MATH4321 — Game Theory

Topic One: Strategies and equilibriums of the games

1.1 Definitions and examples
   - Normal form of a game
   - Extensive form of a game: Russian Roulette
   - Examples: market entry and chain store paradox, dollar auction

1.2 Saddle points and dominant-strategy equilibrium
   - Value of a game under pure strategies
   - Saddle points for zero-sum games
   - Best responses and dominant strategies: Prisoner’s dilemma
   - Iterated dominance: Battle of the Bismarck Sea

1.3 Nash equilibrium models
   - Characterization
   - Examples: Modified prisoner’s dilemma; Cuban crisis; battle of sexes; coordination games; Yom Kippur war
1.1 Definitions and examples

Essential elements of a game

- *Players*: Individuals who make decisions. Each player’s goal is to maximize his utility by choice of actions.

  - Nature is a pseudo-player who takes random actions at specified points in the game with specified probabilities.

- *Action (move)*: Choice $a_i$ made by player $i$.

  - Player $i$’s action set $A_i$ contains the entire set of actions available.

  - An action combination is an ordered set $a = (a_1, a_2, \ldots, a_n)$ of one action for each of the $n$ players in the game.

Even if the players always took the same actions, the random move by Nature means that the model would yield more than just one prediction (different realizations of a game).
The players devise strategies \((s_1, s_2, \ldots, s_n)\) that pick actions depending on the information that has arrived at each moment so as to maximize their payoffs.

By Player \(i\)'s payoff \(\pi_i(s_1, s_2, \ldots, s_n)\), we mean either

(i) The utility player \(i\) receives after all players have picked their strategies and Nature has realized its move, and the game has been played out.

(ii) The expected utility he receives as a function of the mixed s-strategies chosen by himself and the other players.

The two definitions of payoff are different and distinct, one is the actual payoff and the other is the expected payoff.

The combination of strategies is known as the equilibrium. Given an equilibrium, the modeller can see what actions come out of the conjunction of all the players’ plans — outcome of the game.
Examples of games

1. **OPEC members choosing their annual output**
   Saudi Arabia knows that Kuwait’s oil output is based on Kuwait’s forecast of Saudi output. This is the very nature of a game.

2. **General Motors purchasing steel from USX**
   The two companies realize that the quantities of steel traded by each of them affect the price.

3. **A board of directors setting up a stock plan for the CEO**
   The directors choose the plan anticipating the effects on the action of the CEO.

4. **Nuts manufacturer and bolts counterpart deciding whether to use metric or American standards**
   The two manufacturers are not in conflict, but the actions of one do affect the desired actions of the others.
Examples of non-games

1. *The US Air Force hiring jet fighter pilots*

   Each pilot makes his employment decision without paying regard for the impact on the Air Force’s policies.

   This may become a game if the Air Force faces a pilots union.

2. *An electric company deciding whether to build a new power plant given its estimate of demand for electricity in 10 years*

   This is more appropriate for the use of decision theory — the careful analysis of how one makes a decision when faced with uncertainty or an entire sequences of decisions that interact with each other.

   Change to a game if the public utility commission pressures the company to change its generating capacity.
Strategies (action plans)

Player $i$’s strategy $s_i$ is a rule that tells him which action to choose at each instant of the game, given his information set.

An action is *physical* while a strategy is *mental* (unobservable).

Player $i$’s *strategy set* or *strategy space* $S_i = \{s_i\}$ is the set of strategies available to him.

A *strategy profile* $s = (s_1, s_2, \ldots, s_n)$ is an ordered set consisting of one strategy for each of the $n$ players in the game.

Since the information set includes whatever the player knows about the previous actions of other players, the strategy tells him how to react to their actions and what actions to pick in every conceivable situation.
**Payoff or game matrix of a zero sum game**

In a zero sum game, if $a_{ij}$ is the amount received by Player I, then Player II receives $-a_{ij}$.

<table>
<thead>
<tr>
<th>player I</th>
<th>player II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy 1</td>
<td>$a_{11}$</td>
</tr>
<tr>
<td>Strategy 2</td>
<td>$a_{21}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>Strategy $n$</td>
<td>$a_{n1}$</td>
</tr>
</tbody>
</table>

Player I (Row player) wants to choose a strategy to maximize the payoff in the game matrix, while Player II (Column player) wants to minimize the payoff.
Example Cat versus Rat in a maze

If Cat finds Rat, Cat gets 1; and otherwise, Cat gets 0. For example, when Cat chooses $dcba$ and Rat $abcd$, they will meet at an intersection point.
Example - Nim with two piles of two coins

2 × 2 Nim represented in an extensive form - a tree representing the successive moves of players.

- Four pennies are placed in two piles of two pennies each.

- Each player chooses a pile and decides to remove one or two pennies.

- The loser is the one who removes the last penny (pennies).
### Strategies for player I

1. Play (1,2) then, if at (0,2) → (0,1).
2. Play (1,2) then, if at (0,2) → (0,0).
3. Play (0,2).

### Strategies for player II

1. If at (1,2) → (0,2); if at (0,2) → (0,1)
2. If at (1,2) → (1,1); if at (0,2) → (0,1)
3. If at (1,2) → (1,0); if at (0,2) → (0,1)
4. If at (1,2) → (0,2); if at (0,2) → (0,0)
5. If at (1,2) → (1,1); if at (0,2) → (0,0)
6. If at (1,2) → (1,0); if at (0,2) → (0,0)
• Player II would never play Strategy 5 (dominated strategy for Player II).

• No matter what Player I does, Player II wins +1 by playing S-strategy 3.

In that sense, this game is quite boring since Player II always wins. The value of the game is $-1$. 

\[
\begin{array}{c|ccccccc}
\text{player I/player II} & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 \\
2 & -1 & 1 & -1 & -1 & 1 & -1 \\
3 & -1 & -1 & -1 & 1 & 1 & 1 \\
\end{array}
\]
Randomization of strategies

In the Evens or Odds game, each player decides to show 1, 2 or 3 fingers. Player I wins $1 if the sum of fingers is even; otherwise, Player II wins $1. The game matrix is shown below.

<table>
<thead>
<tr>
<th></th>
<th>Odds</th>
</tr>
</thead>
<tbody>
<tr>
<td>I/II</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

How should each player decide what number of fingers to show? If a player always plays the same strategy, then the opponent can always win the game. The only alternative is to mix the strategies. How?

