1. Consider the Cheuk-Vorst algorithm for pricing floating strike lookback put options. The truncated binomial tree for the process is constructed for

\[ Y_t = \frac{S^\text{max}_t}{S_t}, \quad Y_t \geq 1. \]

Let \( \tilde{V}_n^j \) denote the numerical approximation to \( \tilde{V}_n^j = \tilde{V}_n^j/S_n^j \) at the \((n, j)\)th node of the binomial tree for \( Y_t \), where \( Y_t = u^j, j \geq 0 \).

(i) Explain how to derive the binomial formula

\[ \tilde{V}_n^j = e^{-r\Delta t}[p\tilde{V}_{n+1}^{j+1}u + (1-p)\tilde{V}_{n+1}^{j+1}d], \quad j \geq 1. \]

(ii) How to modify the above formula when \( j = 0 \)?

(iii) How to determine the terminal condition in the binomial calculations?

(iv) How to modify the above binomial formula in order to incorporate the American early exercise feature?

2. Explain how to construct the forward shooting grid algorithm for pricing the European style fixed strike lookback call option under (i) continuous monitoring, (ii) discrete monitoring. In particular, explain how to construct the grid function that models the lookback feature.

3. (a) Explain why the stock price process and its underlying volatility process are negatively correlated in general.

(b) We assume the volatility to be function of time (no dependence on the stock price). Recall the formula:

\[ \int_t^{t^*} \sigma^2(u) \, du = \sigma^2_{\text{imp}}(t^*, t)(t - t^*), \]

we would like to obtain the term structure of volatility \( \sigma(t) \) given the implied volatility \( \sigma^2_{\text{imp}}(t^*, t) \) at discrete time points. Suppose we assume \( \sigma(t) \) to be piecewise constant over \((t_{i-1}, t_i)\), where \( \sigma(t) = \sigma_i, t_{i-1} < t < t_i, i = 1, 2, \ldots, n \), show that

\[ \sigma_i = \sqrt{(t_i - t^*)\sigma^2_{\text{imp}}(t^*, t_i) - (t_{i-1} - t^*)\sigma^2_{\text{imp}}(t^*, t_{i-1})}, \quad t_{i-1} < t < t_i. \]

(c) What is the difference between “implied volatility” and “local volatility”?

(d) The usual binomial tree is generated based on constant volatility in the underlying Geometric Brownian motion of the stock price process. The implied binomial tree constructed by the Derman-Kani algorithm uses quite different approach.
(i) Describe very briefly in qualitative terms how the implied binomial tree is constructed as a discrete approximation to the continuous risk neutral process for the underlying asset in a lattice tree that is consistent with the smile effect and the term structure of the implied volatility.

(ii) Is it model free? Why the risk neutral process but not the physical process of the asset price is generated under the Derman-Kani algorithm?

4. (a) Why the implicit scheme is most preferred in usual option pricing calculations when compared to the explicit scheme and the Crank-Nicolson scheme?

\[ \text{Hint} \quad \text{Make comments with reference to computational efficiency, time step restriction and numerical oscillations.} \]

(b) Explain why the usual dynamic programming procedure cannot be applied to implicit schemes for numerical calculations of American option model. Explain the rationale and outline briefly the numerical procedure in the Projected Successive-over-Relaxation method.

5. (a) Explain why the antithetic variates method in pricing a European call option can achieve variance reduction in Monte Carlo simulation.

(b) Let \( c_i \) and \( \tilde{c}_i \) denote the simulated call value in the \( i \)th simulation run. Explain why

\[ \text{var} \left( \frac{c_i + \tilde{c}_i}{2} \right) = \frac{1}{2} \left[ \text{var}(c_i) + \text{cov}(c_i, \tilde{c}_i) \right]. \]

(c) Show that the antithetic variates method improves computational efficiency provided that

\[ \text{cov}(c_i, \tilde{c}_i) < 0. \]

Explain why the above negative correlation property is in general valid.

