Mathematics and Social Choice Theory

Topic 1 – Voting systems and power indexes

1.1 Weighted voting systems and yes-no voting systems
1.2 Power indexes: Shapley-Shubik index and Banzhaf index
1.3 Case studies of power indexes calculations
1.1 Weighted voting systems and yes-no systems

Weighted voting system – individuals/political bodies can cast more ballots than others. For example, voting by stockholders in a corporation, more votes held by countries with stronger economic powers in the International Monetary Authority.

Question: How to define voting power in a weighted voting system?

- It is fallacious to expect that one’s voting power is directly proportional to the number of votes he can deliver.
- Power is not a trivial function of one’s strength as measured by his number of votes.
Define two indices that indicate the real distribution of influence.

- Should be useful in evaluating some of the existing democratic institutions in terms of fairness, concealed biases, etc.

**Remark**

No mathematical theory will suffice to reveal all of the behind-the-scene nuances - influences of party loyalty, persuasion, lobbying, fractionalism, bribes, gratuities, campaign financing, etc.

A new field called GOVERNMETRICS 計量管治學
Weighted majority voting game

\[ [q; w_1, w_2, \cdots, w_n] \]

\( n \) voters, \( w_i \) is the voting weight of player \( i \); \( N = \{1, 2, \cdots, n\} \) be the set of all \( n \) voters.

Let \( S \) be a typical coalition, which is a subset of \( N \). A coalition wins a vote (called winning) whenever

\[ \sum_{i \in S} w_i \geq q. \]

It is natural to require \( q > \frac{1}{2} \sum_{i \in N} w_i \), where \( q \) is called the quota.
Examples

1. \([51; 28, 24, 24, 24]\); 1st voter is much stronger than the last 3 since he needs only one other to pass an issue, while the other three must all combine in order to win.

2. \([51; 26, 26, 26, 22]\), the last player seems powerless since any winning coalition containing him can just as well win without him.

3. \([51; 40, 30, 20, 10]\) and \([51; 30, 25, 25, 20]\) seem identical in terms of voting influence, since the same coalitions are winning in both cases.

4. In the game \([q; 1, 1, \cdots, 1]\), each player has equal power. This is called a pure bargaining game.
5. Games such as $[3; 2, 2, 1]$, $[8; 7, 5, 3]$ and $[51; 49, 48, 3]$ are identical to $[2; 1, 1, 1]$ in terms of power, since they give rise to the same collection of winning coalitions.

6. If we add to the game $[3; 2, 1, 1]$ the rule that player 2 can cast an additional vote in the case of 2 to 2 tie, then it is effectively $[3; 2, 2, 1]$. If player 1 can cast the tie breaker, then it becomes $[3; 3, 1, 1]$ and he is the dictator. He forms a winning coalition by himself.

7. In the game $[50(n−1)+1; 100, 100, \cdots, 100, 1]$, player $n$ has the same power as the others when $n$ is odd; the game is similar to one in which all players have the same weights. For example, when $n = 5$, we have $[201; 100, 100, 100, 100, 1]$. Any 3 of the 5 players can form a winning coalition.
Dummy players

Any winning coalition that contains such an impotent voter could win just as well without him.

Examples

• Player 4 in $[51; 26, 26, 26, 22]$.

• Player $n$ in $[50(n - 1) + 1; 100; 100, \ldots, 100, 1]$ is a dummy when $n$ is even. For example, take $n = 4$, we have $[151; 100, 100, 100, 1]$. Obviously, the last player is a dummy.

• In $[10; 5, 5, 5, 2, 1, 1]$, the 4$^{\text{th}}$ player with 2 votes is a dummy. The 5$^{\text{th}}$ and 6$^{\text{th}}$ players with only one vote are sure to be dummies. The collection of dummies remains to be a dummy collection. This is because one cannot turn a losing coalition into a winning coalition by adding the dummy coalition.
Example

Consider $[16; 12, 6, 6, 4, 3]$, player 5 with 3 votes is a dummy since no subset of the numbers 12, 6, 6, 4 sums to 13, 14 or 15. Therefore, player 5 could never be pivotal in the sense that by adding his vote a coalition would just reach or surpass the quota of 16.

Example

If we add the 7$^{th}$ player with one vote into $[10; 5, 5, 5, 2, 1, 1, 1]$ so that the new game becomes $[10; 5, 5, 5, 2, 1, 1, 1]$, the 4$^{th}$ player in the new voting game is not a dummy since sum of votes of some coalition may assume the value of 8.
Notion of Power

- The index should indicate one’s relative influence, in some numerical way, to bring about the *passage or defeat of some bill*.

- The index should depend upon the number of players involved, on one’s fraction of the total weight, and upon how the remainder of the weight is distributed (critical swing-man in causing a desired outcome).

- A winning coalition is said to be *minimal winning* if no proper subset of it is winning. A coalition that is not winning is called *losing*. Technically, the one who is ‘last’ to join a minimal winning coalition is particularly influential. A voter \( i \) is a dummy if every winning coalition that contains him is also winning without him, that is, he is in no minimal winning coalition. A dummy has ZERO power.
Example

<table>
<thead>
<tr>
<th>Party</th>
<th>Leader</th>
<th>No. of seats</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liberals</td>
<td>Pierre E. Trudeau</td>
<td>109</td>
</tr>
<tr>
<td>Tories</td>
<td>Robert L. Stanfield</td>
<td>107</td>
</tr>
<tr>
<td>New Democrats</td>
<td>David Lewis</td>
<td>31</td>
</tr>
<tr>
<td>Others</td>
<td></td>
<td>17</td>
</tr>
</tbody>
</table>

- Though Liberals has the largest number of seats, none of the parties has the majority. Any two of the three leading parties can form a coalition and obtain the majority, so the first three parties had equal power. Other small parties are all dummies.
Veto Power and Dictators

A player or coalition is said to have *veto power* if no coalition is able to win a ballot without his or their consent. A subset $S$ of voters is a blocking coalition or has *veto power* if its complement $N - S$ is not winning. We also require that $S$ itself be losing in order to be a blocking coalition.

