MATH 4512 — Fundamentals of Mathematical Finance

Topic Four — Bond portfolio management and immunization

4.1 Duration measures and convexity

4.2 Horizon rate of return: return from the bond investment over a time horizon

4.3 Immunization of bond investment
4.1 Duration measures and convexity

Duration

The duration of a bond is the weighted average of the times of payment of all the cash flows generated by the bond, the weights being the proportional shares of the bond’s cash flows in the bond’s present value.

Macauley’s duration: internal rate of return is used

Let \( i \) denote the yield to maturity of the bond. Bond duration is

\[
D = 1 \frac{c/B}{1+i} + 2 \frac{c/B}{(1+i)^2} + \cdots + T \frac{c + B_T}{(1+i)^T} \\
= \frac{1}{B} \sum_{t=1}^{T} \frac{tc_t}{(1+i)^t}
\]  

(D1)
Measure of a bond’s sensitivity to change in yield to maturity

Starting from

\[ B = \sum_{t=1}^{T} c_t (1 + i)^{-t}, \]

\[ \frac{dB}{di} = \sum_{t=1}^{T} (-t)c_t (1 + i)^{-t-1} = -\frac{1}{1 + i} \sum_{t=1}^{T} tc_t (1 + i)^{-t}; \]

\[ \frac{1}{B} \frac{dB}{di} = -\frac{1}{1 + i} \sum_{t=1}^{T} \frac{tc_t (1 + i)^{-t}}{B} = -\frac{D}{1 + i}. \]

Suppose a bond is at par, its coupon is 9%, so YTM = 9%. The duration is found to be 6.99.

Suppose that the yield increases by 1%, then the relative change in bond value is

\[ \frac{\Delta B}{B} \approx \frac{dB}{B} = -\frac{D}{1 + i} di = -\frac{6.99}{1.09} \times 1\% = -6.4\%. \]
Mystery behind the factor $\frac{1}{1+i}$ in modified duration

Suppose there is only one par payment $P$ (as in a discount bond) that is paid at $T$ years, then

$$\frac{dB_{dis}}{di} = \frac{d}{di} \left[ \frac{P}{(1+i)^T} \right] = -\frac{T}{1+i} B_{dis},$$

where $i$ is the interest rate per annum. Obviously, the duration is $T$ in a discount bond.

If the interest rate is compounded $m$ times per year, then

$$\frac{dB_{dis}}{di} = \frac{d}{di} \left[ \frac{P}{(1 + \frac{i}{m})^{mT}} \right] = -\frac{T}{1 + \frac{i}{m}} B_{dis},$$

As $m \to \infty$, which corresponds to continuous compounding, we obtain

$$\frac{dB_{dis}}{B_{dis}} = -T di.$$

The factor $\frac{1}{1+i}$ disappears in continuous compounding. In discrete compounding, the interest rate at time $T$ is applied to the cash flow paid at one compounding period later.
How good is the linear approximation?

\[
\frac{\Delta B}{B} = \frac{B(10\%) - B(9\%)}{B(9\%)} = \frac{93.855 - 100}{100} = -6.145\%.
\]

**Modified duration**

Modified duration = \(D_M = \frac{\text{duration}}{1+i}\)

\[
\frac{\Delta B}{B} \approx \frac{dB}{B} = -D_m \cdot di
\]

\[
\text{var} \left( \frac{dB}{B} \right) = D_m^2 \text{var}(di).
\]

The standard deviation of the relative change in the bond price is a linear function of the standard deviation of the changes in interest rates, the coefficient of proportionality is the modified duration.
Example

A bond with annual coupon 70, par 1000, and yield 5%; duration was calculated at 7.7 years, modified duration \( \frac{77}{1.05} = 7.33 \text{ yr.} \)

A change in yield from 5% to 6% or 4% entails a relative change in the bond price approximately \(-7.33\%\) or \(+7.33\%\), respectively. The modified duration is seen to be the more accurate proportional factor.
Calculation of the duration of a bond with a 7% coupon rate for a yield to maturity $i = 5\%$

<table>
<thead>
<tr>
<th>Time of payment $t$</th>
<th>Cash flow in current value $C_t$</th>
<th>Cash flows in present value $(i = 5%)$ $C_t(1+i)^{-t}$</th>
<th>Share of cash flows in present value in bond's price $C_t(1+i)^{-t}$</th>
<th>Weighted time of payment (col. 1 X col. 4) $t(1+i)^{-t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70</td>
<td>66.67</td>
<td>0.0577</td>
<td>0.0577</td>
</tr>
<tr>
<td>2</td>
<td>70</td>
<td>63.49</td>
<td>0.0550</td>
<td>0.1100</td>
</tr>
<tr>
<td>3</td>
<td>70</td>
<td>60.47</td>
<td>0.0524</td>
<td>0.1571</td>
</tr>
<tr>
<td>4</td>
<td>70</td>
<td>57.59</td>
<td>0.0499</td>
<td>0.1995</td>
</tr>
<tr>
<td>5</td>
<td>70</td>
<td>54.85</td>
<td>0.0475</td>
<td>0.2375</td>
</tr>
<tr>
<td>6</td>
<td>70</td>
<td>52.24</td>
<td>0.0452</td>
<td>0.2715</td>
</tr>
<tr>
<td>7</td>
<td>70</td>
<td>49.75</td>
<td>0.0431</td>
<td>0.3016</td>
</tr>
<tr>
<td>8</td>
<td>70</td>
<td>47.38</td>
<td>0.0410</td>
<td>0.3283</td>
</tr>
<tr>
<td>9</td>
<td>70</td>
<td>45.12</td>
<td>0.0391</td>
<td>0.3517</td>
</tr>
<tr>
<td>10</td>
<td>1070</td>
<td>656.89</td>
<td>0.5690</td>
<td>5.6901</td>
</tr>
<tr>
<td>Total</td>
<td>1700</td>
<td>1154.44</td>
<td>1</td>
<td>$7.705 = \text{duration}$</td>
</tr>
</tbody>
</table>

$\text{bond value} = 1154.44.$
Duration of a bond as the center of gravity of its cash flows in present value (coupon: 7%; interest rate: 5%).
**Duration in terms of coupon rate, maturity and yield**

Recall $D = -\frac{1 + i dB}{B \frac{di}{di}}$ and $B = \frac{c}{i} \left[ 1 - \frac{1}{(1+i)^T} \right] + \frac{B_T}{(1+i)^T}$, we have

$$\frac{B}{B_T} = \frac{1}{i} \left\{ \frac{c}{B_T} \left[ 1 - \frac{1}{(1+i)^T} \right] + \frac{i}{(1+i)^T} \right\}.$$ 

$$\frac{d \ln \left( \frac{B}{B_T} \right)}{di} = \frac{d \ln B}{di} = \frac{1 dB}{B \frac{di}{di}} = \frac{1}{i} \left[ \frac{(c/B_T)T(1+i)^{-T-1} + (1+i)^{-T} + i(1+i)^{-T-1}(-T)}{(c/B_T)[1 - (1+i)^{-T}] + i(1+i)^{-T}} \right].$$

so that

$$D = -\frac{1 + i dB}{B \frac{di}{di}} = 1 + \frac{1}{i} + \frac{T \left( i - \frac{c}{B_T} \right) - (1 + i)}{\frac{c}{B_T}[(1+i)^T - 1] + i}.$$  

(D2)
Duration of a bond as a function of its maturity for various coupon rates ($i = 10\%$).
• The impact of the coupon rate $c/B_T$ and maturity $T$ on duration $D$ can be deduced from the last term.