The equilibrium strategies of player 1 (player 2 will adopt the same strategies due to symmetry) are to play 50% chance of strategy 2 and combined 50% chance of strategy 1 and strategy 3 (since both these two strategies are equivalent in payoff).
Example — Russian Roulette

The two players are faced with a 6-shot pistol loaded with one bullet. Both players put down $1 and Player I goes first.

- At each play of the game, a player has the option of putting an additional $1 into the pot and passing; or not adding to the pot, spinning the chamber and firing at his own head.

- If Player I chooses the option of spinning and survives, then he passes the gun to Player II, who has the same two options. Player II decides what to do, carries it out, and the game ends.
• If Player I fires and survives and then Player II passes, both will split the pot. In effect, Player II will pay Player I $0.5.

• If Player I chooses to pass and Player II chooses to fire, then if Player II survives, he takes the pot.

For Player I, he either spins (I1) or passes (I2).

<table>
<thead>
<tr>
<th></th>
<th>II1</th>
<th>II2</th>
<th>II3</th>
<th>II4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>If I1, then P; If I2, then S.</td>
<td>If I1, then P; If I2, then P.</td>
<td>If I1, then S; If I2, then P.</td>
<td>If I1, then S; If I2, then S.</td>
</tr>
</tbody>
</table>
Nature comes in since the player survives with probability $\frac{5}{6}$ if he chooses to spin. One needs to consider the expected payoff based on the law of probabilities.
The expected payoff to I is

I1 against II3: \[ \frac{5}{6} \left( \frac{5}{6} (0) + \frac{1}{6} (1) \right) + \frac{1}{6} (-1) = -\frac{1}{36}, \text{ and} \]

I2 against II3: \[ \frac{5}{6} (-2) + \frac{1}{6} (1) = -\frac{3}{2}. \]

I1 against II1: \[ \frac{5}{6} \left( \frac{1}{2} \right) + \frac{1}{6} (-1) = \frac{1}{4}, \text{ and} \]

I2 against II1: \[ \frac{5}{6} (-2) + \frac{1}{6} (1) = -\frac{3}{2}. \]
### Game matrix of the Russian Roulette

<table>
<thead>
<tr>
<th>I/II</th>
<th>II1</th>
<th>II2</th>
<th>II3</th>
<th>II4</th>
</tr>
</thead>
<tbody>
<tr>
<td>I1</td>
<td>1</td>
<td>1</td>
<td>1/36</td>
<td>1/36</td>
</tr>
<tr>
<td></td>
<td>4/4</td>
<td></td>
<td>36</td>
<td>36</td>
</tr>
<tr>
<td>I2</td>
<td>-3</td>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2/2</td>
<td></td>
<td>2/2</td>
<td></td>
</tr>
</tbody>
</table>

Player II will only play II3. Player I will play I1 if Player II plays II3. The expected payoff to Player I is $-\frac{1}{36}$.

Even though the two players do not move simultaneously, they choose their strategies simultaneously at the start of the game.
Dry cleaners game — Nonzero sum game

To decide whether to start a dry cleaning store under the presence of an existing dry cleaner.

**Concerns of the NewCleaner**

- Status of economy (normal or recession) - determines the prices that customers are willing to pay for dry cleaning.

- Price war - OldCleaner may respond with a price war.
(a) Normal economy

<table>
<thead>
<tr>
<th></th>
<th>OldCleaner</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low price</td>
</tr>
<tr>
<td><strong>Enter</strong></td>
<td>-100, -50</td>
</tr>
<tr>
<td><strong>NewCleaner</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Stay Out</strong></td>
<td>0, 50</td>
</tr>
</tbody>
</table>

(b) Recession

<table>
<thead>
<tr>
<th></th>
<th>OldCleaner</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low price</td>
</tr>
<tr>
<td><strong>Enter</strong></td>
<td>-160, -110</td>
</tr>
<tr>
<td><strong>NewCleaner</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Stay Out</strong></td>
<td>0, -10</td>
</tr>
</tbody>
</table>

Payoffs to: (NewCleaner, OldCleaner) in thousands of dollars.
Lay out information and actions in an order of play

At the time that it chooses its price, OldCleaner knows NewCleaner’s decision about entry. How about recession?

Nature picks demand: Recession with probability 0.3, or Normal with probability 0.7.

1. If both firms know, Nature moves before NewCleaner.

2. If only OldCleaner knows, Nature moves after NewCleaner.

3. If neither firm knows, Nature moves at the end of the game.

Outcome of the NewCleaner

1. Action: Enter or Stay Out

2. Payoff: one element from the set \{0, 100, −100, 40, −160\}.
Construction of a *decision tree* from the rules of the game — graphical display of the order of play
The strategy set for NewCleaner is Enter, Stay Out.

In a naive approach that does not include consideration of the best strategy of the OldCleaner, we compute the expected payoff to the NewCleaner when the OldCleaner has 50 – 50 chance of choosing High Price or Low Price, where

\[
0.5\left\{ \left[ 0.7(100) + 0.3(40) \right] + \left[ 0.7(-100) + 0.3(-160) \right] \right\} = -18.
\]

This is worse than the zero payoff under Stay Out. Hence, the prediction is that the NewCleaner will stay out.

Should the OldCleaner decide on High Price or Low Price based on probabilistic ground or consider the expected payoff for each strategy?
The Dry Cleaners Game as a game tree.
Suppose NewCleaner has entered, the OldCleaner chooses High Price since

\[
\frac{0.7(100) + 0.3(40)}{0.7(-100) + 0.3(-160)} > 0
\]

Therefore, the OldCleaner chooses High Price with 100% chance (instead of 50 – 50 chance). The NewCleaner predicts an expected payoff of \(0.7(100) + 0.3(40) = 82\) under “Enter”.

The NewCleaner will have a payoff of zero (worse than 82) under “Stay Out”.

The equilibrium strategies of the two players are “Enter for NewCleaner” and “High Price for OldCleaner”.
Chain store paradox

Market entry game

Suppose the opponent does not enter, the status quo gives the monopolist a payoff of 2. If the monopolist does not fight, each store shares a payoff of 1. If a price war is conducted, both stores have negative payoff of $-1$. 

\[
\begin{array}{c}
\text{opponent} \\
\begin{array}{c}
\text{no entry} \\
(2,0) \\
\text{no fight} \\
(1,1) \\
\text{fight} \\
(-1,-1)
\end{array}
\end{array}
\]
Backward induction argument:

In the second step, it is better for the monopolist to choose not to fight for a price war.

