ON HUNTINGTON METHODS OF APPORTIONMENT*

M. L. BALINSKI† AND H. P. YOUNG‡

Abstract. A (generalized) Huntington method for apportioning representatives among states, or seats among parties, is one which distributes seats one by one by using a rank index that determines how deserving a state, or party, is to receive the next available seat. A characterization of these methods is given by two basic properties: consistency and monotonicity.

The arguments used to establish this result are combinatorial in nature and use classical theorems concerning partial orders and their representation by a real-valued function.

1. Introduction and brief history. The apportionment problem is the problem of determining how to divide a given integer number of representatives or delegates proportionally among given constituencies according to their respective sizes. The problem arises in deciding how to distribute a given number of delegates in a legislature among the component states of a country and also in determining how to divide a given number of candidates among the various political parties receiving votes in an election. In the latter guise this is the proportional representation problem.

In the United States the apportionment problem has a long and interesting history stemming from the Constitutional mandate, "Representatives and direct taxes shall be apportioned among the several States . . . according to their respective numbers" (Article I, Section 2). This stipulation led to an early consideration of various methods by which apportionments might be computed. Jefferson, Hamilton, and Webster all actually proposed methods, and many important political figures in United States history concerned themselves with the apportionment problem at regular ten year intervals following each census, thus testifying both to its political importance and its mathematical nontriviality. For an historical account of the problem in the United States see [4], [14]. In Europe, the question of apportionment methods does not seem to have been debated until the second half of the nineteenth century and then in the context of proportional representation (see, e.g., [12]).

Formally, the apportionment problem may be stated as follows. Let \( p = (p_1, p_2, \ldots, p_s) \) be the populations of \( s \) states, where each \( p_i > 0 \) is integer, and let \( h \geq 0 \) be the number of seats in the house to be distributed. The problem is to find, for any \( p \) and all house sizes \( h \geq 0 \), an apportionment for \( h \): an \( s \)-tuple of nonnegative integers \( a = (a_1, \ldots, a_s) \) whose sum is \( h \). A solution of the apportionment problem is a function \( f \) which to every \( p \) and \( h \) associates a unique apportionment method may give several different solutions, for "ties" may occur when using it—for example when two states have identical populations and must share an odd number of seats. It is useful, for this reason, to define an apportionment method \( M \) as a nonempty set of solutions. Two different apportionment solutions \( f \) and \( g \) of a

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method \( \mathbf{M} \) may be identical up to some house \( h \) and then branch, depending on how a particular tie is resolved. The restriction of \( \mathbf{f} \) to the domain \( (p, h') \), \( 0 \leq h' \leq h \), will be called a \textit{solution up to} \( h \), \( \mathbf{f}^h \), and \( \mathbf{f} \) will be called an \textit{extension} of \( \mathbf{f}^h \).

As early as 1792 Thomas Jefferson [10], then Secretary of State, pointed to the advantages of using a method of apportionment after each census, as opposed to relying on ad hoc procedures which are susceptible of endless political argument and manipulation. Moreover, he proposed a general and important method known today as Jefferson's method \( (J) \) [4]. This method, later rediscovered by the Belgian doctor-of-law Victor d'Hondt, has been widely used for the proportional representation problem in Europe [12]. The United States apportionments based on the censuses of 1790 through 1830 were Jefferson's.

In 1792 Alexander Hamilton, then Secretary of the Treasury, proposed the following method [7]. Given the populations \( (p_1, p_2, \cdots, p_s) \) and \( h \), first compute the \textit{exact quota} for each state \( i \), \( p_i h/(\sum_j p_j) = q_i \), and consider the fractional remainders \( d_i = q_i - \lfloor q_i \rfloor \) (where \( \lfloor x \rfloor \) represents the largest integer less than or equal to \( x \)) arranged in descending order, say \( d_{i_1} \geq d_{i_2} \geq \cdots \geq d_{i_s} \). Then Hamilton's \textit{method} is to first give each state \( i \lfloor q_i \rfloor \) seats, and if \( d_i \) is among the first \( d_{i_1} \) terms of the above list then it is given one more, or \( \lfloor q_i \rfloor + 1 \) seats. This method was proposed again after the 1850 census by Representative Samuel F. Vinton of Ohio, and was used (subject to politically motivated amendments) for the censuses of 1850 through 1900 under the name "Vinton's Method of 1850."

A serious difficulty with this method came to light in 1881 when C. W. Seaton, the Chief Clerk of the United States Census Office, discovered that, whereas the Hamilton method, in apportioning 299 seats among the states gave Alabama 8, it gave her only 7 in a house of 300 seats. This phenomenon (which is no isolated quirk of the Hamilton method but in fact occurs frequently) was dubbed the \textit{Alabama paradox}, and was immediately recognized as a critical flaw in the Hamilton method.