• When the coupon rate $\frac{c}{B_T}$ is less than $i$, the numerator may change sign at $T^*$ where

$$T^* \left( i - \frac{c}{B_T} \right) = 1 + i.$$ 

That is, $D$ may assume value above $1 + \frac{1}{i}$ when $T > T^*$. When $\frac{c}{B_T} > i$, $T^*$ does not exist, as $D$ always stays below $1 + \frac{1}{i}$. 
Adding one additional cash flow at \( T+1 \) has two effects on the right end of the scale:
1. It *adds weight* at the end of the scale, first by adding a new coupon at $T + 1$, and second by increasing the length of the lever corresponding to the reimbursement of the principal.

2. It *removes weight* at the end of the scale by replacing $A$ with a smaller quantity, $A' = A/(1 + i)$.

Therefore, the total effect on the center of gravity (on duration) is ambiguous and requires more detailed analysis.
With an increase in coupon rate $c/B_T$, should there always be a definite increase in duration?

- Just looking at eq. (D1): \[ D = \frac{1}{B} \sum_{t=1}^{T} \frac{tc_t}{(1+i)^t}, \] it is not apparent since the bond value $B$ also depends on the coupon rate.

- From eq. (D.2), it is seen that the numerator (denominator) in the last term decreases (increases) with increasing $c/B_T$. Hence, $D$ decreases with increasing $c/B_T$.

- Intuitively, when the coupon rate increases, the weights will be tilted towards the left, and the center of gravity will move to the left.
**Perpetual bond – infinite maturity**

- The last term in eq. (D2) gives the impact of maturity on \( D \). When we consider a perpetual bond, where \( T \to \infty \), the final par payment is immaterial.

- Recall that \( D \) can be thought of as the average time that one has to wait to get back the money from the bond issuer. Interestingly,

\[
D \to 1 + \frac{1}{i} \quad \text{as} \quad T \to \infty.
\]  

(\( D3 \))

When \( i = 10\% \), we have \( D \to 11 \) as \( T \to \infty \).

The inverse of the yield \( i \) is the amount of time that one has to wait to recoup 100\% of your money. Since the payment of coupons is not continuous, one has to wait one extra year, so this leads to \( 1 + \frac{1}{i} \). In Qn 1 of HW 1, when there are \( m \) compounding periods in one year, we have \( D \to \frac{1}{m} + \frac{1}{i} \) as \( T \to \infty \).
**Relationship between duration and maturity**

1. For zero-coupon bonds, duration is always equal to maturity.

   For all coupon-bearing bonds,

   \[
   \text{duration} \to 1 + \frac{1}{i} \quad \text{when maturity increases infinitely.}
   \]

   The limit is independent of the coupon rate.

2. Coupon rate $\geq$ YTM (bonds above par)

   An increase in maturity entails an increase in duration towards the limit $1 + \frac{1}{i}$.

3. Coupon rate $<$ YTM (bonds below par)

   When maturity increases, duration first increases, pass through a maximum and decreases toward the limit $1 + \frac{1}{i}$. 
Relationship between duration and yield

Should the yield to maturity increases, the center of gravity will move to the left and duration will be reduced. Actually

\[
\frac{dD}{di} = -\frac{S}{1+i},
\]

where \(S\) is the dispersion or variance of the payment times of the bond.
Proof

Starting from

\[ D = \frac{1}{B} \sum_{t=1}^{T} t c_t (1 + i)^{-t} \]

\[
\frac{dD}{di} = -\frac{1}{B^2} \left[ \sum_{t=1}^{T} t^2 c_t (1 + i)^{-t-1} B(i) + \sum_{t=1}^{T} t c_t (1 + i)^{-t} B'(i) \right]
\]

\[
= -\frac{1}{1 + i} \left[ \sum_{t=1}^{T} \frac{t^2 c_t (1 + i)^{-t}}{B(i)} + (1 + i) \frac{B'(i) \sum_{t=1}^{T} t c_t (1 + i)^{-t}}{B(i)} \right] - D \frac{D}{D} - D^2
\]
If we write

\[ w_t = \frac{c_t(1 + i)^{-t}}{B(i)} \]

so that

\[ \sum_{t=1}^{T} w_t = 1 \quad \text{and} \quad D = \sum_{t=1}^{T} tw_t. \]

Here, \( w_t \) is the share of the bond’s cash flow \( c_t \) (in the present value) in the bond’s value. The bracket term becomes

\[ \sum_{t=1}^{T} t^2w_t - D^2 = \sum_{t=1}^{T} w_t(t - D)^2, \]

which is equal to the weighted average of the squares of the difference between the times \( t \) and their average \( D \).
We obtain

\[
\frac{dD}{di} = -\frac{1}{1 + i} \sum_{t=1}^{T} w_t (t - D)^2
\]

which is always negative.

- Intuitively, since the discount factor is \((1 + i)^{-t}\), an increase in \(i\) will move the center of gravity to the left, and the duration is reduced.
Fisher-Weil’s duration

- Fisher-Weil’s duration: weights of the times of payment make use of the spot rates pertaining to each term.

- Term structure of the spot interest rates over successive years:

\[ i(0, 1), i(0, 2), \ldots, i(0, t), \ldots, i(0, T). \]

\[
i(z) = \frac{1}{\int_0^{z+dz} e^{i(z)dz}} \]

growth factor over the time interval \([z, z + dz] = e^{i(z)dz} \).
Let $i(z)$ denote the instantaneous forward rate, the growth factor over the period $[0, t]$ based on $i(z)$ is given by

$$e^{\int_0^t i(z) \, dz} = e^{i(0,t) t}$$

so that

$$i(0, t) = \frac{1}{t} \int_0^t i(z) \, dz.$$ 

The spot rate $i(0, t)$ is the average of all implicit forward rates.

Since

$$[i(0, t) + \alpha] t = \int_0^t i(z) \, dz + \int_0^t \alpha \, dz = \int_0^t [i(z) + \alpha] \, dz$$

so increasing $i(0, t)$ by $\alpha$ is equivalent to displacing vertically by $\alpha$ the implicit forward rates structure.
Mystery behind duration

- We would like to understand the financial intuition why duration is the multiplier that relates relative change in bond value and interest rate.

- Under the continuous framework, the bond value $B(\vec{i})$ is given by

$$B(\vec{i}) = \int_0^T c(t) e^{-i(0,t)t} \, dt$$

where $c(t)$ is the cash flow received at time $t$. 
This is considered as a *functional* since this is a relation between a function $\vec{i}$ (term structure of interest rates) and a number $B(\vec{i})$.

Naturally, duration of the bond with the initial term structure is

$$D(\vec{i}) = \frac{1}{B(\vec{i})} \int_0^T tc(t)e^{-i(0,t)t} dt,$$

where $\frac{c(t)e^{-i(0,t)t}}{B(\vec{i})} dt$ represents the weighted present value of cash flow within $(t, t + dt)$.

Suppose the whole term structure of interest rates move up by $\Delta \alpha$, then

$$B(\vec{i} + \Delta \alpha) = \int_0^T c(t)e^{-i(0,t)t}e^{-t\Delta \alpha} dt.$$
Note that when $\Delta \alpha$ is infinitesimally small, we have

$$e^{-t\Delta \alpha} \approx 1 - t\Delta \alpha$$

so that the discounted cash flow $e^{-i(0,t)t}c(t)\,dt$ within $(t, t + dt)$ falls in proportional amount $t\Delta \alpha$.