Knowing that the monopolist chooses not to fight, then the intruder would choose to enter to obtain the payoff of 1 (instead of zero if not entering).

Even if the monopolist were to threaten to fight, this threat would not be credible.

Deterrence theory

Suppose the monopolist has stores in 20 cities, it seems reasonable to argue that by fighting with the first intruder, the monopolist (though with a loss) would prevent other intruders to enter market in other cities.
However, the chain store paradox would contradict the logic of backward induction presented below.

- Consider the last city, the solution (entry, no fight) prevails since there are no future possible games that could change the situation. In the last but one city, since the solution (entry, no fight) to the last game is known anyway, it cannot influence the last but one game, so the same solution (entry, no fight) prevails. Continuing backward in all earlier games, we should obtain the same solution. The same answer remains valid for any number of cities.

Can we assume that all players are rational and perform the perfect reasoning all the way through? Deterrence is a reality since it appears to be reasonable at the initial phase of reasoning.
Dollar auction game — conflict escalation

Auction $1 bill for the highest bidder.

The highest bidder wins $1 less the amount of the bid.

The second highest bidder must pay the auctioneer her bid as well.

- Everyone in the game has the same desire of winning a buck for 10 to 15 cents.

- In real situations, the number of players typically decreases until there are two bidders. The motivation of the remaining two bidders changes from a desire to maximize return to one of minimizing losses. The auction often goes well above $1 (as high as $5 or $6 or even $30).
Analogy in the business world: The fierce competition in a contested merger or acquisition has led many acquiring companies into a deal that made no sense at all - quiet divestment 2 to 3 years later.

Conflict escalation: unconscious transform triggered by the deepening conflict.

Let $s$ denote the number of units in the stakes. For example, if the units are dimes (10 cents) and the stakes one dollar, then $s = 10$.

Let $b$ denote the bankroll in units of money. For example, $b = 120$ corresponds to both bidders having a bankroll of $12$ if the units are dimes.
Game tree analysis for $s = b = 3$

“p” denotes “pass” in the terminal nodes.

We adopt the backward induction procedure and assume rational play.

Rational play means that each is aware that the other is aware that he operates in this rational way, and so on.
The first and second entries are the gain/loss of the first and second players. Once the bid reaches 3, the other player cannot bid higher since \( b = 3 \).

The optimal strategies are for Player 1 to bid one unit and for Player 2 to pass.

Question: what is the optimal opening bid? When \( s = 20 \) and \( b = 100 \), the optimal opening bid is 5, that is, 25 cents.
1.2 Saddle points and dominant-strategy equilibrium

Value of a zero sum game under pure strategies

Given the game matrix \( A = (a_{ij}) \), from Player 1’s perspective, Player 1 assumes that Player 2 chooses a column \( j \) so as to

\[
\text{Minimize } a_{ij} \text{ over } j = 1, 2, \ldots, m
\]

for any given row \( i \) (\( i^{th} \) strategy of Player 1). Player 1 can choose the row \( i \) that will maximize this. That is, Player 1 can guarantee that in the worst scenario he can receive at least

\[
v^- = \max_{i=1,\ldots,n} \min_{j=1,\ldots,m} a_{ij}.
\]

This is the lower value of the game (or Player 1’s game floor).
From Player 2’s perspective, Player 2 assumes that Player 1 chooses a row so as to

$$\text{Maximize } a_{ij} \text{ over } j = 1, 2, \ldots, n$$

for any given column $j = 1, 2, \ldots, m$. Player 2 can choose the column $j$ so as to guarantee a loss of no more than

$$v^+ = \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} a_{ij}.$$

This is the \textit{upper value} of the game (or Player 2’s loss ceiling).

Based on the \textit{minimax criterion}, a player chooses a strategy (among all possible strategies) to minimize the maximum damage the opponent can cause.
In general, $v^- \leq v^+$. However, if there exists a saddle point $(i^*, j^*)$ where row min along row $i^*$ coincides with column max along column $j^*$, then $v^- = v^+$ and this is simply the value of the entry at the saddle point $(i^*, j^*)$. Based on the minimax criterion, both Row and Column players would choose row $i^*$ and column $j^*$, respectively, as their optimal strategies.

\[
\begin{array}{cccc|c}
 a_{11} & a_{12} & \cdots & a_{1m} & \rightarrow \min_j a_{1j} \\
 a_{21} & a_{22} & \cdots & a_{2m} & \rightarrow \min_j a_{2j} \\
 \vdots & \vdots & \cdots & \vdots \\
 a_{n1} & a_{n2} & \cdots & a_{nm} & \rightarrow \min_j a_{nj} \\
\end{array}
\]

\[
\begin{array}{c}
\downarrow & \downarrow & \cdots & \downarrow \\
\max_i a_{i1} & \max_i a_{i2} & \cdots & \max_i a_{im}
\end{array}
\]

$v^- = \text{largest min}$

$v^+ = \text{smallest max}$
Proof of $v^- \leq v^+$

Observe that

$$v^- = \max_i \min_j a_{ij} \leq \max_i a_{ij}.$$  

The above inequality is independent of $j$, so it remains to be valid when we take $\min_j (\max_i a_{ij})$, which is precisely $v^+$. Hence, $v^- \leq v^+$.

The game is said to have a value if $v^- = v^+$, and we write

$$v = v(A) = v^+ = v^-.$$  

In a later lemma, we show that $v^+ = v^-$ is equivalent to have the existence of a saddle point (may not be unique) under pure strategies (players do not randomize the choices of strategies).
Saddle point in pure strategies

We call a particular row $i^*$ and column $j^*$ a saddle point in pure strategies of the game

$$a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j},$$

for all rows $i = 1, 2, \ldots, n$ and columns $j = 1, 2, \ldots, m$.

We can spot a saddle point in a matrix (if there is one) as the entry that is simultaneously the smallest in a row and largest in a column.

In words, $(i^*, j^*)$ is a saddle point if when Player 1 deviates from row $i^*$, but Player 2 still plays $j^*$, then Player 1 will get less (largest value in the strategy chosen by Column). Vice versa, if Player 2 deviates from column $j^*$ but Player 1 sticks with $i^*$, then Player 1 will do better (smallest value in the strategy chosen by Row).