Given that $q > 50\%$ of votes, a player $i$ is a dictator if he forms a winning coalition $\{i\}$ by himself.

- If the dictator says “yes”, then the bill is passed. If the dictator says “no”, then the bill is not passed (veto power).
- If a dictator exists, then all other players are dummies.
Example

Player 1 has veto power in $[51; 50, 49, 1]$ and $[3; 2, 1, 1]$. In the last case, if he is the chairman with additional power to break ties, then the game becomes $[3; 3, 1, 1]$ and now he becomes a dictator.

Example

The ability of an individual to break tie votes in the pure bargaining game

$$\begin{cases} 
\left[ \frac{n}{2} + 1; 1, 1, \ldots, 1 \right] & \text{when } n \text{ is even} \\
\left[ \frac{n+1}{2}; 1, 1, \ldots, 1 \right] & \text{when } n \text{ is odd}
\end{cases}$$

adds power when $n$ is even and adds nothing when $n$ is odd. Actually, when $n$ is odd, tie votes will not occur.
Properties on dummies

A collection of dummies can never turn a losing coalition into a winning coalition.

In other words, is it possible that $S \cup \{D_1, \ldots, D_m\}$ is winning but $S$ is losing? This is not possible since the dummies can be successively deleted while the coalition remains to be winning.

Corollary

If both “$d$” and “$\ell$” are dummies, then the coalition $\{d, \ell\}$ is dummy.
Theorem

In a weighted voting game, let “d” and “ℓ” be two voters with votes $x_d$ and $x_\ell$, respectively. Suppose “d” is a dummy and $x_\ell \leq x_d$, then “ℓ” is also a dummy.

Proof

Assume the contrary. Suppose “ℓ” is not a dummy, then there exists a coalition $S$ such that $S$ is losing but $S \cup \{\ell\}$ is winning. Note that $S$ does not contain the dummy “d” since $S \cup \{\ell\}$ is a minimal winning coalition (recall a winning coalition is minimal if no proper subset of it is winning). Now, $n(S) < q$ while $n(S \cup \{\ell\}) \geq q$. Since $x_\ell \leq x_d$, so $n(S \cup \{d\}) \geq q$, contradicting that “d” is a dummy.
Corollary

The coalition \( \{d, \ell\} \) is dummy, then both “\(d\)” and “\(\ell\)” are dummies.

The result follows immediately from the above Theorem if we treat \( \{d, \ell\} \) as a single voter and observe

\[
n(\{d, \ell\}) \geq \max(x_d, x_\ell).
\]
Yes-no voting system

A yes–no voting system is simply a set of rules that specify exactly which collections of “yea” votes yield passage of a bill.

Under what condition that a yes–no voting system is a weighted system (with real number weights for the voters and a real number quota)?

Example

Bill to be passed: Grades of MATH392 are based on random draw.

Set of players, \( N = \{ \text{professor, tutor, Chan, Lee, Cheung, Wong, Ho} \} \). Veto power is held by “professor” and “tutor or Chan”.

Passage of the bill requires professor, at least one from “tutor and Chan” and number of students must be at least 3.
United States federal system

- 537 voters in this system: 435 House of Representatives, 100 Senate members, Vice-President and President.
- Vice President plays the role of tie breaker in the Senate
- President has veto power that can be overridden by a two-thirds vote of both the House and the Senate.

To pass a bill, it must be supported by

1. 218 or more representatives and 51 or more senators and President

2. 218 or more representatives and 50 senators and both Vice President and President

3. 290 or more representatives and 67 or more senators.
System to amend the Canadian constitution (since 1982)

• In addition to the House of Commons and the Senate, approval by two-thirds majority of provincial legislatures, that is, at least 7 of the 10 Canadian provinces subject to the proviso that the approving provinces have among them at least half of Canada’s population.

• Based on 1961 census
  
  Prince Edward Island (1%)
  Newfoundland (3%)
  New Brunswick (3%)
  Nova Scotia (4%)
  Manitoba (5%)
  Saskatchewan (5%)
  Alberta (7%)
  British Columbia (9%)
  Quebec (29%)
  Ontario (34%)
Swap robustness and trade robustness

Definition

A yes–no voting system is said to be swap robust if a “swap” of players between two winning coalitions leaves at least one of the two coalitions winning.

- Start with two arbitrary winning coalitions $X$ and $Y$ (not necessarily distinct)
  - arbitrary player $x$ in $X$ (not in $Y$) and arbitrary player $y$ in $Y$ (not in $X$)
  - let $X'$ and $Y'$ be the result of exchanging $x$ and $y$
  - either $X'$ or $Y'$ is winning for swap robustness
Proposition
Every weighted voting system is swap robust.

Proof

1. If $x$ and $y$ have the same weight, then both $X'$ and $Y'$ are winning.

2. If $x$ is heavier than $y$, then $Y'$ weighs strictly more than $Y$. The weight of $Y'$ certainly exceeds the quota, and thus $Y'$ is winning.

3. If $y$ is heavier than $x$, similar argument as in (2) holds.

To show a voting system to be not swap robust, we produce two winning coalitions $X$ and $Y$ and a trade between them that renders both losing. Intuitively, $X$ and $Y$ should both be “almost losing” and make both actually losing by a one-for-one trade. Find the appropriate $X$ and $Y$ among the minimal winning coalitions.
Proposition

The US federal system is not swap robust.

Proof

\[ X = \{ \text{President, 51 shortest senators and 218 shortest House Representatives} \} \]

\[ Y = \{ \text{President, 51 tallest senators and 218 tallest House Representatives} \} \]

Let \( x \) be shortest senator and \( y \) be the tallest House Representative.

Both \( X \) and \( Y \) are winning coalitions; \( x \in X \) but \( x \notin Y \); \( y \in Y \) but \( y \notin X \). After swapping, \( X' \) is a losing coalition since it has only 50 senators and \( Y' \) is a losing coalition because it has only 217 Representatives.