Beginning early in this century attention was therefore focused on developing methods that do \textit{not} admit the Alabama paradox, that is, methods that are \textit{monotone} in the sense that \( \mathbf{f}(p, h + 1) \geq \mathbf{f}(p, h) \) for every \( p \) and \( h \). W. F. Willcox [16] generalized an earlier proposal of Webster [15] to obtain a monotone method, known alternately as the method of major fractions or \textit{Webster's method} \( (W) \), which was used in 1911. This method was proposed independently by Sainte-Lagüe in 1910 [13] and has been used in proportional representation systems in Europe. Beginning at about this time E. V. Huntington [9], Professor of Mathematics at Harvard, undertook a formal investigation of monotone methods.

From a computational point of view Huntington's approach may be summarized as follows. Let \( r(p, a) \) be any real-valued function of two variables, called a \textit{rank index}. Then a monotone apportionment method \( \mathbf{M} \) is obtained by taking all apportionment solutions \( \mathbf{f} \) defined recursively as follows:

(i) \[ f_i(p, 0) = 0, \quad 1 \leq i \leq s, \]

(ii) if \( a_i = f_i(p, h) \) is an \( \mathbf{M} \)-apportionment for \( h \), and \( k \) is some one state for which \( r(p_k, a_k) \geq r(p_i, a_i) \) for \( 1 \leq i \leq s \), then \( f_k(p, h + 1) = a_k + 1 \), and \( f_i(p, h + 1) = a_i \) for \( i \neq k \).
The method obtained in this way will be called the *Huntington method based on* \( r(p, a) \), and as a class such methods will be called *Huntington methods* (see [4]). It is obvious that all Huntington methods are monotone. But Huntington himself only considered five particular choices of ranking function—these are listed in Table 1.

<table>
<thead>
<tr>
<th>Method</th>
<th>Rank index</th>
<th>Test of inequality ( (p_i/a_i \equiv p_j/a_j) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smallest Divisors (SD)</td>
<td>( p/a )</td>
<td>( a_i - a_j(p_i/p_j) )</td>
</tr>
<tr>
<td>Harmonic Mean (HM)</td>
<td>( p/[2a(a + 1)/(2a + 1)] )</td>
<td>( p_i/a_i - p_j/a_j )</td>
</tr>
<tr>
<td>Equal Proportions (EP)</td>
<td>( p/(a(a + 1))^{1/2} )</td>
<td>( p_i a_j/p_j a_i - 1 )</td>
</tr>
<tr>
<td>Webster (W)</td>
<td>( p/(a + 1/2) )</td>
<td>( a_i/p_i - a_j/p_j )</td>
</tr>
<tr>
<td>(also known as Major Fractions and Sainte-Lagüe)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jefferson (J)</td>
<td>( p/(a + 1) )</td>
<td>( a_i(p_i/p_j) - a_i )</td>
</tr>
<tr>
<td>(also known as Greatest Divisors and d'Hondt)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

That the five methods discussed by Huntington are, in fact, all different is shown in Table 2 by the apportionments obtained for a house of 36 seats.

<table>
<thead>
<tr>
<th>Party</th>
<th>Votes received</th>
<th>Exact quota</th>
<th>SD</th>
<th>HM</th>
<th>EP</th>
<th>W</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>27,744</td>
<td>9.988</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>B</td>
<td>25,178</td>
<td>9.064</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>19,947</td>
<td>7.181</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>D</td>
<td>14,614</td>
<td>5.261</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>E</td>
<td>9,225</td>
<td>3.321</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>3,292</td>
<td>1.185</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>100,000</td>
<td>36.000</td>
<td>36</td>
<td>36</td>
<td>36</td>
<td>36</td>
<td>36</td>
</tr>
</tbody>
</table>

Huntington derived these five particular methods from certain binary-comparison “tests of inequality.” Given an apportionment \( \mathbf{a} = (a_1, a_2, \ldots, a_s) \) for \( h \) and populations \( \mathbf{p} = (p_1, p_2, \ldots, p_s) \), consider any pair of states \( i, j \) and the numbers \( p_i/a_i \) and \( p_j/a_j \), which represent the average district sizes in states \( i \) and \( j \) respectively. Huntington then argued: “Now in a perfect apportionment, these two numbers would be exactly equal . . . hence, in any practical case, . . . if [the] inequality can be reduced by a transfer of a representative from one state to the other then . . . the transfer should be made . . . The question then comes down to
this: what shall be meant by the inequality between these two numbers?' [9, p. 86]. Huntington then goes on to consider the absolute difference, $|p_i/a_i - p_j/a_j|$, versus the relative difference, 