The corresponding contribution to the relative change in value as normalized by $B(\vec{i})$ is

$$\frac{tc(t)e^{-i(0,t)t}\,dt}{B(\vec{i})}\Delta \alpha.$$  

This is the payment time weighted by discounted cash flow within $(t, t + dt)$ multiplied by the change in interest rate $\Delta \alpha$. 
The total relative change in bond value

\[
\frac{\Delta B}{B} = -\Delta \alpha \left[ \text{weighted average of payment times that are weighted according to present value of cash flow} \right] = -D \Delta \alpha.
\]

In the differential limit, \( \frac{B(\vec{i} + \Delta \alpha) - B(\vec{i})}{B(\vec{i})} \) becomes \( \frac{dB}{B} \) and \( \Delta \alpha \) becomes \( di \), so we obtain

\[
\frac{dB}{B} = -D(\vec{i}) \, di.
\]

• In the discrete case of annual compounding, we need to modify the duration multiplier by multiplying the discount factor \( \frac{1}{1+i} \) over one year so that

\[
\frac{1 \, dB}{B \, di} = -\frac{D}{1+i}.
\]
4.2 Horizon rate of return: return from the bond investment over a time horizon

**Horizon rate of return**, $r_H$ – bond is kept for a time horizon $H$

It is the return that transforms an investment bought today at price $B_0$ by its owner into a future value at horizon $H$, which is $F_H$.

\[ B_0(1 + r_H)^H = F_H \]

or

\[ r_H = \left( \frac{F_H}{B_0} \right)^{1/H} - 1. \]

- All coupons to be paid are supposed to be regularly reinvested until horizon $H$ ($< T$). As a result, we need to evaluate the future rates of interest at which it will be possible to reinvest these coupons in order to obtain a fair estimate of $F_H$. In other words, when interest rates change in the future, the interests earned from reinvesting the coupons will change.
• Suppose a bond investor bought a bond valued at $B(i_0)$ when the interest rate common to all maturities was $i_0$ (flat rate). On the following day, the interest rate moves up to $i$ (parallel shift). The new future value at $H$ given the bond price $B(i)$ at the new interest rate level $i$ is given by $B(i)(1 + i)^H$. To the investor, by paying $B_0$ as the initial investment, the new horizon rate of return is given by $B_0(1 + r_H)^H = B(i)(1 + i)^H$ and so

$$r_H = \left[ \frac{B(i)}{B_0} \right]^{1/H} (1 + i) - 1.$$  

• The impact on the bond value on changing interest rate is spread out in $H$ years. Future cash flows from the bond are compounded annually at the new interest rate $i$. 


• As a function of $i$, the horizon rate of return $r_H$ is a product of a decreasing function $B(i)$ and an increasing function $(1 + i)$. This represents a counterbalance between an immediate capital gain/loss and rate of return on the cashflows from now till $H$.

• Whatever the horizon, the rate of return will always be $i_0$ if $i$ does not move away from this value. In this case,

$$F_H = B_0(1 + i_0)^H = B_0(1 + r_H)^H$$

so that $r_H = i_0$ for any $H$.

• If $H \to \infty$, then $r_H = \left[\frac{B(i)}{B_0}\right]^{1/H} (1 + i) - 1 \to i$. With infinite time of horizon, the immediate change of bond price is immaterial. The horizon rate of return is simply the new prevailing interest rate $i$. 


The table shows the horizon return (in percentage per year) on the investment in a 7% coupon, 10-year maturity bond bought at 1154.44 when interest rates were at 5%, should interest rates move immediately either to 6% or 4%. At $H = 7.7$, $r_H$ increases when $i$ either increases or decreases.

<table>
<thead>
<tr>
<th>Horizon (years)</th>
<th>Interest rates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4%</td>
</tr>
<tr>
<td>1</td>
<td>12.01</td>
</tr>
<tr>
<td>2 increasing</td>
<td>7.93</td>
</tr>
<tr>
<td>3</td>
<td>6.60</td>
</tr>
<tr>
<td>4</td>
<td>5.95</td>
</tr>
<tr>
<td>5</td>
<td>5.55</td>
</tr>
<tr>
<td>6</td>
<td>5.29</td>
</tr>
<tr>
<td>7</td>
<td>5.11</td>
</tr>
<tr>
<td>7.7</td>
<td>5.006</td>
</tr>
<tr>
<td>8</td>
<td>4.97</td>
</tr>
<tr>
<td>9</td>
<td>4.86</td>
</tr>
<tr>
<td>10</td>
<td>4.77</td>
</tr>
<tr>
<td>$\infty$</td>
<td>4.00</td>
</tr>
</tbody>
</table>
Example – Calculation of $r_H$

A 10-year bond with coupon rate of 7% was bought when the interest rates were at 5%. We have $B(i_0) = $1154.44.

Suppose on the next day, the interest rates move up to 6%. The bond drops in value to $1073.60. If he holds his bond for 5 years, and if interest rates stay at 6%, then

$$r_H = \left( \frac{1073.60}{1154.44} \right)^{1/5} \times (1.06) - 1 = 4.47\%.$$
Observation

Though the rate of interest at which the investor can reinvest his coupons (which is now 6%) is higher, his overall performance will be lower than 5% ($r_H$ is only 4.47%).

Counterbalance between

- Capital loss on the bond value due to an increase in the interest rate.
- Gain from the reinvestment of coupons at a higher rate.
• The longer the horizon and the longer the reinvestment of the coupons at a higher rate, the greater the chance that the investor will outperform the initial yield of 5%.

• If the horizon is 7.7 years, then the horizon rate of return will be slightly above 5% (5.006%) whether the interest rate falls to 4% or increases to 6%.

• Comparing the one-year horizon and four-year horizon, if interest rates rise, the four-year horizon return is higher than the one-year return. Here, we assume that bond’s maturity is longer than four years.
1. The capital loss will be less severe with the four-year horizon since the bond’s value will have come back closer to par after four years than after one year.

2. The four-year investor will be able to benefit from the coupon reinvestment at a higher rate.

• For the extreme case of $H \to \infty, r_H = i$. The immediate capital gain/loss is immaterial since all cash flows from the bonds remain the same while they can be reinvested at the rate of return $i$. 
• The horizon rate of return is a decreasing function of $i$ when the horizon is short and an increasing one for long horizons. For a horizon equal to the duration of the bond, the horizon rate of return first decreases, goes through a minimum for $i = i_0$ then increases by $i$.

• There is a critical value for $H$ such that $r_H$ changes from a decreasing function of $i$ to an increasing function of $i$. This critical value is the bond duration. Why?
Dependence of $r_H$ on $i$ with varying $H$
The critical $H$ happens to be $D$. Why?

Relative change in bond value due to change of interest rate of amount $\Delta i$:

$$\frac{\Delta B}{B} \approx -D \times \Delta i$$

- The immediate capital loss of amount $D\Delta i$ is spread over $H$ years.
- The gain in a higher rate of return of the future cash flows is $H\Delta i$ over $H$ years of horizon of investment.
- These two effects are counterbalanced if $H = D$. 
Remark

Suppose an investor is targeting at a time horizon of investment $H$, he or she should choose a bond whose duration equals $H$ so that the horizon rate of return is immunized from any change in the interest rate.

Stronger mathematical result

There exists a horizon $H$ such that $r_H$ always increases when the interest rate moves up or down from the initial value $i_0$. The more precise statement is stated in the following theorem.

