1. Let $c$ be the coupon rate per period and $y$ be the yield per period. There are $m$ periods per year (say, $m = 4$ for quarterly coupon payments), and let $n$ be the number of periods remaining until maturity. Show that the duration $D$ is given by

$$D = \frac{1 + y}{my} - \frac{1 + y + n(c - y)}{mc[(1 + y)^n - 1] + my}.$$  

Here, the yield per year $\lambda$ is given by $my$. Show that, as $T \to \infty$, we obtain

$$D \to \frac{1}{m} + \frac{1}{\lambda}.$$

Remark

The above analytic results are revealed in the following numerical example. Consider the duration calculated for various bonds as shown in the following table, where $\lambda = 0.05$ and $m = 2$. We obtain $D \to \frac{1}{2} + \frac{1}{0.05} = 20.5$.

Duration of a Bond Yielding 5% as Function of Maturity and Coupon Rate

<table>
<thead>
<tr>
<th>Years to maturity</th>
<th>1%</th>
<th>2%</th>
<th>5%</th>
<th>10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.997</td>
<td>0.995</td>
<td>0.988</td>
<td>0.977</td>
</tr>
<tr>
<td>2</td>
<td>1.984</td>
<td>1.969</td>
<td>1.928</td>
<td>1.868</td>
</tr>
<tr>
<td>5</td>
<td>4.875</td>
<td>4.763</td>
<td>4.485</td>
<td>4.156</td>
</tr>
<tr>
<td>10</td>
<td>9.416</td>
<td>8.950</td>
<td>7.989</td>
<td>7.107</td>
</tr>
<tr>
<td>25</td>
<td>20.164</td>
<td>17.715</td>
<td>14.536</td>
<td>12.754</td>
</tr>
<tr>
<td>50</td>
<td>26.666</td>
<td>22.284</td>
<td>18.765</td>
<td>17.384</td>
</tr>
<tr>
<td>100</td>
<td>22.572</td>
<td>21.200</td>
<td>20.363</td>
<td>20.067</td>
</tr>
<tr>
<td>$\infty$</td>
<td>20.500</td>
<td>20.500</td>
<td>20.500</td>
<td>20.500</td>
</tr>
</tbody>
</table>

The table shows that duration does not increase appreciably with maturity. In fact, with fixed yield, duration increases only to a finite limit as maturity is increased.

2. The return-to-maturity expectations hypothesis states that the return generated by holding a bond for term $t$ to $T$ will equal the expected return generated by continually rolling over a bond whose term is a period evenly divisible into $T - t$. Explain why the above relationship can be expressed formally as

$$\frac{1}{B(t, T)} = E_t[(1 + r_t)(1 + \tilde{r}_{t+1}) \cdots (1 + \tilde{r}_{T-1})],$$

where $B(t, T)$ is the time-$t$ price of a discount bond maturing at $T$ and $r_t$ is the one-period spot rate at time $t$. The operator $E_t$ indicates that expectation is taken based on the information available at the current time $t$.

Remark
Suppose the investor starts with one dollar at time $t$ and invests in a discount bond maturing one year later the “deterministic” return is

$$1 + r_t = \frac{1}{B(t, t+1)},$$

where $B(t, t+1)$ is known at time $t$. At time $t+1$, the investor uses the proceed $\frac{1}{B(t, t+1)}$ to invest in a discount bond maturing one year later. The bond price is $\tilde{B}(t+1, t+2)$, which is not known at time $t$. The return over $[t+1, t+2]$ is

$$1 + \tilde{r}_{t+1} = \frac{1}{\tilde{B}(t+1, t+2)},$$

where “tilde” quantities represent stochastic quantities. At time $t+2$, the investor again invests in $\frac{1}{B(t, t+1)\tilde{B}(t+1, t+2)}$ units of discount bond maturing one year later. After $T-t$ years, the random return at time $T$ is

$$(1 + r_t)(1 + \tilde{r}_{t+1}) \cdots (1 + \tilde{r}_{T-1}) = \frac{1}{B(t, t+1)\tilde{B}(t+1, t+2) \cdots \tilde{B}(T-1, T)}.$$

This strategy is like investing in a money market account with annual rolling over.

3. Show that all curves $r_H = r_H(i)$ for various horizons $H (H = 1, 2, ..., \infty)$ go through the point $(i_0, i_0)$. In other words, show that $(i_0, i_0)$ is a fixed point for all curves $r_H(i)$.

4. Suppose that an obligation occurring at a single time period is immunized against interest rate changes with bonds that have only nonnegative cash flows (see p.92-94 in Topic One). Let $P(\lambda)$ be the value of the resulting portfolio, including the obligation, when the interest rate is $r + \lambda$ and $r$ is the current interest rate. Here, $\lambda$ represents the change in the interest rate. By immunization construction, we have set $P(0) = 0$ and $P'(0) = 0$. In this problem, we would like to show that $P(0)$ is a local minimum; that is, $P''(0) \geq 0$. 