When a saddle point exists in pure strategies, if any player deviates from playing his part of the saddle, then the other player can take advantage and improve his payoff since the player is not minimizing the maximum damage made by the opponent.
Consider the following two-person zero sum game

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td>12</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>5</td>
<td>1</td>
<td>7</td>
<td>-20</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>-16</td>
<td>0</td>
<td>0</td>
<td>16</td>
</tr>
</tbody>
</table>

Note that Rose wants the payoff to be large (16 would be the best) while Colin wants the payoff to be small (−20 the smallest).

\[ v^+ = \min(12, 2, 7, 16) = 2 \] and \[ v^- = \max(-1, -20, 2, -16) = 2. \]
Experimental results on people’s choices

<table>
<thead>
<tr>
<th>Rose Strategy</th>
<th>Percent of time</th>
<th>Colin Strategy</th>
<th>Percent of time</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>31%</td>
<td>A</td>
<td>20%</td>
</tr>
<tr>
<td>B</td>
<td>10%</td>
<td>B</td>
<td>51%</td>
</tr>
<tr>
<td>C</td>
<td>49%</td>
<td>C</td>
<td>2%</td>
</tr>
<tr>
<td>D</td>
<td>10%</td>
<td>D</td>
<td>27%</td>
</tr>
<tr>
<td></td>
<td>100%</td>
<td></td>
<td>100%</td>
</tr>
</tbody>
</table>

Apparently, participating players may not practise the minimax criterion. However, the average payoff to Rose is close to the game value of 2.0.
Rose C - Colin B is an equilibrium outcome

If Colin knows or believes that Rose will play Rose C, then Colin would respond with Colin B; similarly, Rose C is Rose’s best response to Colin B.

Once both players are playing these strategies, then neither player has any incentive to move to a different strategy.
A two-person zero sum game may have no saddle point or more than one saddle point.

**Example 2 × 2 Nim**

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td>→ min = -1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td>→ min = -1</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>→ min = -1</td>
</tr>
</tbody>
</table>

The third column is a dominating strategy for the Column player. All entries in the third column are saddle points.
Example

Multiple saddle points

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-20</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>D</td>
<td>-16</td>
<td>0</td>
<td>16</td>
<td>1</td>
</tr>
</tbody>
</table>

All four of the circled outcomes are saddle points. They are the corners of a rectangular block.

Note that the “2” at Rose B - Colin A is not a saddle point. It just happens to have the same value as that of other saddle points but it does not possess the minimax property.
Lemma

A game will have a saddle point in pure strategies if and only if

\[ v^- = \max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v^+ \].

Proof

(i) existence of a saddle point \( \Rightarrow v^+ = v^- \)

Suppose \((i^*, j^*)\) is a saddle point, we have

\[ v^+ = \min_j \max_i a_{ij} \leq \max_i a_{i^*j^*} \leq a_{i^*j^*} \leq \min_j a_{i^*j} \leq \max_i \min_j a_{ij} = v^- \].

However, \( v^- \leq v^+ \) always holds, so we have equality throughout and

\[ v = v^+ = v^- = a_{i^*j^*}. \]
(ii) \( v^+ = v^- \Rightarrow \) existence of a saddle point

On the other hand, suppose \( v^+ = v^- \), so

\[
v^+ = \min_j \max_i a_{ij} = \max_i \min_j a_{ij} = v^-.
\]

Let \( j^* \) be such that \( v^+ = \max_i a_{ij^*} \) and \( i^* \) such that \( v^- = \min_j a_{i^*j} \).

Note that for any \( i = 1, 2, ..., n \) and \( j = 1, 2, ..., m \), we have

\[
a_{i^*j} \geq \min_j a_{i^*j} = v^- = v^+ = \max_i a_{ij^*} \geq a_{ij^*}.
\]

Lastly, taking \( j = j^* \) on the left inequality and \( i = i^* \) on the right, we obtain

\[
a_{i^*j^*} = v^+ = v^-,
\]

and so \( a_{i^*j} \geq a_{i^*j^*} \geq a_{ij^*} \). This satisfies the condition for \((i^*, j^*)\) to be a saddle point.
Lemma

In a two-person zero sum game, suppose \((\sigma_1, \sigma_2)\) and \((\tau_1, \tau_2)\) are two saddle strategies, then \((\sigma_1, \tau_2)\) and \((\tau_1, \sigma_2)\) are also saddle strategies. Also, their payoffs are the same. That is,

\[a_{\sigma_1\sigma_2} = a_{\tau_1\tau_2} = a_{\sigma_1\tau_2} = a_{\tau_1\sigma_2}.\]
Proof

Since \((\sigma_1, \sigma_2)\) is a saddle point, so \(a_{\sigma_1\sigma_2} \geq a_{\tau_1\sigma_2}\) (largest value in a column). Similarly, we have \(a_{\tau_1\sigma_2} \geq a_{\tau_1\tau_2}\) (smallest value in a row). Combining the results, we obtain

\[
a_{\sigma_1\sigma_2} \geq a_{\tau_1\sigma_2} \geq a_{\tau_1\tau_2}.
\]

In a similar manner, moving from \((\sigma_1, \sigma_2)\) to \((\sigma_1, \tau_2)\) along the row \(\sigma_1\) and \((\tau_1, \sigma_2)\) to \((\tau_1, \tau_2)\) along the column \(\tau_2\), we can establish

\[
a_{\sigma_1\sigma_2} \leq a_{\sigma_1\tau_2} \leq a_{\tau_1\tau_2}.
\]

Hence, we obtain equality of the 4 payoffs:

\[
a_{\sigma_1\sigma_2} = a_{\tau_1\tau_2} = a_{\sigma_1\tau_2} = a_{\tau_1\sigma_2}.
\]

For any \(\hat{\sigma}_1\), we have \(a_{\hat{\sigma}_1\sigma_2} \leq a_{\sigma_1\sigma_2} = a_{\tau_1\sigma_2}\); and for any \(\hat{\sigma}_2\), we also have \(a_{\tau_1\hat{\sigma}_2} \geq a_{\tau_1\tau_2} = a_{\tau_1\sigma_2}\). Therefore, \(a_{\tau_1\hat{\sigma}_2} \geq a_{\tau_1\sigma_2} \geq a_{\hat{\sigma}_1\sigma_2}\) and so \((\tau_1, \sigma_2)\) is a saddle strategy. Similarly, we can also establish that \((\sigma_1, \tau_2)\) is a saddle strategy.
Equilibrium

An *equilibrium* $s^* = (s^*_1, s^*_2, ..., s^*_n)$ is a strategy profile consisting of a best strategy for each of the $n$ players in the game.