Theorem

There exists a horizon $H$ such that the rate of return for such a horizon goes through a minimum at point $i_0$. 
Proof

Minimizing $r_H$ is equivalent to minimizing any positive transformation of it, and so it is equivalent to minimizing $F_H$. Consider

$$\frac{dF_H}{di} = \frac{d}{di}[B(i)(1 + i)^H] = B'(i)(1 + i)^H + HB(i)(1 + i)^{H-1},$$

We would like to find $H$ such that the first order condition : $\frac{dF_H}{di} = 0$ at $i = i_0$ is satisfied. This gives

$$B'(i_0)(1 + i_0) + HB(i_0) = 0$$

and

$$H = -\frac{1 + i_0}{B(i_0)}B'(i_0) = \text{duration}.$$

The horizon must be chosen to be equal to the duration at the initial rate of return $i_0$ for $F_H$ to run through a minimum. If otherwise, then $\frac{dF_H}{di} = 0$ at $i = i_0$ cannot be satisfied. This is revealed by the other curves (see P.78) that pass through $i = i_0$ but they are either monotonic increasing or decreasing in $i$. 
Checking the second order condition

We have the first order condition for \( r_H \) to go through a minimum. Since \( \frac{d^2 \ln F_H}{d i^2} \bigg|_{i=i_0} = \frac{1}{F_H} \frac{d^2 F_H}{d i^2} \bigg|_{i=i_0} \), it suffices to show that \( \ln F_H(i) \) has a positive 2\(^{nd} \) derivative.

\[
\frac{d \ln F_H}{d i} = \frac{d}{d i} \ln B(i) + \frac{H}{1 + i} = \frac{1}{B(i)} \frac{d B(i)}{d i} + \frac{H}{1 + i} = \frac{-D + H}{1 + i}
\]

\[
\frac{d^2 \ln F_H}{d i^2} = \frac{1}{(1 + i)^2} \left[ -\frac{dD}{d i} (1 + i) + D - H \right].
\]

Setting \( H = D \), we obtain

\[
\frac{d^2 \ln F_H}{d i^2} = -\frac{1}{1 + i} \frac{dD}{d i} = \frac{S}{(1 + i)^2} > 0.
\]

Therefore, \( F_H \) and \( r_H \) go through a global minimum at point \( i = i_0 \) whenever \( H = D \).
Continuous case

Define $F_H(\vec{i})$ to be the future value when the term structure of interest rates is $\vec{i}$. Recall that the future value based on $\vec{i}$ is given by $F_H(\vec{i}) = B(\vec{i})e^{i(0,H)}$. Under a parallel shift of $\alpha$, the new future value is

$$F_H(\vec{i} + \alpha) = B(\vec{i} + \alpha)e^{i(0,H)+\alpha}H.$$ 

As before, we determine $H$ such that $F_H$ is minimized at $\vec{i}$. We consider

$$\frac{d}{d\alpha} \ln F_H(\vec{i} + \alpha) \bigg|_{\alpha=0} = \frac{d}{d\alpha} \ln B(\vec{i} + \alpha) \bigg|_{\alpha=0} + H = 0$$

so that

$$H = -\frac{1}{B(\vec{i})} \frac{dB(\vec{i})}{d\alpha} = D(\vec{i}).$$

The immunizing horizon must be equal to the duration.
First order condition for immunization

\[ F_H(\vec{i} + \alpha) \]

\[ F_H(\vec{i}) \]

\[ (-) \quad 0 \quad (+) \quad \alpha \]
Second order condition for immunization

First, we consider

\[ \frac{d^2 \ln F_H}{d\alpha^2} = \frac{d}{d\alpha} \left[ \frac{1}{B(\vec{i} + \alpha)} \frac{dB(\vec{i} + \alpha)}{d\alpha} \right] = \frac{d}{d\alpha} [-D(\vec{i} + \alpha)]. \]

The derivative of \( D(\vec{i} + \alpha) \) with respect to \( \alpha \) is equal to

\[
\frac{dD(\vec{i} + \alpha)}{d\alpha} = \frac{d}{d\alpha} \left\{ \frac{1}{B(\vec{i} + \alpha)} \int_0^T tc(t)e^{-[i(0,t)+\alpha]t} dt \right\} = \frac{1}{[B(\vec{i} + \alpha)]^2} \left\{ \int_0^T -t^2c(t)e^{-[i(0,t)+\alpha]t} dt \cdot B(\vec{i} + \alpha) \right. \\
\left. - \int_0^T tc(t)e^{-[i(0,t)+\alpha]t} dt \cdot B'(\vec{i} + \alpha) \right\} = \frac{\int_0^T t^2c(t)e^{-[i(0,t)+\alpha]t} dt}{B(\vec{i} + \alpha)} + \frac{\int_0^T tc(t)e^{-[i(0,t)+\alpha]t} dt \cdot B'(\vec{i} + \alpha)}{B(\vec{i} + \alpha)}
\]
The expression in the second term of the last brace is recognized as $-D^2(\vec{i} + \alpha)$. Simplifying the notation, we may thus write

\[
w(t) = \frac{c(t)e^{-[i(0,t)+\alpha]t}}{B(\vec{i} + \alpha)} \quad \text{with} \quad \int_0^T w(t) \, dt = 1,
\]

and $D(\vec{i} + \alpha) = D(\alpha)$. We obtain

\[
\frac{dD}{d\alpha} = - \left\{ \int_0^T t^2 w(t) \, dt - D^2 \right\}
\]

\[
= - \left\{ \int_0^T t^2 w(t) \, dt - 2D \int_0^T tw(t) \, dt + D^2 \int_0^T w(t) \, dt \right\}
\]

\[
= - \int_0^T w(t)[t^2 - 2tD + D^2] \, dt = - \int_0^T w(t)(t - D)^2 \, dt.
\]
As a happy surprise, the last expression is none other than minus the dispersion (or variance) of the terms of the bond, and such a variance is of course always positive. Denoting the bond’s variance as $S(\vec{i}, \alpha)$, we may write

$$\frac{d^2 \ln F_H}{d\alpha^2} = -\frac{dD}{d\alpha} = S(\vec{i}, \alpha) > 0.$$ 

Consequently, we may conclude that $\ln F_H$ is indeed convex.

- Together with the first order condition, this condition is sufficient for $\ln F_H$ to go through a global minimum at point $\alpha = 0$. 

Main formulas

• Value of a bond in continuous time, with $\vec{i} \equiv i(0, t)$ being the term structure or interest rates:

$$B(\vec{i}) = \int_0^T c(t)e^{-i(0,t)t} \, dt$$

• Duration of the bond:

$$D(\vec{i}) = \frac{1}{B(\vec{i})} \int_0^T tc(t)e^{-i(0,t)t} \, dt$$

• Duration of the bond when $\vec{i}$ receives a (constant) variation $\alpha$:

$$D(\vec{i} + \alpha) = \frac{1}{B(\vec{i} + \alpha)} \int_0^T tc(t)e^{-[i(0,t)+\alpha]t} \, dt$$
- Fundamental property of duration:
  \[-\frac{1}{B(\vec{i})} \frac{dB(\vec{i})}{d\alpha} = \frac{1}{B(\vec{i})} \int_0^T tc(t) e^{-i(0,t)t} dt = D\]

- First order condition for immunization of \( r_H \) against a parallel shift of interest rates:
  \[H = -\frac{1}{B(\vec{i})} \frac{dB(\vec{i})}{d\alpha} = D(\vec{i})\]

- Second order condition for immunization:
  \[\frac{d^2 \ln F_H}{d\alpha^2} = -\frac{d}{d\alpha} [D(\vec{i} + \alpha)] = S(\vec{i}, \alpha) > 0\]
4.3 Immunization of bond investment

• In the case of either a drop or a rise in interest rates, when the horizon was properly chosen, the horizon rate of return for the bond’s owner was about the same as if interest rates had not moved. This horizon is the duration of the bond.

• Immunization is the set of bond management procedures that aim at protecting the investor against changes in interest rates.

• It is dynamic since the passage of time and changes in interest rates will modify the portfolio’s duration by an amount that will not necessarily correspond to the steady and natural decline of the investor’s horizon.
• Even if interest rates do not change, the simple passage of one year will reduce duration of the portfolio by less than one year. The money manager will have to change the composition of the portfolio so that the duration is reduced by a whole year.

• Changes in interest rates will also modify the portfolio’s duration.

• *Immunization* may be defined as the process by which an investor can protect himself against interest rate changes by suitably choosing a bond or a portfolio of bonds such that *its duration is kept equal to his horizon dynamically*. 
Numerical example

A company has an obligation to pay $1 million in 10 years. That is, the future value at the time of horizon 10 years is $1 million. It wishes to invest money now that will be sufficient to meet this obligation. The purchase of a single zero-coupon bond would provide one solution, but such discount bonds are not always available in the required maturities.

