# MATH 4321 – Game Theory

## Mid-term Test, 2017

*Time allowed: 90 minutes* *Instructor: Prof. Y. K. Kwok*

**[points]**

1. Consider the two-person zero-sum game whose game matrix is given by

$$A = \begin{pmatrix} 2 & 8 & 7 \\ 8 & 4 & 7 \\ 12 & -4 & 10 \end{pmatrix}.$$

(a) Show that the third column of $A$ is dominated by a convex combination of the first and second columns, where

$$\text{column } 3 > \lambda \text{ column } 1 + (1-\lambda) \text{ column } 2.$$

Find the range of $\lambda$ such that the above convex combination relation holds. **[2]**

(b) Reduce the matrix and solve for the mixed strategy of the Column player. Find the value of the game. **[6]**

*Hint* Consider $\min_{0 \le y \le 1} \max\{E(1,Y^*), E(2,Y^*), E(3,Y^*)\}$, where $Y^* = (y, 1-y, 0)$. Look for the intersection points of the upper envelope.

2. In a two-person zero-sum game, let $(\sigma_1, \sigma_2)$ and $(\tau_1, \tau_2)$ be the two pure saddle point strategies.

(a) Show that $(\sigma_1, \tau_2)$ and $(\tau_1, \sigma_2)$ are also pure saddle point strategies. **[3]**

(b) Show that the payoffs at all these pure saddle point strategies must be the same. **[3]**


*Hint* Let $A$ be the game matrix. At the saddlepoint $(i^*, j^*)$, the entry $A_{i^*j^*}$ achieves minimum value along row $i^*$ and maximum value along column $j^*$.

3. Let $A$ be the game matrix of a symmetric two-person zero-sum game, where $A = -A^T$. Let $v(A)$ denote the value of the symmetric game.

(a) Show that $E(X, X) = 0$ for any $X$. **[3]**

(b) Show that the value of the symmetric game is always zero. **[4]**

*Hint* Suppose $(X^*, Y^*)$ is a saddlepoint for the game, then

$$E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y) \text{ for any } X \text{ and } Y.$$

4. Use the theory of moves to determine the final outcome of the following two-person nonzero sum game when the initial position is $(1, 3)$.

| | C | N |
|---|---|---|
| C | $(4, 4)$ | $(1, 3)$ |
| N | $(2, 1)$ | $(3, 2)$ |

We assume that the Row player moves first. Either the Row or Column player chooses to stay once the highest payoff of 4 is achieved. **[4]**

*Hint* There may be two possible outcomes, depending on whether a player is aggressive enough to lower her payoff at the current move with anticipation that the opponent will move to a position of better payoff to both in his next move.

5. Two companies both want to take over a sales territory. Each has two strategies: fighting (F) or backing off (B). The payoffs of the nonzero sum game are given by

| | F | B |
|---|---|---|
| F | $(-1, -1)$ | $(2, 0)$ |
| B | $(0, 2)$ | $(0, 0)$ |

(a) State the Equality of Payoff Theorem and use it to find the best responses of both players. **[4]**

(b) Find the two pure Nash equilibriums and the mixed Nash equilibrium. Explain the mechanism how the two players may stumble and settle on the mixed Nash equilibrium. **[4]**

(c) Find the safety value of the Column player. Verify that the safety value is always less than or equal to the expected payoffs of the Column player at the two pure Nash equilibriums and mixed Nash equilibrium.

Present a theoretical proof of the result:

Expected payoff of a player at a Nash equilibrium in a two-person nonzero game is always bounded below by the player's safety value of the game. **[7]**

— *End* —