Corollary The US federal system is not a weighted voting system.
Proposition

The procedure to amend the Canadian Constitution is swap robust (later shown to be not weighted).

Proof

Suppose $X$ and $Y$ to be winning coalitions, $x \in X, x \notin Y$ while $y \in Y$ but $y \notin X$. We must show that at least one of $X'$ and $Y'$ is still a winning coalition. That is, at least one of $X'$ and $Y'$ still satisfies both conditions:

(i) It contains at least 7 provinces.

(ii) The provinces represent at least half of the Canadian population.

The first condition is obvious. If $x$ has more population than $y$, then $Y'$ is a winning coalition. Similar argument when $x$ has a smaller population than $y$. 
Definition (trade robust)

A yes–no voting system is said to be trade robust if an arbitrary exchange of players (a series of trades involving groups of players) among several winning coalitions leaves at least one of the coalitions winning.

1. The exchanges of players are not necessarily one-for-one as they are in swap robustness.

2. The trades may involve more than two coalitions.
Proposition

Every weighted voting system is trade robust.

Proof

• A series of trades among several winning coalitions leaves the number of coalitions to which each voter belongs unchanged. Hence, the total weight of all coalitions added together is unchanged.

• The average weight of these coalitions is unchanged.

• Suppose we start with several winning coalitions in a weighted voting system, then their average weight at least meets quota. After the trades, the average weight of the coalitions is unchanged and at least meets quota. Hence, at least one of the coalitions must meet quota (at least one of the resulting coalitions is winning).
Proposition  The procedure to amend the Canadian Constitution is not trade robust.

Proof

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prince Edward Island (1%)</td>
<td>New Brunswick (3%)</td>
</tr>
<tr>
<td>Newfoundland (3%)</td>
<td>Nova Scotia (4%)</td>
</tr>
<tr>
<td>Manitoba (5%)</td>
<td>Manitoba (5%)</td>
</tr>
<tr>
<td>Saskatchewan (5%)</td>
<td>Saskatchewan (5%)</td>
</tr>
<tr>
<td>Alberta (7%)</td>
<td>Alberta (7%)</td>
</tr>
<tr>
<td>British Columbia (9%)</td>
<td>British Columbia (9%)</td>
</tr>
<tr>
<td>Quebec (29%)</td>
<td>Ontario (34%)</td>
</tr>
</tbody>
</table>

• Let $X'$ and $Y'$ be obtained by trading Prince Edward Island and Newfoundland for Ontario.

• $X'$ is a losing coalition because it has too few provinces (having given up two provinces in exchange for one).

• $Y'$ is a losing coalition because the eight provinces in $Y'$ represent less than half of Canada’s population.
**Corollary**

The procedure to amend the Canadian Constitution is not a weighted voting system.

**Theorem** (proof omitted)

A yes–no voting system is weighted if and only if it is trade robust.

**Minority veto**

For example, we have a majority group of 5 voters and a minority group of 3 voters. The passage requires not only approval of at least 5 out of the eight voters, but also approval of at least 2 of the 3 minority voters. This system is swap robust but not trade robust.

**Hint** Consider how to make one coalition to have more total number of voters but do not have enough minority voters while the other coalition has less total number of voters but more minority voters after the trades.
Illustration

Majority group of 5 voters minority group of 3 voters

To form a winning coalition, we must have

- at least 5 out of 8 total voters
- at least 2 out of 3 minority voters

\[ X = \{M_1, M_2, M_3, m_1, m_2\} \]  \hspace{1cm} \text{Both } X \& Y \text{ are winning coalitions}
\[ Y = \{M_3, M_4, M_5, m_2, m_3\} \]  \hspace{1cm} \text{Both } X' \& Y' \text{ are losing coalitions}
\[ X' = \{M_1, M_2, M_3, M_4, M_5, m_2\} \]
\[ Y' = \{M_3, m_1, m_2, m_3\} \]

Hence, the system is NOT trade robust.

Swap robustness is easily seen since the requirement of 5 votes out of 8 voters is satisfied under one-for-one trade and the minority requirement is always satisfied by at least one of the coalitions after one-for-one trade.
Intersection of weighted voting systems and dimension theory

The procedure to amend the Canadian Constitution can be constructed by "putting together" two weighted systems.

- $W_1 = \text{collection of coalitions with 7 or more provinces}$
- $W_2 = \text{collection of coalitions representing at least half of Canada's population}$

A coalition is winning if and only if it is winning in both System I and System II.

$$W = W_1 \cap W_2.$$  

Question How to construct a non-weighted system from weighted ones?
Proposition

Suppose $S$ is a yes–no voting system for the set $V$ of voters, and let $m$ be the number of losing coalitions in $S$. Then it is possible to find $m$ weighted voting systems with the same set $V$ of voters such that a coalition is winning in $S$ if and only if it is winning in every one of these $m$ weighted systems.

Proof

For each losing coalition $L$ in $S$, we construct a weighted voting system. Let $|L|$ be the number of voters in $L$. Every voter in $L$ is given weight $-1$. Every voter not in $L$ is given weight 1. Quota is set at $-|L| + 1$. The possibility that a coalition is losing if and only if it contains exactly all the voters in this particular losing coalition $L$. Then every other coalition is a winning coalition in this weighted voting system.
• If a coalition is winning in $S$, then it is winning in each of these weighted systems.

• Conversely, if a coalition is winning in each of these weighted systems, then it is a winning coalition in $S$ (a losing coalition in $S$ must lose in one of these weighted voting systems).

Unfortunately, this procedure is an enormously inefficient since there may be too many losing coalitions.

**Definition**

A yes–no voting system is said to be dimension $k$ if and only if it can be represented as the intersection of exactly $k$ weighted voting systems, but not as the intersection of $k−1$ weighted voting systems. For example, the procedure to amend the Canadian Constitution is of dimension 2 since the passage requires two separate weighted voting systems with regard to “population” and “provinces”.
Proposition

The US federal system has dimension 2.