<table>
<thead>
<tr>
<th></th>
<th>coupon rate</th>
<th>maturity</th>
<th>price</th>
<th>yield</th>
<th>duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>bond 1</td>
<td>6%</td>
<td>30 yr</td>
<td>69.04</td>
<td>9%</td>
<td>11.44</td>
</tr>
<tr>
<td>bond 2</td>
<td>11%</td>
<td>10 yr</td>
<td>113.01</td>
<td>9%</td>
<td>6.54</td>
</tr>
<tr>
<td>bond 3</td>
<td>9%</td>
<td>20 yr</td>
<td>100.00</td>
<td>9%</td>
<td>9.61</td>
</tr>
</tbody>
</table>

- The above 3 bonds all have the yield of 9%. Present value of obligation at 9% yield is $414,643.
• Since bond 2 and bond 3 have their duration shorter than 10 years, it is not possible to attain a portfolio with duration 10 years using these two bonds. A bond with a longer maturity is required (say, bond 1). The coupons received are reinvested earning rate of return at the prevailing yield.

Suppose we use bond 1 and bond 2 of notional amount $V_1$ and $V_2$ in the portfolio, by matching the present value and duration, we obtain

\[
V_1 + V_2 = PV = $414,643
\]
\[
D_1V_1 + D_2V_2 = 10 \times PV = $4,146,430
\]

giving

\[
V_1 = $292,788.64\quad \text{and}\quad V_2 = $121,854.78.
\]
What would happen when we have a sudden change in the prevailing yield?

<table>
<thead>
<tr>
<th>Yield</th>
<th>9.0</th>
<th>8.0</th>
<th>10.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Price</td>
<td>69.04</td>
<td>77.38</td>
<td>62.14</td>
</tr>
<tr>
<td>shares</td>
<td>4241</td>
<td>4241</td>
<td>4241</td>
</tr>
<tr>
<td>value</td>
<td>292,798.64</td>
<td>328,168.58</td>
<td>263,535.74</td>
</tr>
<tr>
<td>Bond 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Price</td>
<td>113.01</td>
<td>120.39</td>
<td>106.23</td>
</tr>
<tr>
<td>shares</td>
<td>1078</td>
<td>1078</td>
<td>1078</td>
</tr>
<tr>
<td>value</td>
<td>121,824.78</td>
<td>129,780.42</td>
<td>114,515.94</td>
</tr>
<tr>
<td>Obligation value</td>
<td>414,642.86</td>
<td>456,386.95</td>
<td>376,889.48</td>
</tr>
<tr>
<td>Surplus</td>
<td>-19.44</td>
<td>1,562.05</td>
<td>1,162.20</td>
</tr>
</tbody>
</table>

- Surplus at 8% yield = $328,168.58 + 129,780.42 − 456,386.95 = 1,562.05.$
Observation: At different yields (8% and 10%), the value of the portfolio almost agrees with that of obligation (at the new yield).

Difficulties

• It is quite unrealistic to assume that both the long- and short-duration bonds can be found with identical yields. Usually longer-maturity bonds have higher yields.

• When interest rates change, it is unlikely that the yields on all bonds will change by the same amount.
Convexity and its uses in bond portfolio management

<table>
<thead>
<tr>
<th></th>
<th>Coupon rate</th>
<th>maturity</th>
<th>price</th>
<th>yield to maturity</th>
<th>duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond A</td>
<td>9%</td>
<td>10 years</td>
<td>$1,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bond B</td>
<td>3.1%</td>
<td>8 years</td>
<td>$673</td>
<td>9%</td>
<td>6.99 years</td>
</tr>
</tbody>
</table>

Bond $B$ is found such that it has the same duration and YTM as Bond $A$. Bond $B$ has coupon rate 3.1% and maturity equals 8 years. Its price is $673.

- Portfolio $\alpha$ consists of 673 units of Bond $A$ ($673,000$)
- Portfolio $\beta$ consists of 1,000 units of Bond $B$ ($673,000$)
Would an investor be indifferent to these two portfolios since they are worth exactly the same, offer the same YTM and have the same duration (apparently faced with the same interest rate risk)?

Recall the formula:

\[ \text{Convexity} = \frac{1}{B(i)} \frac{d}{di} \left( \frac{dB}{di} \right) = \frac{1}{B} \frac{d^2B}{di^2}. \]

What makes a bond more convex than the other one if they have the same duration? The key is the dispersion of payment times.

\[ \text{Convexity} = \frac{\text{dispersion} + \text{duration} \ (\text{duration} + 1)}{(1 + i)^2}. \]

- The convexity has the second order effect on bond portfolio management.

- Higher convexity is resulted with higher dispersion of payment times, say, in bonds with longer maturities.
The effect of a greater convexity for bond A (longer maturity) than for bond B enhances an investment in A compared to an investment in B in the event of change in interest rates. Investment in A will gain more value than investment in B if interest rates drop and it will lose less value if interest rates rise.
\[ C = \frac{1}{B} \frac{d^2 B}{di^2} = \frac{1}{B(1+i)^2} \sum_{t=1}^{T} t(t+1)c_t(1+i)^{-t} \]

**Calculation of the convexity of bond A** (coupon: 9%; yield to maturity: 10 years)

<table>
<thead>
<tr>
<th>Time of payment</th>
<th>Cash flow in nominal value ( c_t )</th>
<th>Share of the discounted cash flows in bond’s value ( c_t(1+i)^{-t}/B )</th>
<th>( t(t+1) ) times share of discounted cash flows ( = (2) \times (4) )</th>
<th>( t(t+1) ) times share of discounted cash flows ( = (2) \times (4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>( t(t+1) )</td>
<td>( t(t+1) )</td>
<td>( t(t+1) )</td>
<td>( t(t+1) )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>9</td>
<td>0.0826</td>
<td>0.165</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>9</td>
<td>0.0758</td>
<td>0.456</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>9</td>
<td>0.0695</td>
<td>0.834</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>9</td>
<td>0.0638</td>
<td>1.275</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>9</td>
<td>0.0585</td>
<td>1.755</td>
</tr>
<tr>
<td>6</td>
<td>42</td>
<td>9</td>
<td>0.0537</td>
<td>2.254</td>
</tr>
<tr>
<td>7</td>
<td>56</td>
<td>9</td>
<td>0.0492</td>
<td>2.757</td>
</tr>
<tr>
<td>8</td>
<td>72</td>
<td>9</td>
<td>0.0452</td>
<td>3.252</td>
</tr>
<tr>
<td>9</td>
<td>90</td>
<td>9</td>
<td>0.0414</td>
<td>3.729</td>
</tr>
<tr>
<td>10</td>
<td>110</td>
<td>109</td>
<td>0.4604</td>
<td>50.647</td>
</tr>
</tbody>
</table>

Convexity: total of \( (5) \times \frac{1}{(1.09)^2} = 56.5 \) (years\(^2\)
**Improvement in the measurement of a bond’s price change by using convexity**

<table>
<thead>
<tr>
<th>Change in the rate of interest</th>
<th>Change in the bond's price in linear approximation (using duration)</th>
<th>Change in the bond's price in quadratic approximation (using both duration and convexity)</th>
<th>Change in the bond's price in exact value</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1%</td>
<td>-6.4%</td>
<td>-6.135%</td>
<td>-6.14%</td>
</tr>
<tr>
<td>-1%</td>
<td>+6.4%</td>
<td>+6.700%</td>
<td>+6.71%</td>
</tr>
</tbody>
</table>
By Taylor series, the relative increase in the bond’s value is given in quadratic approximation by

\[
\frac{\Delta B}{B} \approx \frac{1}{B} \frac{dB}{di} + \frac{1}{2} \frac{d^2B}{di^2} (di)^2
\]

\[
= -6.4176 + 0.2825 = -6.135%.
\]

On the other hand, suppose \(i\) decreases by 1%. With \(di\) equal to \(-1\%\), we obtain

\[
\frac{\Delta B}{B} \approx +6.4176 + 0.2825 = 6.700%.
\]

- Most financial services provide the value of convexity for bonds by dividing the value of \(\frac{1}{B} \frac{d^2B}{di^2}\) by 200, served as the adjustment to the modified duration based on the change of interest rate of 1%.