$$\frac{|p_i/a_i - p_j/a_j|}{\min \{p_i/a_i, p_j/a_j\}}.$$

Assume that $i$ and $j$ are chosen so that $p_j/a_j \geq p_i/a_i$; then the relative difference is $p_j/a_j - p_i/a_i - 1$. Suppose the relative difference is chosen as the "right" measure of inequality. Then it is easily shown that $a = (a_1, \ldots, a_s)$ is an apportionment such that no transfer can be made between two states that reduces the amount of inequality, if and only if, for all $i$ and $j$,

$$p_i/(a_i(a_i - 1))^{1/2} \geq p_i/(a_i(a_i + 1))^{1/2},$$

which holds if and only if $(a_1, a_2, \ldots, a_s)$ is obtained as a Huntington method solution with $r(p, a) = p/(a(a + 1))^{1/2}$, that is, $\text{EP}$ [4]. Similarly, the test $p_i/a_i - p_j/p_j$ leads to the Harmonic Mean method. On the other hand, one could just as well begin by comparing the numbers $a_i/p_i$ and $a_j/p_j$, or $a_i$ and $a_i(p_i/p_j)$ or $a_j(p_i/p_j)$ and $a_i$, whose differences lead to $\text{W}$, $\text{SD}$, and $\text{J}$ respectively, and whose relative differences all result in $\text{EP}$ [9]. It is interesting to note in this context that Huntington's approach to $\text{J}$ was quite different from Jefferson's; moreover Huntington was apparently not aware of Jefferson's proposal.

Huntington's goal was to show that $\text{EP}$ is the best of the five methods, because it is based on what he felt was the most natural measure of difference—namely, the relative difference. In this he was supported by two select committees which reported to the President of the National Academy of Sciences, one in 1929 [5] and one in 1948 [11]. These reports both argued for $\text{EP}$ because, of the "now known" methods which are "unambiguous" and monotone, $\text{EP}$ satisfies a test which seems to be preferable to others and yields apportionments which are "neutral . . . with respect to emphasis on larger and smaller states" [5]. The existence of monotone methods based on rank-indices other than Huntington's five had apparently escaped observation.

2. The two basic properties. By his tests of inequality Huntington restricted the field to five particular methods, but did not convincingly single out any one method as unequivocally "best." Here we ask, what are the essential properties that distinguish the class of Huntington methods from all others? The answer is surprisingly simple.

The first basic property of Huntington methods—monotonicity—has already been mentioned: it was, indeed, the fundamental motivation for these methods. But the Huntington methods are not the only monotone methods—for example the Quota method is a monotone method that is not a Huntington method [1], [4].

A further consideration of monotonicity reveals a second basic property that we call in this context consistency. If $M$ is any monotone method, and $f$ is a solution of $M$, then for any given populations $p$ the operation of $f$ can be fully described by specifying, for each $h$, which state gets the "next" (i.e., $(h + 1)$st) seat. For in going
from \( f(p, h) \) to \( f(p, h + 1) \), exactly one state must get one more seat while all the others stay the same. Why does some state \( i \), having population \( p_i \) and current apportionment \( a_i = f_i(p, h) \) get the \((h + 1)\)st seat instead of some other state \( j \) with population \( p_j \) and apportionment \( a_j = f_j(p, h) \)? Evidently because state \( i \) "deserves" it more than \( j \). In comparing the relative claim for an extra seat between any two states \( i \) and \( j \), the only relevant data should be their populations \( p_i \) and \( p_j \), and their current numbers of seats \( a_i \) and \( a_j \). That is, \( M \) defines a partial relation \( \succeq \) on the set \( X \) of pairs of integers \((p, a), p > 0, a \geq 0\), as follows:

\[
(p, a) \succeq (q, b) \text{ if and only if there is some } p, h \text{ and some } i, j \text{ such that } \\
(1) \quad p_i = p, \quad p_j = q, \quad f_i(p, h) = a, \quad f_j(p, h) = b \quad \text{ and } \quad f_i(p, h + 1) = a + 1, \quad f_j(p, h + 1) = b.
\]

In this case we say \((p, a)\) has weak priority over \((q, b)\). It should be noted that if \((p, a) \succeq (q, b)\) by some \( M \) then this implies there is a problem with populations \( p = (\cdots, p, \cdots, q, \cdots) \) and some \( h \) at which \( M \) gives \( a \) seats to the state with population \( p \) and \( b \) seats to the state with population \( q \). If \((p, a) \succeq (q, b)\) and not \((q, b) \succeq (p, a)\) we write \((p, a) \succ (q, b)\), whereas if \((p, a) \succeq (q, b)\) and \((q, b) \succeq (p, a)\) we write \((p, a) \sim (q, b)\) and say \((p, a)\) and \((q, b)\) are tied.