System I: quota = 67

- weight 0 to House Representative
- weight 1 to Senator
- weight 0.5 to Vice President
- weight 16.5 to President

System II: quota = 290

- weight 1 to House Representative
- weight 0 to Senator
- weight 0 to Vice President
- weight 72 to President

System I is meant for the Senate (veto power of the President and tie breaker role of the Vice President; sum of weights of P & VP = 17 and weight of $P \geq 16$). System II is meant for the House, with veto power of the President.
Let $X$ be a minimal winning coalition. Then $X$ is one of the following 3 kinds of coalitions:

1. $X$ consists of 218 House members, 51 Senators and the President

2. $X$ consists of 218 House members, 50 Senators, the Vice President and the President

3. $X$ consists of 290 House members and 67 Senators.

All these 3 kinds of coalitions achieve quota in both Systems I and II.
Conversely, how to find the minimal winning coalitions that satisfy both weighted voting systems?

*Hint* (With the President)

In System I, weight of the President is 16.5, so the other members of $X$ must contribute at least weight 50.5 to the total System I. $X$ must contain either 51 (or more) senators or at least 50 senators and the Vice President.

Looking at System II, which is at least 290 including the 72 contributed by the President. So $X$ must also contain at least $290 - 72 = 218$ House Representatives.
1.2 Power indexes

1. One looks at all possible orderings of the $n$ players, and consider this as all of the potential ways of building up toward a winning coalition. For each one of these permutations, some unique player joins and thereby turns a losing coalition into a winning one, and this voter is called the *pivot*.

2. In the sequence of player $x_1, x_2, \ldots, x_{i-1}, x_i, \ldots, x_n$, \{x_1, x_2, \ldots, x_i\} is a winning coalition but \{x_1, x_2, \ldots, x_{i-1}\} is losing, then $i$ is in the *pivotal position*.

3. What is the probability that a particular voter is the pivot? The expected frequency with which a voter is the pivot, over all possible assignments of the voters, is taken to be a good indication of his voting power.
Example

The 24 permutations of the four players 1, 2, 3 and 4 in the weighted majority game $[51; 40, 30, 20, 10]$ are listed below. The "*" indicates which player is pivotal in each alignment.

\[
\begin{align*}
1 & \ 2*3 \ 4 & \ 2 & \ 1*3 \ 4 & \ 3 & \ 1*2 \ 4 & \ 4 & \ 12* \ 3 \\
1 & \ 2*4 \ 3 & \ 2 & \ 1*4 \ 3 & \ 3 & \ 1*4 \ 2 & \ 4 & \ 1 \ 3*2 \\
1 & \ 3*2 \ 4 & \ 2 & \ 3 \ 1*4 & \ 3 & \ 2 \ 1*4 & \ 4 & \ 2 \ 1*3 \\
1 & \ 3*4 \ 2 & \ 2 & \ 3 \ 4* \ 1 & \ 3 & \ 2*4 \ 1 & \ 4 & \ 2*3 \ 1 \\
1 & \ 4 \ 2* \ 3 & \ 2 & \ 4 \ 1* \ 3 & \ 3 & \ 4 \ 1* \ 2 & \ 4 & \ 3 \ 1* \ 2 \\
1 & \ 4 \ 3* \ 2 & \ 2 & \ 4 \ 3* \ 1 & \ 3 & \ 4 \ 2* \ 1 & \ 4 & \ 3 \ 2* \ 1 \\
\end{align*}
\]

For Player 1 winning coalitions consisting of 2 players. 

winning coalitions consisting of 3 players.
Shapley-Shubik power index or voting value for the $j^{th}$ player is

$$\phi_j = \frac{\text{number of sequences in which player } j \text{ is a pivot}}{n!}$$

and we write $\phi = (\phi_1, \cdots, \phi_n)$.

Here, we assume that each of the $n!$ alignments is *equiprobable*.

The power index can be expressed as

$$\phi_j = \sum \frac{(S - 1)!(n - S)!}{n!} \quad \left(\text{with } \sum_{i \in N} \phi_i = 1\right)$$

where $S = |S| = \text{number of voters in set } S$. The summation is taken over all winning coalitions $S$ for which $S - \{i\}$ is losing.
In the above example

$$
\phi_1 = 3 \frac{(3 - 1)!(4 - 3)!}{4!} + 2 \frac{(2 - 1)!(4 - 2)!}{4!} = \frac{10}{24}.
$$

Player 1 is pivotal in 3 coalitions (namely \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}) consisting of 3 players and in 2 coalitions (namely \{1, 2\}, \{1, 3\}) consisting of two players. For player 2, she is pivotal in \{1, 2, 3\} and \{1, 2, 4\} with 3 players and \{1, 2\} with 2 players. Therefore

$$
\phi_2 = 2 \frac{(3 - 1)!(4 - 3)!}{4!} + \frac{(2 - 1)!(4 - 2)!}{4!} = \frac{6}{24}.
$$

We have

$$
\phi = \frac{(10, 6, 6, 2)}{24}.
$$
Example – A bloc versus singles

Suppose we have $n$ players and that a single block of size $b$ forms. Consider the resulting weighted voting system: $[q; b, 1, 1, \ldots, 1]$. $n - b$ of these

- $n - b + 1$ is just the number of distinct orderings. The $b$ bloc will be pivotal precisely when the initial sequence of ones is of length at least $q - b$ but not more than $q - 1$.

- The $b$ bloc is pivotal when the initial sequence of ones is any of the following lengths:

  $$q - 1, q - 2, \ldots, q - b.$$
Note that there are $n - b$ ones available, so the above statement is valid provided that $n - b \geq q - 1$ and $q > b$. Under the assumption of $b < q \leq n - b + 1$, there are $b$ possible initial sequences of ones that make the bloc pivotal, so

\[
\text{Shapley-Shubik index of the block of size } b = \frac{\text{number of orderings in which } b \text{ is pivotal}}{\text{total number of distinct orderings}} = \frac{b}{n - b + 1}.
\]

The Shapley-Shubik index is higher than the percentage of votes of $b/n$. 
Banzhaf index

- Consider all significant combinations of “yes” or “no” votes, rather than permutations of the players as in the Shapley-Shubik index.
- A player is said to be marginal, or a swing or critical, in a given combination of “yes” and “no” if he can change the outcome.
- Let $b_i$ be the number of voting combinations in which voter $i$ is marginal; then $\beta_i = \frac{b_i}{\sum b_i}$. 
Assuming that all voting combinations are equally probable.