- In the numerical example, \(di = 0.01\), \(C = \frac{1}{B} \frac{d^2B}{di^2} = 56.6\), so \(\frac{1}{B} \frac{d^2B}{di^2} \frac{1}{200} = 0.2825\). When this is added or subtracted from the modified duration, we obtain \(-6.135\%\) and \(+6.700\%\).
Linear and quadratic approximations of a bond’s value
The linear approximation is given by the following equation:

\[
\frac{\Delta B}{B_0} = \frac{1}{B_0} \frac{dB}{di} di = -D_m di, \quad D_m = \frac{\text{duration}}{1 + i}.
\]

Replacing \(\Delta B\) by \(B - B_0\) and \(di\) by \(i - i_0\), we have

\[
B(i) = B_0[(1 + D_m i_0) - D_m i].
\]

The quadratic approximation is given by

\[
\frac{\Delta B}{B_0} = \frac{1}{B_0} \frac{dB}{di} di + \frac{1}{2} \frac{1}{B_0} \frac{d^2B}{di^2} (di)^2 = -D_m di + \frac{1}{2} C(di)^2.
\]
What makes a bond convex?

To make the convexity \( \frac{1}{B} \frac{d^2B}{di^2} \) appear, we consider the derivative of \( D \) and equate the result to \(-\frac{S}{1+i}\). Now

\[
D = -\frac{1+i}{B(i)}B'(i) \quad \text{so} \quad \frac{dD}{di} = -\left[ \frac{B - (1+i)B'}{B^2} \right] B'(i) - \frac{1+i}{B}B''.
\]

Recall \( \frac{dD}{di} = -\frac{S}{1+i} \) so that

\[
\frac{1}{B}(1 + D)B'(i) + \frac{1+i}{B}B'' = \frac{S}{1+i}.
\]

Writing \( \frac{B'}{B} = -\frac{D}{1+i} \) and \( \frac{B''}{B} = C \) so that

\[-D(D+1) + (1+i)^2C = S.
\]

Finally, we obtain

\[
C = \frac{S + D(D+1)}{(1+i)^2}.
\]
Yield curve strategies

• Seek to capitalize on investors’ market expectations based on the short-term movements in yields.

• Source of return depends on the maturity of the securities in the portfolio since different parts of the yield curve respond differently to the same economic stock (like the operation twist in September 2011 where the US Fed bought long-term treasury bonds and sell short-term treasury bonds).

In most circumstances, yield curve is upward sloping with maturity and eventually level off at sufficiently high value of maturity. How to choose the spacing of the maturity of bonds in the portfolio?
Bullet strategy

Maturity of the securities are highly concentrated around one maturity date.

Barbell strategy

Maturity of the securities are concentrated at two extreme maturities.
<table>
<thead>
<tr>
<th>Bond</th>
<th>Coupon</th>
<th>Maturity</th>
<th>Price</th>
<th>YTM</th>
<th>Duration</th>
<th>Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8.5%</td>
<td>5</td>
<td>100</td>
<td>8.50</td>
<td>4.005</td>
<td>19.8164</td>
</tr>
<tr>
<td>B</td>
<td>9.5%</td>
<td>20</td>
<td>100</td>
<td>9.50</td>
<td>8.882</td>
<td>124.1702</td>
</tr>
<tr>
<td>C</td>
<td>9.25%</td>
<td>10</td>
<td>100</td>
<td>9.25</td>
<td>6.434</td>
<td>55.4506</td>
</tr>
</tbody>
</table>

In general, yield increases with maturity while the increase in convexity is more significant with increasing maturity. There is a 75 bps increase from 5-year maturity to 10-year maturity but only a 25 bps increase from 10-year maturity to 20-year maturity.

- **Bullet portfolio**: 100% bond C
- **Barbell portfolio**: 50.2% bond A and 49.8% bond B
  
  duration of barbell portfolio = \(0.502 \times 4.005 + 0.498 \times 8.882\)
  
  \[= 6.434\]
  
  convexity of barbell portfolio = \(0.502 \times 19.8164 + 0.498 \times 124.1702\)
  
  \[= 71.7846\].
**Yield**

As the bond prices are equal to par, so one can deduce that YTM = coupon.

portfolio yield for the barbell portfolio

\[ = 0.502 \times 8.5\% + 0.498 \times 9.5\% = 8.998\% < \text{yield of bond C} \]

**Duration**

Both strategies have the same duration.

**Convexity**

convexity of barbell > convexity of bullet

The lower value of yield for the barbell portfolio is a reflection of the level off effect of yield at higher maturity. The barbell strategy gives up yield in order to achieve a higher convexity.
Assume a 6-month investment horizon

1. Yield curve shifts in a parallel fashion

   When the change in yield $\Delta \lambda < 100$ basis points, the bullet portfolio outperforms the barbell portfolio; vice versa if otherwise.

   If $\lambda$ shifts parallel in a small amount, the portfolio with less convexity provides a better total return. The change in yield has to be more significant in order that the high convexity portfolio can outperform.

2. Non-parallel shift (flattening of the yield curve)

   \[
   \Delta \lambda \text{ of bond } A = \Delta \lambda \text{ of bond } C + 45 \text{ bps} \\
   \Delta \lambda \text{ of bond } B = \Delta \lambda \text{ of bond } C - 15 \text{ bps}
   \]

   The barbell strategy always outperforms the bullet strategy. This is due to the yield pickup for shorter-maturity bonds.
3. Non-parallel shift (steepening of the yield curve)

\[ \Delta \lambda \text{ of bond } A = \Delta \lambda \text{ of bond } C - 25 \text{ bps} \]
\[ \Delta \lambda \text{ of bond } B = \Delta \lambda \text{ of bond } C + 25 \text{ bps} \]

The bullet portfolio outperforms the barbell portfolio as long as the yield on bond \( C \) does not rise by more than 250 bps or fall by more than 325 bps.

Conclusion

Barbell portfolio with higher convexity may outperform only when the yield change is significant. The performance depends on the magnitude of the change in yields and how the yield curve shifts.

Barbell strategy (higher convexity + lower yield) versus bullet strategy (lower convexity + higher yield).
Summary on the yield curve strategies

The barbell strategy gives up yield in order to achieve a higher convexity (drops less if yield increases and increases more if yield decreases).

- **Yield curve shifts in parallel**
  Yield change has to be significant in order that convexity can pick up the loss in yield.

- **Flattening of the yield curve**
  The yield loss becomes less significant under barbell strategy. It becomes easier for convexity to work to compensate for the loss in yield.

- **Steepening of yield curve**
  This works against barbell strategy since the loss in yield is even more significant.
Convexity in bond management

• Search for convexity that improves the investor’s performance.

• *Active management*
  Bonds were bought with a short-term horizon, expecting fall in interest rates. When interest rates fall, the rate of return will be higher with more convex bonds or portfolio. If interest rates move in the opposite direction, the loss will be smaller for highly convex bonds.

• *Defensive management (immunization)*
  Any movement in interest rates that occur immediately after the purchase of the portfolio will translate into higher returns if the portfolio is highly convex.
Comparing two coupon-bearing bonds with differing maturities

- Coupon rate is set from zero to 15%. Note that increasing the coupon rate decreases both the duration and convexity.
- Yield to maturity is set at 9%.