It is natural, from the context of apportionment itself, to require that the relation \( \succeq \) satisfy:

\[
(2) \quad \text{if } (p, a) \sim (q, b) \text{, then } M \text{ should be "indifferent" between them, that is, whenever for some } p \text{ and } h, f_i(p, h) = a, f_j(p, h) = b, p_i = p, \text{ and } p_j = q, \text{ if } f \text{ gives the } (h + 1)\text{st seat to state } i \text{ then there should be an alternate solution } g \in M \text{ that is identical with } f \text{ up to } h \text{ (i.e. } g^h = f^h), \text{ but which gives the } (h + 1)\text{st seat instead to state } j.
\]

Any method \( M \) having property (2) will be said to be consistent. Basically, consistency means that if \((p, a) \sim (q, b)\), then any two states with populations \( p \) and \( q \) and apportionments \( a \) and \( b \) are equally deserving in terms of the operation of the method \( M \).

The thrust of consistency is two-fold. First, it means that the priority for an extra seat between any two states is independent of the status of other states. Second, it implies that the priority between two seats cannot strictly change in different problems. The condition seems eminently acceptable for apportionment—as versus the debatable "independence of irrelevant alternatives" of social choice theory—because it is the very nature of apportionment for a state to compare its representation with that of other states (or, indeed, in proportional representation systems for a party to compare its number of seats with that of the other parties).

It is immediately obvious that any Huntington method is both house-monotone and consistent, with \((p, a) \sim (q, b)\) when \( r(p, a) = r(q, b) \).

Of course, there are other very natural properties that we might wish the priority relation to satisfy, e.g., transitivity. Remarkably enough, however, it turns out that something sufficiently close to transitivity—namely, acyclicity—is implied by the two conditions of monotonicity and consistency. Indeed, these two properties precisely characterize the class of Huntington methods.
3. The characterization.

Theorem. An apportionment method \( M \) is monotone and consistent if and only if it is a Huntington method.

The proof of this theorem needs the following key lemma concerning the relation \( \simeq \). It is proved in the next section.

**Lemma 1.** Let \( \simeq \) be the priority relation of a monotone and consistent apportionment method \( M \). If \((p_1, a_1) \simeq \cdots \simeq (p_k, a_k)\) then not \((p_k, a_k) > (p_1, a_1)\).

Let \( \approx \) be the union of the reflexive identity relation on \( X \) and the transitive closure of \( \sim \) on \( X \), so that \( \approx \) is an equivalence relation. Let \( \bar{X} = X/\approx \) and define \( \mu \) on \( \bar{X} \times \bar{X} \) by \((y, z) \in \mu\) if and only if \((p, a) \sim (q, b)\) for some \((p, a) \in y\) and some \((q, b) \in z\).

We claim that \( \mu \) is acyclic. If not, then there is a sequence \( y^1, y^2, \cdots, y^k \in \bar{X}, k \geq 2\), such that \((y^1, y^2) \in \mu, (y^2, y^3) \in \mu, \cdots, (y^k, y^1) \in \mu\). Hence there are equivalence class representatives \((p^1, a^1) \in y^1\) and \((q^1, b^1) \in y^1\) such that

\[
(p^1, a^1) > (q^2, b^2), (p^2, a^2) > (q^3, b^3), \cdots, (p^k, a^k) > (q^1, b^1).
\]

If \((p^i, a^i) \sim (q^i, b^i)\) for each \( i, 1 \leq i \leq k \), so that either \((p^i, a^i) = (q^i, b^i)\) or else there is a chain in \( X \) such that \((p^i, a^i) \sim \cdots \sim (p^n, a^n) = (q^1, b^1)\). From these and (3) we immediately derive a chain that contradicts Lemma 1. Hence \( \mu \) is acyclic, and in particular asymmetric and irreflexive.

Since \( \mu \) is acyclic, its transitive closure \( \mu' \) is a strict partial order, and therefore, since \( \bar{X} \) is countable, there exists a real-valued function \( \phi \) on \( \bar{X} \) such that \( \phi(y) > \phi(z) \) whenever \((y, z) \in \mu'\). This result is well-known in the literature; see for example [6].

Suppose now that \( M \) is monotone and consistent. With \( \phi \) as just given, define the rank-index \( r(p, a) = \phi(y) \) for all \((p, a) \in y\) and all \( y \in \bar{X}\). We claim that \( M \) is the Huntington method based on \( r \). If not, there exists \( f \in M \) and \( p, h \) such that \( f(p, h) = a_i, f(p, h) = a_i \) and \( r(p_i, a_i) > r(p_i, a_i) \), a contradiction.

4. Establishing acyclicity. In effect, given a sequence \((p_1, a_1) \simeq \cdots \simeq (p_k, a_k)\), what we would like to do is construct a solution \( f \) such that for \( 1 \leq i \leq k \), \( f_i((p_1, \cdots, p_k), h) = a_i \) where \( h = \sum a_i \). This turns out to be technically quite involved. A particular stumbling block is that \( \simeq \) is only a partial relation, so that not every two pairs \((p, a)\) and \((q, b)\) are comparable.