2
Assume a yearly compounding convention. The discount factor at time $t$ is

$$d_t(\lambda) = (1 + r + \lambda)^{-t}.$$

Let $d_t = d_t(0)$. For convenience, we assume that the obligation has magnitude 1 and is due at time $\bar{t}$. The conditions for immunization are then given by

$$P(0) = \sum_t c_t d_t - d_{\bar{t}} = 0$$

$$P'(0)(1 + r) = \sum_t t c_t d_t - \bar{t} d_{\bar{t}} = 0.$$

Here, the summation is taken over the various bonds in the immunized portfolio with varying maturity $t$.

(a) Show that for all values of $\alpha$ and $\beta$ there holds

$$P''(0)(1 + r)^2 = \sum_t (t^2 + \alpha t + \beta) c_t d_t - (\bar{t}^2 + \alpha \bar{t} + \beta) d_{\bar{t}}.$$

(b) Since $t^2 + \alpha t + \beta$ is a quadratic equation in $t$, one can always choose $\alpha$ and $\beta$ such that the function $t^2 + \alpha t + \beta$ has a minimum at $\bar{t}$ and has a value of 1 there. Use these results to conclude that $P''(0) \geq 0$.

5. Looking at the figure below, you will notice that the quadratic approximation curve of the bond’s value lies between the bond’s value curve and the tangent line to the left of the tangency point and outside (above) these lines to the right of the tangency point. The fact that a quadratic (and hence “better”) approximation behaves like this is not intuitive: we would tend to think that a “better approximation” would always lie between the exact value curve and its linear approximation. How can you explain this apparently non-intuitive result?

![Linear and quadratic approximations of a bond’s value.](image)

6. Consider the following two bonds:

<table>
<thead>
<tr>
<th></th>
<th>Bond $A$</th>
<th>Bond $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>15 years</td>
<td>11 years</td>
</tr>
<tr>
<td>Coupon rate</td>
<td>10%</td>
<td>5%</td>
</tr>
<tr>
<td>Par value</td>
<td>$1000$</td>
<td>$1000$</td>
</tr>
</tbody>
</table>
(a) The current yield to maturity is taken to be 12%. Determine the convexity of each bond.

(b) Suppose you have a defensive strategy, and that you want to immunize the investor. What is each bond’s rate of return at horizon $H = D$ if interest rates keep jumping from 12% to either 10% or 14%?

(c) By examining the rates of return of the two bonds under an increase or decrease of interest rates, and different choices of horizon, which bond would you choose?

7. An investor is considering the purchase of 10-year U.S. Treasury bonds and plans to hold them to maturity. Federal taxes on coupons must be paid during the year they are received, and tax must also be paid on the capital gain realized at maturity (defined as the difference between face value and original price). This investor’s federal tax bracket rate is $r = 30\%$, as it is for most individuals. There are two bonds with par 100 that meet the investor’s requirements. Bond 1 is a 10-year, 10% bond with a price (in decimal form) of $P_1 = 92.21$. Bond 2 is a 10-year, 7% bond with a price of $P_2 = 75.84$. Based on the price information contained in these two bonds, the investor would like to compute the theoretical price of a hypothetical 10-year zero-coupon bond that had no coupon payments and required tax payment only at maturity equal in amount to 30% of the realized capital gain (the par value minus the original price). This theoretical price should be such that the price of this bond and those of bonds 1 and 2 are mutually consistent on an after-tax basis. Find this theoretical price, and show that it does not depend on the tax rate $t$. Assume all cash flows occur at the end of each year.

8. Orange County managed an investment pool into which several municipalities made short-term investments. A total of $7.5$ billion was invested in this pool, and this money was used to purchase securities. Using these securities as collateral, the pool borrowed $12.5$ billion from Wall Street brokerages, and these funds were used to purchase additional securities. The $20$ billion total was invested primarily in long-term fixed-income securities to obtain a higher yield than the short-term alternatives. Furthermore, as interest rates slowly declined, as they did in 1992-1994, an even greater return was obtained. Things fell apart in 1994, when interest rates rose sharply.

Hypothetically, assume that initially the duration of the invested portfolio was 10 years, the short-term rate was 6%, the average coupon interest on the portfolio was 8.5% of face value, the cost of Wall Street money was 7%, and short-term interest rates were falling at $\frac{1}{2}\%$ per year.

(a) What was the rate of return that pool investors obtained during this early period? Does it compare favorably with 6% that these investors would have obtained by investing normally in short-term securities?

(b) When interest rates had fallen two percentage points and began increasing at 2% per year, what rate of return was obtained by the pool?

Additional assumptions made in the calculations:

(a) Assume the bond portfolio is restructured annually to maintain a duration of 10 years.

(b) Assume the value of money borrowed is maintained at $12.5$ billion every year.

(c) Assume Orange County makes interest on borrowed fund at the rate which prevailed at the beginning of the given year.
Hints

• In the first year, the coupon rate was 8.5%. With a duration of 10 years, the change in portfolio value due to change in interest rate is given by

\[-\frac{duration}{1 + i} \cdot P \cdot change\ in\ i.\]

The interest cost of borrowing of 12.5 billion per annum with cost of fund 7% is given by 12.5×(0.07).

• In the fifth year, the coupon rate became 6.5% while the change in interest rate is 2%. The cost of fund became 5%.