The game is \([51; 40, 30, 20, 10]\). For the second case, if Player 1 changes from \(Y\) to \(N\), then the outcome changes from “Pass” to “Fail”.

<table>
<thead>
<tr>
<th>Players</th>
<th>Pass/Fail</th>
<th>Marginal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y Y Y Y</td>
<td>P</td>
<td></td>
</tr>
<tr>
<td>Y Y Y N</td>
<td>P</td>
<td>X</td>
</tr>
<tr>
<td>Y Y N Y</td>
<td>P</td>
<td>X X</td>
</tr>
<tr>
<td>Y N Y Y</td>
<td>P</td>
<td>X X</td>
</tr>
<tr>
<td>N Y Y Y</td>
<td>P</td>
<td>X X X</td>
</tr>
<tr>
<td>Y Y N N</td>
<td>P</td>
<td>X X</td>
</tr>
<tr>
<td>Y N Y N</td>
<td>P</td>
<td>X X</td>
</tr>
<tr>
<td>Y N Y N</td>
<td>P</td>
<td>X X</td>
</tr>
</tbody>
</table>
Looking at $YYNN$ (pass) and $NYNN$ (fail), Player 1 can serve as the defector who gives the swing from Pass to Fail in the first case and Fail to Pass in the second case. We expect that the number of swings of winning into losing effected by a particular player is the same as the number of swings of losing into winning by the same player.
Example

Sometimes symmetry can save us writing out all $n!$ orderings. For example, consider the weighted majority game

$$[5; 3, 2, 1, 1, 1, 1].$$

Since the “1” players are all alike, we need to write out only $6 \cdot 5 = 30$ distinct orderings (instead of $6! = 720$):

$$
\begin{array}{ccccccc}
321111 & 231111 & 213111 & 211311 & 211131 & 211113 \\
312111 & 132111 & 123111 & 121311 & 121131 & 121113 \\
311211 & 131211 & 113211 & 112311 & 112131 & 112113 \\
311121 & 131121 & 113121 & 111321 & 111231 & 111213 \\
311112 & 131112 & 113112 & 111312 & 111132 & 111123
\end{array}
$$
Notice that the 1's pivot 12/30 of the time, but since there are four of them, each 1 pivots only 3/30 of the time. We get

\[
\text{Shapley-Shubik index} \quad \phi = \left( \frac{12}{30}, \frac{6}{30}, \frac{3}{30}, \frac{3}{30}, \frac{3}{30}, \frac{3}{30} \right) = (0.4, 0.2, 0.1, 0.1, 0.1, 0.1).
\]

Power as measured by the Shapley-Shubik index in a weighted voting game is not proportional to the number of votes cast. For instance, the player with \( \frac{3}{9} = 33\frac{1}{3}\% \) of the votes has 40% of the power.
Use the same game for the computation of the *Banzhaf index*

<table>
<thead>
<tr>
<th>Types of winning coalitions with</th>
<th>Number of ways this can occur</th>
<th>Number of swings for:</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 votes: 32</td>
<td>1</td>
<td>3 2 1</td>
</tr>
<tr>
<td>311</td>
<td>$6 = 4C_2$</td>
<td>6</td>
</tr>
<tr>
<td>2111</td>
<td>$4 = 4C_3$</td>
<td>4</td>
</tr>
<tr>
<td>6 votes: 321</td>
<td>$4 = 4C_1$</td>
<td>4</td>
</tr>
<tr>
<td>3111</td>
<td>$4 = 4C_3$</td>
<td>4</td>
</tr>
<tr>
<td>21111</td>
<td>$1 = 4C_4$</td>
<td>1</td>
</tr>
<tr>
<td>7 votes: 3211</td>
<td>$6 = 4C_2$</td>
<td>6</td>
</tr>
<tr>
<td>311111</td>
<td>$1 = 4C_1$</td>
<td>1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>22 10 24</strong></td>
</tr>
</tbody>
</table>
Remark It suffices to consider the swings only in winning coalitions in the calculation of the Banzhaf index. A defector that turns a winning coalition into a losing coalition also gives the symmetric swing that turns a losing coalition into a winning coalition.

We do not need include those winning coalitions of 8 or 9 votes, since not even the player with 3 votes can be critical to them. The numbers in the second column are from the theory of combinations. For instance, how many ways could you choose 311 from 321111?

\[
\beta = \left(\frac{22}{56}, \frac{10}{56}, \frac{6}{56}, \frac{6}{56}, \frac{6}{56}, \frac{6}{56}\right)
\approx (0.392, 0.178, 0.107, 0.107, 0.107, 0.107).
\]

Comparing this with \(\phi\), we see that the two indices turn out to be quite close in this case, with \(\beta\) giving slightly less power to the two large players and slightly more to the small players.
1.3 Case studies of power indexes calculations

United Nations Security Council

1. Big “five” – permanent member 常任理事国 each has veto power; ten “small” countries whose membership rotates.

2. It takes 9 votes, the “big five” plus at least 4 others to carry an issue.

The game is [39; 7, 7, 7, 7, 7, 1, 1, … , 1]. Why? Let $x$ be the weight of any of the permanent member and $q$ be the quota. Then

$$4x + 10 < q \quad \text{and} \quad q \leq 5x + 4$$

so that $4x + 10 < 5x + 4$ giving $x > 6$. Taking $x = 7$, we then have $38 < q \leq 39$. We take $q = 39$. 
3. A “small” country $i$ can be pivotal in a winning coalition if and only if $S$ contains exactly 9 countries including the big “five”. There are $\binom{9}{3}$ such different $S$ that contain $i$, and for each such $S$, the corresponding coefficient in the Shapley-Shubik formula for this 15-person game is $\frac{(9-1)!(15-9)!}{15!}$. Hence, $\phi_S = \binom{9}{3} \times \frac{8!6!}{15!} \approx 0.001863$. Any “big-five” has index $\phi_b = \frac{1 - 10\phi_S}{5} = 0.1963$.