Let us say that a sequence \( S = ((p_1, a_1), (p_2, a_2), \cdots, (p_k, a_k)) \) is constructible, written \( C((p_1, a_1), (p_2, a_2), \cdots, (p_k, a_k)) \), if there exists an \( f \in M \) such that for some \( q = (q_1, \cdots, q_k) \), \( s \geq k \), satisfying \( q_1 = p_1, q_2 = p_2, \cdots, q_k = p_k \) and some \( h \) we have \( f_i((q, h) = a_i \) for \( 1 \leq i \leq k \). It follows that if \( S \) is constructible, then it is constructible for the population vector \((p_1, p_2, \cdots, p_k)\) since consistency permits one to imitate the solution for \( q \) restricted to these populations. Also, if it is known that \((p, a) \simeq (q, b)\) by some \( M \) then, of course, we have \( C((p, a), (q, b)) \).

Suppose that \( C((p, a), (q, b)) \), and let \( f \in M \) and \( p, h \) be such that \( p_i = p, f_i((p, h)) = a \) and \( p_i = q, f_i((p, h)) = b \). Consider the sequence of pairs \((f_i((p, 0)), \)
that is, the record of how states $i$ and $j$ went from zero seats each to an apportionment of $a$ and $b$ respectively. After eliminating redundant elements from this sequence we obtain the history $H(a, b)$ for $p, q$. Evidently, since $f$ is house-monotone, any element $(x_1, x_2) \in H(a, b)$ satisfies $0 \leq x_1 \leq a$, $0 \leq x_2 \leq b$, and if $(x_1, x_2) \neq (a, b)$ the successor of $(x_1, x_2)$ is either $(x_1 + 1, x_2)$ or $(x_1, x_2 + 1)$. Note that if $(x_1 + 1, x_2)$ follows $(x_1, x_2)$ then $(p, x_1) \succeq (q, x_2)$, and if $(x_1, x_2 + 1)$ follows $(x_1, x_2)$ then $(q, x_2) \succeq (p, x_1)$. We represent $H(a, b)$ by a tableau of form:

\[
\begin{array}{c|c|c|c|c|c}
  p & 0 & \cdots & x_1 & \cdots & a \\
  q & 0 & \cdots & x_2 & \cdots & b \\
\end{array} \quad : \quad H(a, b).
\]

Any sequence $S = ((p_{i1}, a_{i1}), \ldots, (p_{ik}, a_{ik}))$ such that $(p_{i1}, a_{i1}) \succeq (p_{i2}, a_{i2}) \succeq \cdots \succeq (p_{ik}, a_{ik})$ is called a cycle, and $S$ is a strict cycle if at least one of the relations $\succeq$ is satisfied as $\succ$.

The proof of Lemma 1 now proceeds by several sublemmas. Throughout, the relation $\succeq$ is that given to us by the method $M$.

**Lemma 1a. No strict cycle is constructible.**

**Proof.** Suppose that $(p_1, a_1) \succeq (p_2, a_2) \succeq \cdots \succeq (p_k, a_k)$ is a strict cycle and constructible. Then for some $f \in M$ we would have $a_i = f_i(p_1, \ldots, p_k, h)$ where $h = \sum a_i$. Let $i$ be such that $f_i((p_1, \ldots, p_k), h + 1) = a_i + 1$. Then $(p_i, a_i) \succeq (p_{i-1}, a_{i-1})$ whereas by assumption $(p_{i-1}, a_{i-1}) \succeq (p_i, a_i)$, hence $(p_i, a_i) \succeq (p_{i-1}, a_{i-1})$ (if $i = 1$ let $i - 1$ always mean $k$). Therefore, by consistency there exists an extension $g$ of $f^k$ such that $g$ gives the $(h + 1)$st seat to state $i - 1$. Continuing in this manner we establish that $(p_i, a_i) \succeq (p_{i-1}, a_{i-1})$, $1 \leq i \leq k$. But this contradicts the assumption that for some $i$, $(p_{i-1}, a_{i-1}) \succ (p_i, a_i)$. Hence $S$ is not constructible.

**Lemma 1b. If $(p, a) \succeq (q, b)$ then $C((q, b), (q, b))$.**

**Proof.** Since $(p, a) \succeq (q, b)$ we must have $C((p, a), (q, b))$.

Let $H(a, b)$ be a particular history for $p, q$. Define $p = (p, q, q)$. Consider the largest house $h \leq a + b + b = h^0$ for which there exists an $M$-apportionment $x = (x_1, x_2, x_3)$ satisfying

\[
\begin{align*}
  (4) & \quad x_1 \leq a, \quad x_2 \leq b, \quad x_3 \leq b, \\
  (5) & \quad (x_1, x_2) \in H(a, b) \quad \text{and} \quad (x_1, x_3) \in H(a, b).
\end{align*}
\]

\[
\begin{array}{c|c|c|c|c|c}
  p & 0 & \cdots & x_1 & \cdots \\
  q & 0 & \cdots & x_2 & \cdots \\
  q & 0 & \cdots & x_3 & \cdots \\
\end{array} \quad 0 \cdots h = x_1 + x_2 + x_3 \cdots
\]
Without loss of generality take $x_3 \geq x_2$.