**Characteristics of bonds A and B**

<table>
<thead>
<tr>
<th></th>
<th>Bond A</th>
<th>Bond B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>10 years</td>
<td>20 years</td>
</tr>
<tr>
<td>Coupon</td>
<td>1</td>
<td>13.5</td>
</tr>
<tr>
<td>Duration</td>
<td>9.31 years</td>
<td>9.31 years</td>
</tr>
<tr>
<td>Convexity</td>
<td>84.34 years²</td>
<td>115.97 years²</td>
</tr>
</tbody>
</table>

Bond A is way below par (48.66) and Bond B is above par (141.08). They have the same duration but differing convexities.

- Both duration and convexity are decreasing functions of the coupon rate.
• The 20-year bond (B) can be made to have the same duration as that of 10-year bond (A) by setting a very high coupon rate. Bond B still has a higher convexity.

• Set the horizon $H$ to be the common duration of 9.31 years.

The horizon rates of return for bonds A and B move up even $i$ increases or decreases from $i_0 = 9\%$ (see Table on P.73). Comparing the future value at $H = 9.31$ for the same initial value of $1,000,000$, a difference of $14,023$ is resulted under different convexities.
## Duration and convexity for two types of bonds

<table>
<thead>
<tr>
<th>Coupon (c)</th>
<th>Type I bond</th>
<th>Type II bond</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Maturity: 10 years</td>
<td>Maturity: 20 years</td>
</tr>
<tr>
<td></td>
<td>Duration (years)</td>
<td>Convexity (years²)</td>
</tr>
<tr>
<td>0</td>
<td>10.00</td>
<td>92.58</td>
</tr>
<tr>
<td>1</td>
<td>9.31</td>
<td>84.34</td>
</tr>
<tr>
<td>2</td>
<td>8.79</td>
<td>78.02</td>
</tr>
<tr>
<td>3</td>
<td>8.37</td>
<td>73.02</td>
</tr>
<tr>
<td>4</td>
<td>8.03</td>
<td>68.96</td>
</tr>
<tr>
<td>5</td>
<td>7.75</td>
<td>65.60</td>
</tr>
<tr>
<td>6</td>
<td>7.52</td>
<td>62.79</td>
</tr>
<tr>
<td>7</td>
<td>7.32</td>
<td>60.38</td>
</tr>
<tr>
<td>8</td>
<td>7.15</td>
<td>58.31</td>
</tr>
<tr>
<td>9</td>
<td>6.99</td>
<td>56.50</td>
</tr>
<tr>
<td>10</td>
<td>6.86</td>
<td>54.90</td>
</tr>
<tr>
<td>11</td>
<td>6.76</td>
<td>53.49</td>
</tr>
<tr>
<td>12</td>
<td>6.64</td>
<td>52.23</td>
</tr>
<tr>
<td>13</td>
<td>6.55</td>
<td>51.10</td>
</tr>
<tr>
<td>13.5</td>
<td>6.50</td>
<td>50.58</td>
</tr>
<tr>
<td>14</td>
<td>6.46</td>
<td>50.08</td>
</tr>
<tr>
<td>15</td>
<td>6.38</td>
<td>49.16</td>
</tr>
</tbody>
</table>

\[ D = -\frac{1+i dB}{B di} \]

\[ CONV = \frac{1}{B} \frac{d^2 B}{di^2} \]
• Suppose that the initial rates are 9% and that they quickly move up by 1 or 2% or drop by the same amount. The rate of return above 9% for the more convex bond is tenfold that of the bond with lower convexity.

\[
\text{Horizon rates of return for } A \text{ and } B \text{ with } H = 9.31 \text{ years when the rates move quickly from 9% to another value and stay there}
\]

<table>
<thead>
<tr>
<th>Scenario</th>
<th>( i = 7% )</th>
<th>( i = 8% )</th>
<th>( i = 9% )</th>
<th>( i = 10% )</th>
<th>( i = 11% )</th>
</tr>
</thead>
</table>

Suppose the same current value of 1 million.

• investment in \( A \): \( 1,000,000(1 + 0.9008)^{9.31} = 2,232,222 \)
• investment in \( B \): \( 1,000,000(1 + 0.9085)^{9.31} = 2,246,245 \)

which implies a difference of $14,023 in the future value for no trouble at all, except looking up the value of convexity.
• Two short horizons have been chosen: $H = 1$ and $H = 2$.

• Shorter horizon, the gain of horizon rate of return of Bond $B$ is more significant.

• With an increase in interest rates from 9% to 11%, the convexity of $B$ will cushion the loss from 6.2% to 5.7%.
Horizon rates of return when \( i \) takes a new value immediately after the purchase of bond A and bond B (in percentage per year)

<table>
<thead>
<tr>
<th>Horizon (in years) and rates of return for A and B</th>
<th>Scenario</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( i = 7% )</td>
</tr>
<tr>
<td>( H = 1 )</td>
<td></td>
</tr>
<tr>
<td>( R^A )</td>
<td>27.2</td>
</tr>
<tr>
<td>( R^B )</td>
<td>28.1</td>
</tr>
<tr>
<td>( H = 2 )</td>
<td></td>
</tr>
<tr>
<td>( R^A )</td>
<td>16.7</td>
</tr>
<tr>
<td>( R^B )</td>
<td>17.1</td>
</tr>
</tbody>
</table>
Another numerical example

<table>
<thead>
<tr>
<th></th>
<th>Bond A (zero-coupon)</th>
<th>Bond B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>Maturity</td>
<td>10.58 years</td>
<td>25 years</td>
</tr>
<tr>
<td>Duration</td>
<td>10.58 years</td>
<td>10.58 years</td>
</tr>
<tr>
<td>Convexity</td>
<td>103.12 years²</td>
<td>159.17 years²</td>
</tr>
</tbody>
</table>

Rates of return of A and B when $i$ moves from $i = 9\%$ to another value after the bond has been bought (horizon is set to be the same as the duration)

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$i = 7%$</th>
<th>$i = 8%$</th>
<th>$i = 9%$</th>
<th>$i = 10%$</th>
<th>$i = 11%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond A (zero-coupon)</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>Bond B</td>
<td>9.14</td>
<td>9.04</td>
<td>9</td>
<td>9.02</td>
<td>9.08</td>
</tr>
</tbody>
</table>

There is no change in $r_H$ for the zero-coupon bond when $H = D$ since the future value at time $H$ is always equal to the par value.
Rates of return when \( i \) takes a new value immediately after the purchase of bond A or bond B (in percentage per year)

<table>
<thead>
<tr>
<th>Horizon and rates of return for A and B</th>
<th>Scenario</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( i = 7% )</td>
</tr>
<tr>
<td>( H = 1 )</td>
<td>( R^A )</td>
</tr>
<tr>
<td></td>
<td>30.2 19.1 9.0</td>
</tr>
<tr>
<td></td>
<td>31.9 19.5 9.0</td>
</tr>
</tbody>
</table>
Looking for convexity in building a bond portfolio

Suppose we are unable to find a bond with the same duration and higher convexity as the one we are considering buying. We may build a portfolio that have the same duration but higher convexity.