An equilibrium concept or solution concept

$$F : \{s_1, s_2, ..., s_n, \pi_1, \pi_2, ..., \pi_n\} \rightarrow s^*$$

is a rule (mapping) that defines an equilibrium based on the possible strategy profiles and the payoff functions.

*Uniqueness*

Obtained solution concepts do not guarantee uniqueness. Non-unique equilibrium may mean the players will pick one of the two strategy profiles A or B, not others, but we cannot say whether A or B is more likely.

*Non-existence:* The game has no equilibrium at all.
Best response to other players’ strategies

Player $i$’s best response or best reply to the strategies $s_{-i}$ chosen by the other players is the strategy $s_i^*$ that yields him the greatest payoff, where

$$
\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i', s_{-i}), \ \forall \ s_i' \neq s_i^*;
$$

with strict inequality for at least one $s_i'$.

The best response is strongly best if no other strategies are equally good, and weakly best otherwise.

Each part of a saddle is the best response to the others.

Dominated and dominant strategies

The strategy $s_i^d$ is dominated if there exists a single $s_i'$ such that

$$
\pi_i(s_i^d, s_{-i}) < \pi_i(s_i', s_{-i}), \ \forall \ s_{-i}.
$$

That is, $s_i^d$ is strictly inferior to (dominated by) some other strategy $s_i'$. 
There may exist some strategy that beats every other strategy.

The strategy \( s_i^* \) is a *dominant strategy* if

\[
\pi_i(s_i^*, s_{-i}) > \pi_i(s_i', s_{-i}), \quad \forall s_i' \neq s_i^*.
\]

That is, \( s_i^* \) is a player’s strictly best response to any strategies the other players might pick.

A *dominant-strategy equilibrium* is a strategy profile consisting of each player’s dominant strategy. A player’s dominant strategy is his strictly best response even to wildly irrational actions of the other players.

Outcome \( X \) is said to be strongly Pareto-dominating outcome \( Y \) if all players have higher payoff under outcome \( X \).

A non-dominated outcome is Pareto optimal, defined as one such that there is no other outcome where some players can increase their payoffs without decreasing the payoff of other players.
Prisoner's dilemma — Noncooperative game with conflict

• If each prisoner tries to blame the other, each is sentenced to 8 years in prison.

• If both remain silent, each is sentenced to one year.

• If just one blames the other, he is released but the silent prisoner is sentenced to 10 years.
Each player has 2 possible actions: *Confess* (blaming the other) and *Deny* (silent).

<table>
<thead>
<tr>
<th></th>
<th>Deny</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deny</td>
<td>-1, -1</td>
<td>-10, 0</td>
</tr>
<tr>
<td>Confess</td>
<td>0, -10</td>
<td>-8, -8</td>
</tr>
</tbody>
</table>

Payoffs to: (Row, Column).

Each player has a dominant strategy of Confess since the payoff under Confess is better than Deny under all strategies played by the other player.

The dominant-strategy equilibrium is (Confess, Confess).
The result is robust to substantial changes in the model. As the equilibrium is a dominant-strategy equilibrium, the information structure of the game does not matter.

- If Column is allowed to know Row’s move before taking his own, the equilibrium is unchanged. Row still chooses Confess, knowing that Column will surely choose Confess afterwards.

- What difference would it make if the two prisoners could talk to each other before making their decisions?

  If promises are not biding, though the two prisoners might agree to Deny, they would Confess anyway when the time came to choose actions.
Arm race as Prisoner’s dilemma

Two players: US & USSR

Possible strategies: armed or disarmed

Rank the four possibilities (most preferred to least preferred)

1. Highly positive (payoff = 3)
   Self-armed and other’s unilaterally disarmed (military superiority)

2. Moderately positive (payoff = 1)
   Mutual disarmament (parity without economic hardship)

3. Moderately negative (payoff = −1)
   An arm race (parity but with economic hardship)

4. Highly negative (payoff = −3)
   Self-disarmed and other’s unilaterally armed (military inferiority)
1. Ordinal payoffs rather than cardinal payoffs
   The payoff values reflect only the order of preference as opposed to the absolute magnitude of one’s preference.

2. The play of the game consists of a single move. Both players choose the strategies simultaneously and independently

Both Soviet and US have dominant strategies leading to “arm-arm” outcome that is strictly worse (for both players) than the “disarm-disarm” outcome via mutual operation.
Iterated dominance: Battle of the Bismarck Sea

General Imamura has been ordered to transport Japanese troops across the Bismarck Sea to New Guinea in the South Pacific in 1943.

General Kenney wants to bomb the troop transports.

Imamura must choose between a shorter northern route or a longer southern route to New Guinea. Since the southern route is longer, the potential number of days of bombing is larger than that of the northern route counterpart.

Kenney must decide where to send his planes to look for the Japanese. If Kenney sends his plane to the wrong route, he can recall them but the number of days of bombing is reduced.

Players: Kenney and Imamura; action set = \{North, South\}. 
Payoffs to Kenney in the two-person zero sum game

<table>
<thead>
<tr>
<th></th>
<th>North</th>
<th>South</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Kenney</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><em>North</em></td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td><em>South</em></td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

**Battle of the Bismarck Sea**

**Imamura**

Neither player has a dominant strategy.

- Kenney would choose *North* if he thought Immamura would choose *North*, but *South* if he thought Immamura would choose *South*.
- Immamura would choose North if he thought Kenney would choose South and he would be *indifferent* between actions if he thought Kenney would choose North.
We may manage to find a plausible equilibrium using the concept of “weak dominance”.

- Strategy $s_i'$ is weakly dominated if there exists $s_i''$ such that

\[
\pi_i(s_i'', s_{-i}) \geq \pi_i(s_i', s_{-i}), \forall s_{-i};
\]

\[
\pi_i(s_i'', s_{-i}) > \pi_i(s_i', s_{-i}) \text{ for some } s_{-i}.
\]

- Similarly, a weakly dominant strategy is one that is always at least as good as every other strategy and better than some.