6. In the participating policy model, there are two state variables. The dynamics of the asset \( A(t) \) under a risk neutral measure \( Q \) is governed by the Geometric Brownian motion:

\[ dA(t) = rA(t) \, dt + \sigma A(t) \, dW^Q(t). \]

The updating of the policy account \( P(t) \) is based on

\[ P(t^+) = P(t^-) + \max(r_G P(t^-), \alpha \{ [A(t) - P(t^-)] - \gamma P(t^-) \}) \]

across a fixing date, where

\[ r_G : \text{guaranteed rate of return} \]
\[ \gamma : \text{target buffer ratio} \]
\[ \alpha : \text{distribution ratio}. \]

Explain the key considerations in the incorporation of the jump condition (crediting mechanism) in the finite difference calculations.

\[ \text{Hint} \quad \text{Write } P(t^+) = \tilde{j} \Delta P \text{ and } P(t^-) = j \Delta P. \text{ Let } A = i \Delta A, \text{ where } \Delta A \text{ is the stepwidth for } A. \text{ Explain why} \]

\[ \tilde{j} = j + \max \left\{ r_G j, \alpha \left[ \left( \frac{\Delta A}{\Delta P} - j \right) - \gamma_j \right] \right\}. \]
7. To incorporate the interaction between holder’s conversion right and issuer’s calling right in the numerical calculations of a convertible bond using the lattice tree method, the following dynamic programming procedure is adopted:

$$\min\left(\max(\text{roll, conv}), \max\left(\text{call, conv}\right)\right),$$

where

- conv = conversion value
- call = call price
- roll = value given by rollback (neither converted nor recalled).

(i) Explain the financial rationale behind the above dynamic programming procedure. [3]
(ii) Another dynamic programming procedure that can be used is specified as

$$\max\left(\min(\text{roll, call}), \text{conv}\right).$$

Explain the financial intuition for the new procedure. Also, illustrate the equivalence of the two dynamic programming procedures. [4]

8. Consider the pricing of a discretely monitored fixed strike Asian call option with fixing dates: $t_1, t_2, \ldots, t_n$. Define the running sum of discretely sampled stock prices up to time $t$ by

$$I(t) = \sum_{i=1}^{m(t)} S(t_i),$$

where $m(t) = \sup\{1 \leq i \leq n; t_i \leq t\}$. Define the stochastic process

$$x(t) = \frac{\frac{1}{n} I(t) - K}{S_t},$$

where $K$ is the fixed strike of the Asian call option.

(i) Explain why $x_t$ jumps by a deterministic amount $\frac{1}{n}$ when the calendar time moves across a fixing date. [2]
(ii) Explain why the crossing of $x_t$ across $x = 0$ can only occur at one of the fixing dates. [2]
(iii) The process $x_t$ is Markovian. Is $x_t$ always increasing in $t$? [3]

9. Consider the pricing of a discretely sampled fixed strike lookback call option with fixing dates: $t_1, t_2, \ldots, t_n$, and strike price $K$. Define the discretely monitored realized maximum of the stock price $S(t)$ by

$$\overline{S}(t) = \sup_{1 \leq i \leq m(t)} S(t_i),$$

and

$$x(t) = \frac{\overline{S}(t)}{S(t)} \text{ for } t \geq t_1.$$

(i) Explain why $x_t$ jumps by $(1 - x(t^-))^+$ across a fixing date $t$. [2]
Hint Note that the jump may be zero or equals $1 - x(t^-)$.

(ii) Define

$$f(x(t), t) = E^Q[e^{-q(T-t)}x(T)|x(t)]$$

Where $Q'$ is the share measure. Suppose $t_i$ is the first fixing date after $t$ such that $S$ is staying at or above $K$. Let $F(S(t), t)$ denote the value function of the discrete fixed strike lookback call option. Here, $q$ denotes the constant dividend yield.

Explain why

$$F(S(t_i), t_i) = S(t_i)f(1, t_i) - e^{-r(T-t_i)}K,$$

where $r$ is the riskfree interest rate.

Hint Explain the appearance of the factor $S(t_i)$, the value of the argument for $x$ in $f(x(t), t)$ and the form of the strike term.

(iii) For $t \geq 0$ with $S(t) < K$, explain why

$$F(S(t), t) = E^Q[e^{-r(\tau^*-t)}\{S(\tau^*)f(1, \tau^*) - e^{-r(T-\tau^*)}K\}1_{\tau^* \leq t_n}|S(t)],$$

where $Q$ is the risk neutral measure and $\tau^*$ is the first passage time defined by

$$\tau^* = \inf_{i=1,2,...,n} \{t_i : S(t_i) \geq K\}.\quad [3]$$

Hint Explain why the pricing problem becomes a first passage time problem like that of an up-and-in barrier option. Also, why the risk neutral measure $Q$ is adopted here.

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