**Case 1.** $x_3 > x_2$. Then $H(a, b)$ has form

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| $p$ | | | | | | | | | : $H(a, b)$.
| $q$ | | | | | | | | | 

In particular $(x_1, x_2 + 1) \in H(a, b)$ so

(6) $(q, x_2) \succeq (p, x_1)$.

If also $(q, x_2) \succeq (q, x_3)$, then there exists an apportionment for $h + 1$

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and since $x_2 < x_3 \leq b$, (4) and (5) are satisfied for the larger house $h + 1$, a contradiction.

Otherwise $(q, x_3) > (q, x_2)$, so by Lemma 1a

(7) $(q, x_3) > (p, x_1)$.

**Case 1a.** If also $x_3 < b$, then $(x_1, x_3 + 1) \in H(a_1, a_2)$ and

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is an $M$-apportionment for $h + 1$ satisfying (4) and (5), a contradiction.

Hence $x_3 = b$. We cannot also have $x_1 < a$, because then the history $H(a, b)$ would imply that $(x_1 + 1, b) \in H(a, b)$, so that $(p, x_1) \geq (q, x_3) = (q, b)$, contrary to (7).

**Case 1b.** $x_3 = b$ and $x_1 = a$. Then we have $(q, x_2) \succeq (p, a)$ by (6) and $(p, a) \succeq (q, b)$ by hypothesis of the lemma, so

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is an apportionment for $h + 1$ satisfying (4) and (5), a contradiction.
Case 2. $x_3 = x_2$. If $x_1 < a$ and $x_2 = x_3 < b$, then the successor of $(x_1, x_2)$ in the history $H(a, b)$ determines whether state 1 or state 2 gets the $(h + 1)$st seat, and in either case (4) and (5) are satisfied, a contradiction. If $x_1 < a$ and $x_2 = x_3 = b$ then $(p, x_1) \succeq (q, b)$ by the history and $(x_1 + 1, b, b)$ is an $M$-apportionment for $h + 1$ satisfying (4) and (5), again a contradiction. Finally, if $x_1 = a$ and $x_2 = x_3 < b$ then $(a, x_2 + 1, x_2)$ is an $M$-apportionment for $h + 1$ satisfying (4) and (5), which is a contradiction once again.

It follows that we must have $h = a + b + b$, proving the lemma.

**Lemma 1c.** If $C((q, b), (q, b'))$ then for any $b'$, $0 \leq b' \leq b$, there exists a sequence $b' = b_0 \leq b_1 \leq \cdots \leq b_k = b$ such that $C((q, b_{i-1}), (q, b_i))$ for $1 \leq i \leq k$ and $(q, b_0) \succeq (q, b_1) \succeq \cdots \succeq (q, b_k)$.

**Proof.** The proof is by induction on $b - b'$, the result for $b = b'$ being trivial. Let $H(b, b)$ be any history for $q, q$. Then there exists a pair $(x, y) \in H(b, b)$ such that $x = b'$ or $y = b'$. Choose any such pair $(x, y)$ with $x + y$ maximum. Say without loss of generality, that $x = b'$. Then by choice of $(x, y)$, $y > b'$ and $(q, b') \not\succeq (q, y)$. Set $b_1 = y$. If $b_1 = b$ we are done; otherwise $b_1 < b$ and we argue as with $b'$ to find a $b_2 > b_1$ such that $(q, b_1) \succeq (q, b_2)$ and so forth. This completes the proof of Lemma 1c.

For any sequence $S$ of pairs $(p_1, a_1), (p_2, a_2), \ldots, (p_k, a_k)$ define $b_s$ to be the maximum of the integers $a_i$, $1 \leq i \leq k$, and define $n_s$ to be the number of such that $a_i = b_s$. We say that sequence $S$ precedes $T$, written $S \ll T$ if either $b_s < b_T$ or $b_s = b_T$ and $n_s < n_T$.

Clearly any sequence, other than a trivial one of form $S = ((p, 0))$, has a predecessor.

Suppose, contrary to Lemma 1, that $(p_1, a_1) \succeq \cdots \succeq (p_i, a_i) \succeq (p_k, a_k)$ is a strict cycle $S$. By Lemma 1a, $S$ is not constructible, hence in particular $b_S > 0$. We may therefore assume inductively that $S$ is the "first" strict cycle, i.e. that $T \ll S$ for no strict cycle $T$. Also, we may assume (by relabeling if necessary) that $a_2 = b_S$. We shall now derive a contradiction.

