < t < t_i$, $i > 1$. At $t = t_i^-$, suppose $S(t_i) > \bar{S}(t_i) = \bar{S}(t_{i-1})$, this corresponds to

$$x(t_i^-) = \frac{\bar{S}(t_i^-)}{S(t_i^-)} < 1 \Leftrightarrow y(t_i^-) < 0.$$ 

Right at the moment $t = t_i$, $y$ jumps to the “zero” value. By continuity, $f(y, t_i^-) = f(0, t_i)$, $y < 0$. When $y \geq 0$, we expect to have continuity of $f$ across $t_i$. The auxiliary conditions become

$$f(y, t_i^-) = \begin{cases} 
    f(0, t_i) & \text{if } y < 0 \\
    f(y, t_i^-) & \text{if } y \geq 0
\end{cases}, \quad i = n, n - 1, \ldots, 1;$$

$$f(y, t_{n+1}^-) = f(T, y) = (e^y - \alpha)^+. $$

The formulation for floating strike Lookback option is very similar to that of the fixed strike Lookback option under $\bar{S}(t) \geq K$, except that the terminal condition at $T$ is changed to $(e^y - \alpha)^+$. 
Remark
All options with $\alpha \leq 1$ are guaranteed to be “in-the-money”. For $\alpha < 1$, the European option contract has a value that equals the value of the counterpart with $\alpha = 1$ plus $S(t)(1 - \alpha)e^{-q(T-t)}$. This is easily seen since

$$F(t) = e^{-r(T-t)}E_t^Q[\overline{S}(T) - \alpha S(T)]$$
$$= e^{-r(T-t)}E_t^Q \left[ \{\overline{S}(T) - S(T)\} + (1 - \alpha)S(T) \right]$$
$$= e^{-r(T-t)}E_t^Q [\overline{S}(T) - S(T)] + (1 - \alpha)S(t)e^{-q(T-t)}.$$

In other words, once the lookback option price for $\alpha = 1$ is known, the option price for $\alpha < 1$ can be obtained by adding an extra term $(1 - \alpha)S(t)e^{-q(T-t)}$. 
We perform the usual finite difference calculations as those of the vanilla option models, except that we impose the auxiliary condition:

\[ f(y, t_i^-) = f(0, t_i) \text{ for } y < 0 \]

for \( i = n, n - 1, \ldots, 1 \).
The parameters are: \( r = 0.05, \ q = 0.0, \ \sigma = 0.2, \ T = 1.0, \ t = 0.0, \ n = 10, \ S(0) = 100.0, \ t_i = 0.1i. \) MC refers to Monte Carlo solution, and FD refers to finite-difference solution. The different \( I \)'s refer to the number of time steps and also to the number of steps in the \( y \)-direction. The Monte Carlo prices are based on \( 10^5 \) simulations with a control variate technique. The standard deviation of the Monte Carlo option prices is approximately \( 3.0 \times 10^{-3} \). Reported CPU times are for all 9 strikes.

<table>
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</tbody>
</table>

- Option prices drop drastically with increasing value of \( \alpha \).