4. Old Security Council before 1963, which was

$$[27; 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1, 1].$$

How to guess the voting system?

Answer for $\phi$: $\phi_b = \frac{1}{5} \cdot \frac{76}{77}; \phi_S = \frac{1}{6} \cdot \frac{1}{77}$. 
Canadian Constitutional Amendment

Investigate the voting powers exhibited in a 10-person game between the provinces, and to compare the results with the provincial populations.

The winning coalitions or those with veto power can be described as follows. In order for passage, approval is required of

(a) any province that has (or ever had) 25% of the population,
(b) at least two of the four Atlantic provinces, and
(c) at least two of the four western provinces that currently contain together at least 50% of the total western population.
Recall that a blocking coalition (holding veto power) is a subset of players whose complement is not winning. Using the current population figures, this means that veto power is held by

(d) Ontario ($O$) and Quebec ($Q$),

(e) any three of the four Atlantic ($A$) provinces (New Brunswick ($NB$), Nova Scotia ($NS$), Prince Edward Island ($PEI$), and Newfoundland ($N$)),

(f) British Columbia ($BC$) plus any one of the three prairie ($P$) provinces [Alberta ($AL$), Saskatchewan ($S$), and Manitoba ($M$)], and

(g) the three prairie provinces taken together.
In calculating the Shapley-Shubik index of Quebec or Ontario, it is necessary to list all possible winning coalitions since any of these winning coalitions must contain Quebec and Ontario.

### Winning Provincial Coalitions

<table>
<thead>
<tr>
<th>Type</th>
<th>S</th>
<th>s</th>
<th>No. of such S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1P, 2A, BC, Q, O</td>
<td>6</td>
<td>18</td>
</tr>
<tr>
<td>2</td>
<td>2P, 2A, BC, Q, O</td>
<td>7</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>3P, 2A, Q, O</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>1P, 3A, BC, Q, O</td>
<td>7</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>3P, 2A, BC, Q, O</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>2P, 3A, BC, Q, O</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>3P, 3A, Q, O</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>1P, 4A, BC, Q, O</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>3P, 3A, BC, Q, O</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>10</td>
<td>2P, 4A, BC, Q, O</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>3P, 4A, Q, O</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>3P, 4A, BC, Q, O</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

Total: 88
Ontario’s Shapley-Shubik index

\[ \varphi_O = \frac{[18(5!4!) + 36(6!3!) + 25(7!2!) + 8(8!1!) + 1(9!0!)]}{(10!)} = \frac{53}{168} \]

- There are 18 winning coalitions that contain 6 provinces. In order that Ontario serves as the pivotal player, 5 provinces are in front of her and 4 provinces are behind her. This explains why there are altogether 18(5!4!) permutations of 6-province winning coalitions.

- Ontario and Quebec are equivalent in terms of influential power (though their populations are different).
**British Columbia**

Listing of all winning coalitions that upon deleting British Columbia the corresponding coalition becomes losing. These are the winning coalitions that British Columbia can serve as the pivotal player.

<table>
<thead>
<tr>
<th>Type</th>
<th>S</th>
<th>s</th>
<th>No. of such S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1P, 2A, BC, Q, O</td>
<td>6</td>
<td>(3C_1 \times 4C_2 = 18)</td>
</tr>
<tr>
<td>2</td>
<td>1P, 3A, BC, Q, O</td>
<td>7</td>
<td>(3C_1 \times 4C_3 = 12)</td>
</tr>
<tr>
<td>3</td>
<td>1P, 4A, BC, Q, O</td>
<td>8</td>
<td>(3C_1 \times 4C_4 = 3)</td>
</tr>
<tr>
<td>4</td>
<td>2P, 2A, BC, Q, O</td>
<td>7</td>
<td>(3C_2 \times 4C_2 = 18)</td>
</tr>
<tr>
<td>5</td>
<td>2P, 3A, BC, Q, O</td>
<td>8</td>
<td>(3C_2 \times 4C_3 = 12)</td>
</tr>
<tr>
<td>6</td>
<td>2P, 4A, BC, Q, O</td>
<td>9</td>
<td>(3C_2 \times 4C_4 = 3)</td>
</tr>
</tbody>
</table>

- Note that we exclude those coalitions with 3 prairie provinces since the deletion of British Columbia does not cause the coalition to become losing.

\[
\phi_{BC} = \frac{18(5!4!) + 30(6!3!) + 15(7!2!) + 3(8!1!)}{10!}.
\]
Atlantic provinces

We consider winning coalitions that contain a particular Atlantic province and one of the three other Atlantic provinces.