For each $i$, $1 \leq i \leq k - 1$, the fact that $(p_i, a_i) \succeq (p_{i+1}, a_{i+1})$ implies $C((p_i, a_i), (p_{i+1}, a_{i+1}))$; hence for each such $i$ choose a history $H(a_i, a_{i+1})$ for $p_i, p_{i+1}$. Letting $p = (p_1, p_2, \ldots, p_k)$ and $h^0 = \sum_i a_i$, consider the largest $h' \leq h^0$ for which there exists $f \in M$ satisfying

$$(8) \quad (f_i(p, h), f_{i+1}(p, h)) \in H(a_i, a_{i+1}) \quad \text{for all} \ i, 1 \leq i \leq k - 1, \text{and all} \ h \leq h'.$$

Clearly $h'$ exists and $h' \geq 0$. Moreover, if $h' = h^0$ then $S$ is constructible, a contradiction. Therefore, $h' < h^0$. Let $x_i = f_i(p, h^0)$ for each $i$; in particular, by (8) we have $x_i \leq a_i$. Let $V$ be the set of all pairs $(p_i, x_i)$, $1 \leq i \leq k$. Any two elements of $V$ are comparable relative to $\succeq$ because $V$ has actually been constructed; for the same reason there are no strict cycles in $V$. Second, define $E = \{(p_i, x_i) \in V : x_i = a_i\}$.

We now construct a partial relation $R$ on $V$ as follows: for $1 \leq i \leq k - 1$ if $(x_i, x_{i+1}) \leq (a_i, a_{i+1})$ and the successor of $(x_i, x_{i+1})$ in $H(a_i, a_{i+1})$ is $(x_i + 1, x_{i+1})$ then $(p_i, x_i) R (p_{i+1}, x_{i+1})$ whereas if the successor is $(x_i, x_{i+1} + 1)$ then $(p_{i+1}, x_{i+1}) R (p_i, x_i)$. These are all the relations in $R$. The significance of $R$ is the following: if $(p_i, x_i) \in V - E$ is undominated relative to $R$ then the successive pairs from the sequence $(x_1, x_2, \ldots, x_i + 1, x_{i+1}, \ldots, x_k)$ are again members of the histories $H(a_1, a_2), \ldots, H(a_{k-1}, a_k)$. 
Notice that \( v R w \) implies \( v \geq w \) for any \( v, w \in V \). Further, comparable pairs under \( R \) form a forest (in fact, a forest in which no vertex has degree greater than two) on the vertex set \( V \). Since, by definition, we never have \((p_i, x_i) R (p_j, x_j)\) for any \((p_i, x_i) \in E\) it follows that the set of \( R\)-undominated elements in \( V - E \) is nonempty. Let \( V^R = \{ v \in V - E: \text{not } w R v \text{ for any } w \in V \} \neq \emptyset \).

Finally, let \( v^* = (p_i, x_i) \) be a maximum element of \( V^R \) relative to \( \geq \). Notice that \( v^* \) cannot also be maximum in \( V \) relative to \( \geq \), for if it were then by consistency there would exist an \( M\)-apportionment for \( h' + 1 \) giving the \((h' + 1)\)st seat to state \( l \). Moreover, this would agree with the given histories, contradicting our assumption on \( h' \).

We claim

\[ (9) \quad (p_1, x_1), (p_2, x_2) \in E. \]

For any \( w_0 \in V - E \) there is a chain \( w_n R w_{n-1} R \cdots R w_0 \) in \( V - E \) such that \( w_n \in V^R \), hence \( v^* \geq w_n \geq \cdots \geq w_0 \). In particular, \( E \) cannot be empty, else \( v^* \) would be maximum in \( V \). Suppose (9) is false, and let \((p_i, x_i) = (p_i, a_i)\) be any element of \( E \). First, if \( i \neq 1 \), let \( j \) be the largest index less than \( i \) such that \((p_j, x_j) \in E\). (Such a \( j \) always exists by the assumption that (9) is false.) Then \((x_j, a_{j+1}) \in H(a_j, a_{j+1})\), hence \((p_j, x_j) \geq (p_{j+1}, a_{j+1})\). Moreover, by assumption on \( S \), \((p_{j+1}, a_{j+1}) \geq (p_{j+2}, a_{j+2}) \geq \cdots \geq (p_i, a_i)\). But \((p_i, x_i), (p_{j+1}, a_{j+1}), \ldots, (p_i, a_i)\) has been constructed, so by Lemma 1a it could not be a strict cycle. Hence \((p_j, x_j) \geq (p_i, a_i)\). Second, if \( i = 1 \) then \((p_2, a_2) \in E\) implies \((a_1, x_2) \in H(a_1, a_2)\) so \((p_2, x_2) \geq (p_1, a_1)\). Since in the above argument \((p_i, x_i)\) was arbitrary in \( E \), it follows that for all \((p_i, a_i) \in E\) there exists \((p_i, x_i) \in E\) such that \((p_i, x_i) \geq (p_i, a_i)\). But then \( v^* \) would be maximum in \( V \), a contradiction. Thus (9) is established.