An *iterated-dominance equilibrium* is a strategy profile found by deleting a weakly dominated strategy from the strategy set of one of the players, recalculating to find which remaining strategies are weakly dominated, deleting one of them, and continuing the process until only one strategy remains for each player. Note that iterated-dominance equilibrium may not exist.
• Using the iterated-dominance equilibrium concept, Kenney then decides that Imamura will pick *North* because it is weakly dominant, so Kenney eliminates “Imamura chooses South” from consideration.

• Having deleted one column of the payoff table, Kenney has a strongly dominant strategy: he chooses *North*, which achieves payoff strictly greater than South.

• The strategy profile (North, North) is an iterated dominance equilibrium, and indeed (North, North) was the outcome in 1943.

How about consider modifying the *order of play* or the *information structure* in the Battle of the Bismarck Sea?
Since once Kenney has chosen North, Imamura is indifferent between North and South. The outcomes are dependent on which player moves first.

1. If *Kenney moved first*, (North, North) would remain an equilibrium, but (North, South) would also become one. The payoffs would be the same for both equilibriums, but the outcomes would be different.

2. If Imamura moved first, (North, North) would be the only equilibrium.
1.3 Nash equilibrium models

Nash equilibrium

For the vast majority of games, which lack even iterated dominance equilibriums, modellers use Nash equilibrium concept to justify rationality of players’ actions.

• Nash equilibrium obviously has less rationality when compared to dominant-strategy equilibrium but its existence appears more often in game models.

Definition

The strategy profile $s^*$ is a Nash equilibrium if no player has incentive to deviate from his strategy given that the other players do not deviate. That is

$$\pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i', s_{-i}^*), \ \forall s_i';$$

with strict inequality for at least one $s_i'$.

• To define strong Nash equilibrium, we make the inequality strict.
In other words, these are the mutual best responses. The action profile $a^*$ is a Nash equilibrium of a game with ordinal preference if and only if every player’s action is a best response to the other players' actions:

$$a^*_i \text{ is in player } i\text{'s best response to } a^*_{-i}, \ B_i(a^*_{-i}), \text{ for every player } i.$$  

In other words, if the equilibrium strategies of the other players remain unchanged, the player does not benefit by changing his strategy.

To show that the strategy profile (Confess, Confess) in the Prisoner’s Dilemma is a Nash equilibrium, we test whether each player’s strategy is a best response to the other’s strategies (mutual best responses).

- If Row chooses Confess, then Confess is the best response of Column.

- By symmetry, Confess is the best response of Row if Column chooses Confess.
Every dominant-strategy equilibrium is a Nash equilibrium, but not every Nash equilibrium is a dominant-strategy equilibrium.

- If a strategy is dominant, it is a best response to any strategies the other players pick, including their equilibrium strategies.
- If a strategy is part of a Nash equilibrium, it needs only be a best response to the other players' equilibrium strategies.

in a similar manner, one can show that if a strategy profile $s$ weakly Pareto-dominates all other strategy profiles, then it must be a Nash equilibrium.

How to predict which Nash equilibrium may be more preferred when two Nash equilibria exist?

One may use an equilibrium refinement — adding conditions to the basis equilibrium concept until only one strategy profile satisfies the refined equilibrium concept.
Consider the modified Prisoner’s Dilemma (Modeller’s Dilemma) where the payoff to (Deny, Deny) is (0, 0) instead of (−1, −1), assuming that the police does not have enough evidence to convict the prisoners of even a minor offense if neither prisoner confesses.

- It does not have a dominant-strategy equilibrium.
• It does have a weak dominant-strategy equilibrium since Confess is still a weakly dominant strategy for each player.

• (Confess, Confess) is an iterated dominance equilibrium and it is a strong Nash equilibrium. However, (Deny, Deny) is a weak Nash equilibrium and its outcome is Pareto-superior (strongly Pareto-dominating the other Nash equilibrium). Though (Confess, Confess) is a dominant-strategy equilibrium, it does not weakly Pareto-dominates all other strategy profiles.

**Choices for equilibrium refinement**

1. Insist on a strong equilibrium, so rule out (Deny, Deny).

2. Rule out Nash equilibriums that are Pareto-dominated by other Nash equilibriums, and end up with (Deny, Deny).
Examples

(A, A) is a Nash equilibrium since neither player would gain by unilaterally changing his strategy of A while the other player remains playing A. Note that A is a dominant strategy for Rose but not for Colin. Also, (A, A) is Pareto efficient (optimal).
There is no Nash equilibrium. None of the outcomes is Pareto-dominating. Also, none of the strategies is dominant.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rose</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>(2, 4)</td>
<td>(1, 0)</td>
</tr>
<tr>
<td>B</td>
<td>(3, 1)</td>
<td>(0, 4)</td>
</tr>
</tbody>
</table>

Both (5, 2) and (2, 5) are Nash equilibriums. They are also Pareto efficient (optimal). However, none of the strategies is dominant.
Stag hunt

Each hunter has two options: stag or hare.

- If all hunters pursue the stag, they catch the animal and share it equally.
- If any hunter devotes his energy to catch a hare, the stag escapes, and the hare belongs to the defecting hunter alone.
- Each hunter prefers a share of the stag to a hare.

<table>
<thead>
<tr>
<th></th>
<th>Stag</th>
<th>Hare</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stag</td>
<td>2, 2</td>
<td>0, 1</td>
</tr>
<tr>
<td>Hare</td>
<td>1, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>
The stag hunt game is used to model social cooperation (also called *trust dilemma*) since the defecting hunter is guaranteed to have payoff one (though leaving the other player with zero payoff) while cooperation leads to higher payoff. Apparently, (Hare, Hare) is *risk dominant*.

The two-player stag Hunt has two Nash equilibriums:

(Stag, Stag) and (Hare, Hare).

One of these equilibriums is better in payoffs for both players than the other. This fact has no bearing on the equilibrium status of (Hare, Hare) as a Nash equilibrium. Recall the condition for a Nash equilibrium is that a single player cannot gain by deviating, given the other player’s equilibrium strategy. In other words, we simply require a Nash equilibrium to be immune to any unilateral deviation.