$$\phi_{Asp} = \frac{9(5!4!) + 12(6!3!) + 3(7!2!)}{10!}.$$
**Prairie provinces**

We consider winning coalitions that contain

(i) a particular Prairie province and British Columbia

(ii) a particular Prairie province and two other prairie provinces

<table>
<thead>
<tr>
<th>Type</th>
<th>$S$</th>
<th>$s$</th>
<th>No. of such $S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$P_{sp}, 2A, BC, Q, O$</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>$P_{sp}, 3A, BC, Q, O$</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>$P_{sp}, 4A, BC, Q, O$</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$P_{sp}, 2P, 2A, Q, O$</td>
<td>7</td>
<td>6</td>
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<tr>
<td>5</td>
<td>$P_{sp}, 2P, 3A, Q, O$</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>$P_{sp}, 2P, 4A, Q, O$</td>
<td>9</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
\phi_{P_{sp}} = \frac{6(5!4!) + 10(6!3!) + 5(7!2!) + 8!1!}{10!}.
\]
<table>
<thead>
<tr>
<th>Province</th>
<th>$\varphi$ (in %)</th>
<th>% Population</th>
<th>$\varphi$/Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>BC</td>
<td>12.50</td>
<td>9.38</td>
<td>1.334</td>
</tr>
<tr>
<td>AL</td>
<td>4.17</td>
<td>7.33</td>
<td>0.570</td>
</tr>
<tr>
<td>S</td>
<td>4.17</td>
<td>4.79</td>
<td>0.872</td>
</tr>
<tr>
<td>M</td>
<td>4.17</td>
<td>4.82</td>
<td>0.865</td>
</tr>
<tr>
<td>(4 Western)</td>
<td>(25.01)</td>
<td>(26.32)</td>
<td>(0.952)</td>
</tr>
<tr>
<td>O</td>
<td>31.55</td>
<td>34.85</td>
<td>0.905</td>
</tr>
<tr>
<td>Q</td>
<td>31.55</td>
<td>28.94</td>
<td>1.092</td>
</tr>
<tr>
<td>NB</td>
<td>2.98</td>
<td>3.09</td>
<td>0.965</td>
</tr>
<tr>
<td>NS</td>
<td>2.98</td>
<td>3.79</td>
<td>0.786</td>
</tr>
<tr>
<td>PEI</td>
<td>2.98</td>
<td>0.54</td>
<td>5.53</td>
</tr>
<tr>
<td>N</td>
<td>2.98</td>
<td>2.47</td>
<td>1.208</td>
</tr>
<tr>
<td>(4 Atlantic)</td>
<td>(11.92)</td>
<td>(9.89)</td>
<td>(1.206)</td>
</tr>
</tbody>
</table>

- British Columbia has a higher index value per capita compared to other Western provinces.
Power of the major stockholders

• Consider a corporation with one major stockholder $X$ who controls 40% of the stock, and suppose the remainder is split evenly among 60 other stockholders, each having 1%.

• There are 61 players. Since the 60 minor stockholders are symmetric, there are only 61 distinct orderings, depending only on the position of $X$.

• Of these 61 orderings, $X$ will pivot if he appears in positions 12 – 51 inclusive (if we assume that approval must be by an amount just over 50%), i.e. $40/61$ of the time.
• Now suppose $X$ still controls 40% of the stock, but the remainder is split evenly among 600 other stockholders, each controlling 0.1%.

• Of the 601 distinct orderings, $X$ will pivot if he appears in positions 102 – 501, i.e., $400/601$ of the time. Clearly, as the number of minor stockholders gets very large, $X$’s share of the power (as measured by Shapley-Shubik index) approaches $2/3$. 
Oceanic weighted voting game

- Let there be one major player $X$ controlling 40% of the vote, with the remaining 60% held by an infinite “ocean” of minor voters.

- Think of the minor voters lined up as points in a line segment of length 0.6, as they come to join a coalition in support of some proposal.

- Voter $X$ can join at any point along this line segment. He will pivot if he joins after 0.1 and before (or at) 0.5. His Shapley-Shubik index is

$$\phi_X = \frac{\text{Length of segment in which } X \text{ pivots}}{\text{Total length of segment}} = \frac{0.5 - 0.1}{0.6} = \frac{2}{3}.$$
Example

There are two major voters and an ocean of minor voters. Suppose voter $X$ holds $3/9$ of the total vote, and voter $Y$ holds $2/9$, with the other $4/9$ held by the ocean of minor voters. The minor voters line up along a line segment of length $4/9$. $X$ and $Y$ can join at any point along this line segment:
• We can represent geometrically the positions at which $X$ and $Y$ join by giving a single point in a square of side $4/9$, whose horizontal coordinate is $X$’s position and whose vertical coordinate is $Y$’s:

• The point is above the diagonal of the square if $X$ joins before $Y$, and below the diagonal if $Y$ joins before $X$. 
• Which points in the square correspond to orderings for which $X$ or $Y$ pivots?

• Divide the square into regions where $X$ pivots, $Y$ pivots, or voters in the ocean ($O$) pivot:
$X$ joins before $Y$ (points that lie above the diagonal $\Leftrightarrow a < b$)

1. If $X$ joins before $3/18$, it can never be pivotal since

\[ a + \frac{3}{9} < \frac{1}{2} \quad \text{for } a < \frac{3}{18}. \]

$Y$ can be pivotal provided that $b < 3/18$.

2. The oceanic voters all combined together cannot pass the bill since they hold $4/9$ which is less than 50% of the votes. If $X$ joins after $1/2 - \frac{3}{9} = \frac{3}{18}$, then $X$ pivots.
Calculate the Shapley-Shubik indices for $X$ or $Y$ by calculating the area of the region in which $X$ or $Y$ pivots and dividing by the total area of the square.

$$\phi_X = \frac{(5/18)^2}{(4/9)^2} = \frac{25}{64} \approx 0.391$$

$$\phi_Y = \frac{(3/18)^2}{(4/9)^2} = \frac{9}{64} \approx 0.141.$$  

with the other $30/64$ being shared by the players in the ocean. Interestingly, the major stockholder $X$ has higher power relative to his percentage holding. The gain comes at the expense of $Y$. (做老大總是最好)

- If there are three major players in an oceanic game, we represent orderings as points in a cube, and calculate the volumes of the regions where each of the major players pivots.
Shapley-Shubik index of the President

Consider a mini-federal system: 6 Senators, 6 House Representatives and the President. Passage in the mini-federal system requires two-thirds of both houses or half of each house and the President. Assuming there is no tie breaker (Vice President) in the Senate.

Under what conditions that the President is pivotal in an ordering of the 13 voters in the mini-federal system? He must be preceded by at least 3 members of the House and at least 3 members of the Senate, but by fewer than 4 members of at least one of the two chambers.
1. Three House members and three senators precede the president in the ordering.

2. Three House members and four senators precede the president in the ordering.

3. Three House members and five senators precede the president in the ordering.

4. Three House members and six senators precede the president in the ordering.
5. Four House members and three senators precede the president in the ordering.