Since \( v^* = (p_i, x_i) \) cannot be maximum in \( V \), but \( v^* \geq w_0 \) for all \( w_0 \in V - E \), there must exist \( w \in E \), say \( w = (p_i, a_i) \), such that \( w \geq v^* \). Suppose that \( j \geq l \). Observe that since \((p_{l-1}, a_{l-1}) \geq (p_l, a_l)\), Lemma 1b tells us that \( C((p_l, a_l), (p_i, a_i))\). Hence, by Lemma 1c, there exists a sequence \( x_t = a_t^1 \leq a_t^2 \leq \cdots \leq a_t^n = a_t \) such that \((p_t, a_t^1) \geq (p_t, a_t^2) \geq \cdots \geq (p_t, a_t^n)\). Then \((p_t, a_t) > (p_t, a_t^1) \geq (p_t, a_t^2) \geq \cdots \geq (p_t, a_t^n) \geq (p_t, a_t) \geq (p_{t+1}, a_{t+1}) \geq \cdots \geq (p_i, a_i)\) is a strict cycle \( T \) that does not include the pair \((p_2, a_2)\), hence does not contain as many values as did the original cycle \( S \). Since \( a_2 \) was chosen to be the maximum value of \( a_t \) in \( S \), \( T \ll S \), contradicting the inductive hypothesis that no strict cycle precedes \( S \).

Suppose, then, that \( j < l \). Let \( t \) be the largest index less than \( j \) (if such exist) such that \( x_t < a_t \). Then \((x_t, a_{t+1}) \in H(a_t, a_{t+1})\) since \( x_{t+1} = a_{t+1} \) and so \((p_t, x_t) \geq (p_{t+1}, a_{t+1}) \geq \cdots \geq (p_i, a_i) \geq (p_t, x_t)\). Since this sequence has been constructed we have \((p_t, x_t) > (p_t, x_t)\). But this implies \( v^* = (p_i, x_i) \) is not a maximum in \( V - E \), a contradiction. So \( x_t = a_t \) for \( 1 \leq t \leq j \), and \((p_1, a_1) \geq (p_2, a_2) \geq \cdots \geq (p_i, a_i) \geq (p_t, x_t)\). But this sequence has been constructed so \((p_1, a_1) \geq (p_t, x_t)\). Thus, as before, \((p_1, a_1) > (p_t, x_t) = (p_t, a_t^1) \geq \cdots \geq (p_t, a_t^n) \geq (p_t, a_t) \geq (p_{t+1}, a_{t+1}) \geq \cdots \geq (p_i, a_i) \geq (p_1, a_1)\) is a strict cycle \( T \) (not necessarily constructed) with \( T \ll S \). This contradiction concludes the proof of Lemma 1 and hence of the theorem.

5. Further axiomatic characterizations. This paper has shown how the five methods discussed by Huntington find their place in an axiomatic setting which
uniquely characterizes the class of "generalized" Huntington methods by two basic properties: monotonicity and consistency.

Particular Huntington methods may be uniquely characterized by various additional axioms. A method \( M \) is said to be the unique one satisfying given properties if any other set \( M' \) of solutions having these properties is a set of \( M' \)-solutions, i.e. \( M' \subseteq M \). One of the most fundamental types of axioms not considered by Huntington is that an apportionment should not differ from the exact quotas by one whole integer or more. A method is said to satisfy quota if any apportionment \( (a_1, a_2, \cdots, a_s) \) for \( (p_1, p_2, \cdots, p_s) \) at house \( h \) has the property \( \lfloor q_i \rfloor \leq a_i \leq \lceil q_i \rceil \) where \( q_i \) is the exact quota of state \( i \). A method is said to satisfy upper quota if \( a_i \leq \lceil q_i \rceil \) for all apportionments \( a_i \) and to satisfy lower quota if \( a_i \geq \lfloor q_i \rfloor \). It may then be shown that \( J \) (Jefferson) is the unique monotone, consistent method satisfying lower quota [3]. Also, SD (Smallest Divisors) is the unique monotone, consistent method satisfying upper quota [3]. Since SD and J are not the same method (e.g., see Table 2) it follows, in particular, that there is no monotone consistent method satisfying quota.

In view of the desirability of monotonicity and satisfying quota as properties of an apportionment method, it is natural to ask whether there exists any method which obeys both properties. There is; moreover, if consistency is weakened to "consistently satisfying quota," then there exists a unique method, the Quota method, satisfying the three properties [1], [4].

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REFERENCES