- Additional features not modeled by the strategic game may make the more desirable equilibrium to become focal (more likely to occur than the other equilibrium). For example, the one that is Pareto-superior is more preferred.
Game of Chicken — Cuban Missile Crisis

In October 1962, the US and USSR came close to a nuclear confrontation (President Kennedy estimated the probability of a nuclear war to be between $\frac{1}{3}$ to $\frac{1}{2}$).

- Why did the USSR attempt to place offensive missile in Cuba?

- Why did the US choose to respond to the Soviet missile emplacement with a blockade of Cuba?

- Why did the Soviet Union decide to withdraw the missile?

We may treat the crisis as a contest between a challenger and a defender.
• Why did the challenger (CH) attempt to change the status quo (SQ)?
• How did the other side (defender D) respond to the challenge with a threat? If D does not resist, then the game closes (C).
• Why did one side or the other back down (BD) or the crisis ended in a war (W)?
The payoffs may be modeled like those of the Game of Chicken (opposing drivers maintain a head-on collision course until at least one of them swerves out of the way).

<table>
<thead>
<tr>
<th></th>
<th>Withdraw missiles</th>
<th>Maintain missiles</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>U.S.</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Give up option to invade Cuba</td>
<td>(3, 3)</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>Invade Cuba</td>
<td>(4, 2)</td>
<td>(1, 1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Soviet Union</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Withdraw missiles</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maintain missiles</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Neither C (cooperation) nor N (non-cooperation) is a dominant strategy for both players.

Both (2, 4) and (4, 2) are Nash equilibriums.

The outcome (3, 3) is most desirable and it is unstable since it is not a Nash equilibrium. Only a fear of the (1, 1) outcome would prevent both players from trying for the (4, 2) or (2, 4) outcomes.
Other forms of payoffs are possible depending on different assumptions of political scenarios.

(4, 4) is the Nash equilibrium.
No Nash equilibrium. Under this set of assumed payoff, Soviet would be more aggressive since N is Soviet’s dominant strategy. As a result, payoff of US is lowered.

<table>
<thead>
<tr>
<th></th>
<th>Soviets</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
<td>N</td>
</tr>
<tr>
<td>U.S.</td>
<td>(4, 3)</td>
<td>(1, 4)</td>
</tr>
<tr>
<td>N</td>
<td>(3, 1)</td>
<td>(2, 2)</td>
</tr>
</tbody>
</table>

U.S. fear of (Soviet) preferences for deterrence model

No Nash equilibrium. N is US’s dominant strategy.

<table>
<thead>
<tr>
<th></th>
<th>Soviets</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
</tr>
<tr>
<td>U.S.</td>
<td>(3, 4)</td>
</tr>
<tr>
<td>N</td>
<td>(4, 1)</td>
</tr>
</tbody>
</table>

Soviet fear of (U.S.) preferences for deterrence model
Battle of the Sexes — Compromise and cooperation

<table>
<thead>
<tr>
<th>Woman</th>
<th>Prize Fight</th>
<th>Ballet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prize Fight</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Man</td>
<td>↑</td>
<td>↓</td>
</tr>
<tr>
<td>Ballet</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

Payoffs to: (Man, Woman). Arrows show how a player can increase his payoff.

Two Nash equilibriums

1. Strategy profile (Prize Fight, Prize Fight)
   Given that the man chooses Prize Fight, so does the woman.
   Given that the woman chooses Prize Fight, so does the man.

2. Strategy Profile (Ballet, Ballet)
   Both Nash equilibriums are Pareto-efficient, where no player gains without another player losing. Comparing (2, 1) and (1, 2), when man's payoff increases from 1 to 2, woman's payoff decreases from 2 to 1, and vice versa.
Nature of the game: Failure to cooperate creates lower payoff to both players. However, any cooperative agreement results in unequal distribution of payoff (lower payoff for the player who compromises).

How do the players know which Nash equilibrium to choose? Who moves first is important — first mover advantage. If the man could buy the fight ticket in advance, his commitment would induce the woman to go to the fight.

Economic analogy: Choice of an industry-wide standard when both firms have different preferences but both want a common standard to encourage consumers to buy the product.

Communication between the couple

- If they do not communicate beforehand, the man might go to the ballet and the woman to the fight, each mistaken about the other’s beliefs. Repeating the game night after night, eventually they settle on one of the Nash equilibria with given distribution (see the discussion of mixed strategy Nash equilibrium in Topic 2).
Coordination games — coordinate on one of the multiple Nash equilibriums

One can use the size of the payoffs to choose between Nash equilibriums, where the Pareto-superior equilibrium is more preferred.

Players Smith and Jones decide whether to design the computers they sell to use large or small floppy disks.

<table>
<thead>
<tr>
<th></th>
<th>Large</th>
<th>Small</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large</td>
<td>2, 2</td>
<td>-1, -1</td>
</tr>
<tr>
<td>Small</td>
<td>-1, -1</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

Payoffs to: (Smith, Jones). Arrows show how a player can increase his payoff.

Is the Pareto-efficient equilibrium still more plausible when pre-game communication is not possible? This is really one of psychology rather than economics.
Note that (2, 2) Pareto-dominates all other strategy profiles. However, “Large” is not a dominant strategy for both players, so it is not a dominant-strategy equilibrium. In order that (Large, Large) is a dominant-strategy equilibrium, we require (Large, Small) provides a better payoff to Row than that from (Small, Small), but this is not satisfied.

*Dangerous coordination* — extreme out-of-equilibrium payoff

\[
\begin{array}{c|cc}
 & \text{Large} & \text{Small} \\
\hline
\text{Large} & 2, 2 & -1000, -1 \\
\text{Small} & -1, -1 & 1, 1 \\
\end{array}
\]

*Payoffs to: (Smith, Jones). Arrows show how a player can increase his payoff.*

In real case, the Pareto-dominated equilibrium (Small, Small) was played out. If the assumptions are weakened and Smith cannot trust Jones to be rational, Smith will be reluctant to pick “Large” since his payoff if Jones pick “Small” is then −1,000. Smith suffers greatly if the agreement of coordination is not honored by Jones.
Game tree analysis of the Yom Kippur War (1973)

- Israel fought against Egypt and Syria and gained upper hand within a few days.

- Soviet Union seriously considered intervening on behalf of Egypt and Syria, but they were aware of the US option of intervention. However, the US was faced with the Watergate scandal at home.