6. Five House members and three senators precede the president in the ordering.

7. Six House members and three senators precede the president in the ordering.
For example, the first ordering can be built in a four-step process:

Step 1: Choose three of the six House members to precede the president in the ordering.

Step 2: Choose three of the six senators to precede the president in the ordering.

Step 3: Choose an ordering of the six people from steps 1 and 2 who will precede the president.

Step 4: Choose an ordering of the six people (the remaining House members and senators) who will come after the president.

The total number of orderings = \( \binom{6}{3} \binom{6}{3} 6!6! \).
• A similar argument yields a similar expression for the number of orderings that arise in the other six entries on the list.

• The sum of these seven expressions gives us the total number of orderings of the thirteen voters for which the president is pivotal.

• To obtain the Shapley-Shubik index of the president in this mini-federal system, we simply divide that result by 13!, giving

\[
\frac{\binom{6}{3} \binom{6}{3} 6!6! + 2 \binom{6}{3} \binom{6}{4} 7!5! + 2 \binom{6}{3} \binom{6}{5} 8!4! + 2 \binom{6}{3} \binom{6}{6} 9!3!}{13!}.
\]
**Actual federal system** (with the Vice President ignored) When the number of Representatives $\geq 290$ and the number of Senator $\geq 67$, the President cannot be pivotal.

\[
\binom{435}{218} \left[ \binom{100}{51} (218 + 51)! (535 - 218 - 51)! + \cdots \\
+ \binom{100}{100} (218 + 100)! (535 - 218 - 100)! \right] \\
+ \cdots \\
+ \binom{435}{289} \left[ \binom{100}{51} (289 + 51)! (535 - 289 - 51)! + \cdots \\
+ \binom{100}{100} (289 + 100)! (535 - 289 - 100)! \right] \\
+ \binom{435}{290} \left[ \binom{100}{51} (290 + 51)! (535 - 290 - 51)! + \cdots \\
+ \binom{100}{66} (290 + 66)! (535 - 290 - 66)! \right] \\
+ \cdots \\
+ \binom{435}{435} \left[ \binom{100}{51} (435 + 51)! (535 - 435 - 51)! + \cdots \\
+ \binom{100}{66} (435 + 66)! (535 - 435 - 66)! \right]
\]

When divided by $536!$, we obtain the Shapley-Shubik index of the President as $\phi = 0.16047$. 
<table>
<thead>
<tr>
<th>House</th>
<th>Senate</th>
</tr>
</thead>
<tbody>
<tr>
<td>218</td>
<td>51 to 100</td>
</tr>
<tr>
<td>289</td>
<td>51 to 100</td>
</tr>
<tr>
<td>290</td>
<td>51 to 66</td>
</tr>
<tr>
<td>435</td>
<td>51 to 66</td>
</tr>
</tbody>
</table>
With the inclusion of the Vice President, we need to add the case where 50 Senators and the Vice President say “yes”. The following terms should be added:

\[
\binom{435}{218}\binom{100}{50}(218 + 51)!(535 − 218 − 50)!
\]
\[
+ \binom{435}{219}\binom{100}{50}(219 + 51)!(535 − 219 − 50)!
\]
\[
+ \cdots + \binom{435}{435}\binom{100}{50}(435 + 51)!(535 − 435 − 50)!
\]

The denominator is modified to be 537!

- For the first term, we choose 218 Representatives from 435 of them and 50 Senators from 100 of them. There are 218 + 50 + 1 “yes” voters and 535 − 218 − 50 “no” voters. We have (218 + 50 + 1)! orderings before the President and (535 − 218 − 50)! orderings after the President.
Banzhaf index of the President

Let $S$ denote the number of coalitions within the Senate that contain more than two-thirds of the members of the Senate:

$$S = \binom{100}{67} + \cdots + \binom{100}{100}.$$ 

Let $s$ denote the number of coalitions within the Senate that contain equal and more than one-half of the members of the Senate:

$$s = \binom{100}{50} + \binom{100}{51} + \cdots + \binom{100}{100}.$$ 

Let $H$ denote the number of coalitions within the House that contain more than two-thirds of the members of the House:

$$H = \binom{435}{290} + \cdots + \binom{435}{435}.$$
Let $h$ denote the number of coalitions within the House that contain more than one-half of the members of the House:

$$h = \binom{435}{218} + \cdots + \binom{435}{435}.$$

- We count the number of winning coalitions with the President such that the defection of the President turns winning into losing. Write $N_P = \text{total number of winning coalitions in which the president is critical}.$

$$N_P = \left[ \binom{435}{218} + \cdots + \binom{435}{289} \right] \left[ \binom{100}{50} + \binom{100}{51} + \cdots + \binom{100}{100} \right]$$

$$+ \left[ \binom{435}{290} + \cdots + \binom{435}{435} \right] \left[ \binom{100}{50} + \binom{100}{51} + \cdots + \binom{100}{66} \right]$$

$$= (h - H) \times s + H \times (s - S) = h \times s - H \times S.$$

- The total number of winning coalitions in which the Vice President is critical $= N_V = \binom{100}{50} \times h.$
Given a particular senator, we find the number of winning coalitions such that this senator is critical. Without the President, the number of critical swings effected by this senator is \( \binom{99}{66} \times H \); and with the President, (but without the Vice president), the number of critical swing is \( \binom{99}{50} \times h \). The total number of winning coalitions to which the chosen senator is critical is \( N_S = \binom{99}{66} \times H + \binom{99}{50} \times h + \binom{99}{49} \times h \). The last term corresponds to the presence of both the President and Vice President.

In a similar manner, the total number of winning coalitions to which a particular Representative is critical is \( N_R = \binom{434}{289} \times S + \binom{434}{217} \times s \).
Total number of critical swings by all players

\[ N = 100 \times N_S + 435 \times N_R + N_P + N_V \]

Banzhaf index for any senator

\[ \text{number of winning coalitions to which the chosen senator is critical} \]

\[ \text{total number of critical swings} \]

\[ = \frac{N_S}{N} \]

Banzhaf index for the President \( = \frac{N_P}{N} \).