How did USSR and US rank the different outcomes, and was each aware of the other’s preferences?
Payoff values as viewed by the Soviets

<table>
<thead>
<tr>
<th></th>
<th>Seek diplomatic solution</th>
<th>Supply Egypt and Syria with military aid</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>US</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cooperate with the Soviet initiative (nonintervention)</td>
<td>C</td>
<td>(3, 3)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>(4, 1)</td>
</tr>
<tr>
<td><strong>Soviets</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2, 4)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 2)</td>
</tr>
</tbody>
</table>

Soviets thought that the US would rank the payoff to herself of outcome (C, N) to be ahead of that of outcome (N, N).

- Strategy N is a dominant strategy for the Soviets but US has no dominant strategy.
Nixon’s immediate goal is to convince the Soviets that the correct model is in fact, Prisoner’s Dilemma. If we assume simultaneous moves of 6th players, then strategy N is a dominant strategy for both US and Soviets.

Payoff to US of outcome (N, N) is ahead of that of (C, N).

“Deliberate overreaction” by the US — placed the US forces on worldwide alert.
Theory of moves (dynamic approach to games)

History told us that the two Great Powers wound up at the (3, 3) outcome. This is not a Nash equilibrium, but this is the starting position taken up by both countries since the game should begin with mutual non-intervention. The question is whether or not either side should change its status quo of C (non-intervention) to N (intervention).

The optimal strategies would change if the two players do not have to choose simultaneously.

- Both players make an initial simultaneous choice of C or N (initial position of the game).
- Row has the choice of staying or changing his strategy.
- Column has the same choices as did Row in Step 2.

Remark: Analysis is analogous if Column moves first.
There are alternatives to standard game theory, particularly in looking at the process by which outcomes are chosen, thereby making the analysis more dynamic. Players think ahead about the consequences of all of the participants’ moves and countermoves when formulating plans.

The two players continue alternately. The game ends if

Column’s turn to move but the position is (−, 4) or Row’s turn to move but the position is (4, −). However, the game does not end immediately if the initial position is (4, −).

- Either Row or Column chooses to stay once the highest payoff of 4 is achieved, with the exception of the first stay by Row. We give Column a chance to move even if Row declines the chance to switch strategy on his first move.

We consider separately the 4 possible initial positions in the game, perform a game-tree analysis and find the corresponding outcome.
Case 1: Initial position is (3, 3)

Row makes the first move.

\[
\begin{array}{c|cc}
 & C & N \\
\hline
C & (3, 3) & (1, 4) \\
N & (4, 1) & (2, 2) \\
\end{array}
\]
Starting from the most bottom nodes, cancel the possibilities that would not be undertaken by a player due to unfavorable outcome. The only possibility left is "Stay at (3, 3)".
Case 2: Initial position is (2, 2)
Case 3: Initial position is (1, 4)

Col stays with his 4

Row

Col

Row

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>(3,3)</td>
<td>(1,4)</td>
</tr>
<tr>
<td>N</td>
<td>(4,1)</td>
<td>(2,2)</td>
</tr>
</tbody>
</table>
Case 4: Initial position is (4, 1)

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>(3, 3)</td>
<td>(1, 4)</td>
</tr>
<tr>
<td>N</td>
<td>(4, 1)</td>
<td>(2, 2)</td>
</tr>
</tbody>
</table>
**Summary:** (Row goes first)

<table>
<thead>
<tr>
<th>Initial position</th>
<th>Final outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 3)</td>
<td>→ (3, 3)</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>→ (2, 2)</td>
</tr>
<tr>
<td>(1, 4)</td>
<td>→ (2, 2)</td>
</tr>
<tr>
<td>(4, 1)</td>
<td>→ (3, 3)</td>
</tr>
</tbody>
</table>

Theory of moves postulates that players think ahead not just to the immediate consequences of making moves but also to the consequences of countermoves to these moves, counter-counter-moves, and so on. It helps farsighted players resolve the dilemmas in games like Prisoner’s Dilemma and Chicken.
Analysis of strategies

Obviously, both Row and Column prefer (3, 3) as the final outcome instead of (2, 2). Therefore, they both want (4, 1) or (3, 3) as the initial position.

Column alone can guarantee this simply by choosing C as his initial strategy. So, “C” is a dominant strategy of Column. Similar result is obtained even when Column goes first, where “C” is ended up to be a dominant strategy of Row.

Back to the Yom Kippur War: Given that mutual non-intervention is the initial position, neither side elected to change its initial choice of strategy.
Second-price sealed-bid auction (avoiding bidding escalation)

There are \( n \) bidders that submit a single bid for an object.

Let \( v_i \) be the \( i^{th} \) player’s bid, \( i = 1, 2, ..., n \). The winner of the auction is the bidder with the highest bid but he pays not the bid he submitted but the next highest bid.

Vickrey (1996 Nobel prize winner in economics) shows that the strategy of bidding one’s true valuation for the object being sold weakly dominates every other bidding strategy.

By definition, one bidding strategy is said to weakly dominate another if it is at least as good in payoff as the other in every scenario, and strictly better than the other in at least one scenario.
Assuming for simplicity that draws are excluded (the two highest bids are equal), we would like to show that truth telling for all players is the Nash equilibrium since every player’s action weakly dominates all his other actions.

We compare the strategy $t_i$ (truth telling) of player $i$ with other strategies $v_i$. Consider the following two scenarios.

(i) Suppose player $i$ wins the object with the bid $t_i$, and the second highest bid that he actually has to pay is $u \leq t_i$. The gain is $t_i - u$.

If his bid were $v_i \geq u$, he would get the same win for the same price. If his bid were $v_i < u$, then he would lose the auction, thus getting worse payoff than with $t_i$. 

$u$ is the second highest bid 
$t_i$ is the truth telling bid
(ii) Suppose player $i$ loses the auction (payoff is zero) with the bid $t_i$, and suppose the highest bid was $h > t_i$.

If player $i$ bid $v_i \leq h$, then he loses anyway, thus changing nothing. If his bid $v_i > h$, then he wins the auction but has to pay $h > t_i$, thus having negative payoff which is again worse than bidding $t_i$.

A player would not be worst off with the strategy of honest bid no matter how irrational the other bidders happen to be.

The second-price sealed-bid auction is considered to be more fair since it encourages buyers to bid what they think an item is worth instead of escalating into bidding wars.