<table>
<thead>
<tr>
<th></th>
<th>Price</th>
<th>duration (years)</th>
<th>Convexity (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond 1</td>
<td>$105.96</td>
<td>5</td>
<td>28.62</td>
</tr>
<tr>
<td>Bond 2</td>
<td>$102.80</td>
<td>1</td>
<td>1.75</td>
</tr>
<tr>
<td>Bond 3</td>
<td>$97.91</td>
<td>9</td>
<td>95.72</td>
</tr>
<tr>
<td>Portfolio</td>
<td>$105.96</td>
<td>5</td>
<td>48.73</td>
</tr>
</tbody>
</table>

Portfolio consists of 0.5153 units of Bond 2 and 0.5411 units of Bond 3.

\[
N_2B_2 + N_3B_3 = B_1 \quad \text{and} \quad \frac{N_2B_2}{B_1}D_2 + \frac{N_3B_3}{B_1}D_3 = D_1.
\]
Values of bond 1 and of portfolio $P$ for various values of interest rate

<table>
<thead>
<tr>
<th>$i$</th>
<th>$B_1(i)$</th>
<th>$B_P(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4%</td>
<td>122.28</td>
<td>123.48</td>
</tr>
<tr>
<td>5%</td>
<td>116.50</td>
<td>116.99</td>
</tr>
<tr>
<td>6%</td>
<td>111.06</td>
<td>111.18</td>
</tr>
<tr>
<td>7%</td>
<td>105.96</td>
<td>105.96</td>
</tr>
<tr>
<td>8%</td>
<td>101.16</td>
<td>101.25</td>
</tr>
<tr>
<td>9%</td>
<td>96.64</td>
<td>97.00</td>
</tr>
<tr>
<td>10%</td>
<td>92.38</td>
<td>93.15</td>
</tr>
</tbody>
</table>

$B_P(i)$ always achieves higher value than $B_1(i)$ under varying values of $i$. 
5-year horizon rates of return for bond 1 and portfolio

<table>
<thead>
<tr>
<th>$i$</th>
<th>$r_{H=5}^1$</th>
<th>$r_{H=5}^P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4%</td>
<td>7.023%</td>
<td>7.230%</td>
</tr>
<tr>
<td>5%</td>
<td>7.010%</td>
<td>7.100%</td>
</tr>
<tr>
<td>6%</td>
<td>7.003%</td>
<td>7.025%</td>
</tr>
<tr>
<td>7%</td>
<td>7%</td>
<td>7%</td>
</tr>
<tr>
<td>8%</td>
<td>7.003%</td>
<td>7.024%</td>
</tr>
<tr>
<td>9%</td>
<td>7.010%</td>
<td>7.092%</td>
</tr>
<tr>
<td>10%</td>
<td>7.023%</td>
<td>7.203%</td>
</tr>
</tbody>
</table>

$r_{H=5}^1 < r_{H=5}^P$ for all varying values of $i$. 
Asset and liabilities management

• How should a pension fund, or an insurance company, set up its asset portfolio in such a way as to be practically certain that it will be able to meet its payment obligations in the future?

Redington conditions

Assume that \( L_t, t = 1, \ldots, T \) and \( A_t, t = 1, \ldots, T \) are known. Interest rate term structure is flat, equal to \( i \).

\[
L = \sum_{t=1}^{T} \frac{L_t}{(1+i)^t} \quad \text{and} \quad A = \sum_{t=1}^{T} \frac{A_t}{(1+i)^t}.
\]

\( N = A - L = 0 \) initially.
1. How should one choose the structure of the assets such that this net value does not change in the event of a change in interest rate?

First order condition (first Redington property):

\[N = A - L\] to be insensitive to \(i\).

Set

\[
\frac{dN}{di} = \frac{1}{1 + i} \sum_{t=1}^{T} t(L_t - A_t)(1+i)^{-t} = \frac{1}{1 + i}(D_L L - D_A A)
\]

\[= \frac{A}{1 + i}(D_L - D_A) = 0, \text{ (since } L = A),\]

where

\[D_L = \sum_{t=1}^{T} \frac{tL_t}{L} \frac{1}{(1+i)^t} \text{ and } D_A = \sum_{t=1}^{T} \frac{tA_t}{L} \frac{1}{(1+i)^t}.\]
2. In order that \( N \) remains positive, a sufficient condition is given by \( N(i) \) being a convex function of \( i \) within that interval. This is captured by

Second Redington condition:

\[
\frac{d^2 A}{d i^2} > \frac{d^2 L}{d i^2}.
\]

Once the duration is given, convexity depends positively on the dispersion \( S \) of the cash flows. Therefore, a sufficient condition for the second Redington condition is that the dispersion of the inflows from the assets is larger than that of the outflows to the liabilities.
Example (Savings and Loan Associations in US in early 1980s)

They had deposits with short maturities (duration) while their loans to mortgage developers had very long durations, since they financed mainly housing projects. Their assets are loans to housing projects while their liabilities are deposits.

- When the interest rates climbed sharply, the net worth of the Savings and Loans Associations fell drastically.

In this case, even the first Redington condition was not met. This spelled disaster.
Example (Net initial position of the financial firm is sound)

- Asset: Investing $1 million in a 20-year, 8.5% coupon bond.
- Liability: Financed with a 9-year loan carrying an 8% interest rate.

Same initial value of the asset and liability. Recall the duration formula:

\[ D = \frac{1}{i} + \theta + \frac{N}{m} \left( i - \frac{c}{B_T} \right) - \left( 1 + \frac{i}{m} \right) \frac{c}{B_T} \left[ (1 + \frac{i}{m})^N - 1 \right] + \frac{i}{i}. \]

where

- \( \theta = \) time to wait for the next coupon to be paid \((0 \leq \theta \leq 1)\)
- \( m = \) number of times a payment is made within one year
- \( N = \) total number of coupons remaining to be paid.

Here, \( \theta = \frac{1}{2}, \ m = 2; \ N = 40 \) for the 20-year bond and \( N = 18 \) for the 9-year loan.
We obtain the respective duration of the asset and liability as

\[ D_A = 9.944 \text{ years} \quad \text{and} \quad D_L = 6.583 \text{ years} \]

so that the modified durations are

\[ D_{mA} = \frac{D_A}{1 + i_A} = 9.165 \text{ years} \quad \text{and} \quad D_{mL} = \frac{D_L}{1 + i_L} = 6.095 \text{ years}. \]

Note that

\[ \frac{\Delta V_A}{V_A} \approx \frac{dV_A}{V_A} = -D_{mA} \, di_A \quad \text{and} \quad \frac{\Delta V_L}{V_L} \approx \frac{dV_L}{V_L} = -D_{mL} \, di_L \]
so that

\[ \Delta V_P = \Delta V_A - \Delta V_L \approx -(D_{mA}V_A \, di_A - D_{mL}V_L \, di_L). \]

Suppose \( i_A \) and \( i_L \) receive the same increment and \( V_A = V_L \), we have

\[ \Delta V_P \approx -(D_{mA} - D_{mL})A \, di = -3.070V_A \, di. \]

Based on the linear approximation, if interest rates increase by 1\%, the net value of the project diminishes by 3.070\% of the asset. Its risk exposure presents a net duration of \( D_A - D_L = 3.361 \).
Remarks

1. Immunization is a short-term series of measures destined to match sensitivities of assets and liabilities. As time passes, these sensitivities continue to change since the duration does not generally decrease in the same amount as the planning horizon.

2. Whenever interest rates change, the duration also changes.

3. So far we have considered flat term structures and parallel displacements of them. More refined duration measures and analysis are required if we do not face such flat structures.

4. Financial manager may also want to pay special attention to the convexity of his assets and liabilities as well.
Measuring the riskiness of foreign currency - denominated bonds

Let $B$ be the value of a foreign bond in foreign currency, $e$ be the exchange rate (value of one unit of foreign currency in domestic currency), $V$ be the value of the foreign bond in domestic currency. We have

$$V = eB$$

so that

$$\frac{dV}{V} = \frac{dB}{B} + \frac{de}{e}.$$ 

All the three relative changes are random variables. Observe that

$$\frac{dB}{B} = \frac{1}{B} \frac{dB}{di} = -\frac{D}{1+i} di.$$
so that

$$\frac{dV}{V} = -\frac{D}{1 + i} di + \frac{de}{e}.$$ 

Also, we can deduce that

$$\text{var} \left( \frac{dV}{V} \right) = \left( \frac{D}{1 + i} \right)^2 \text{var}(di) + \text{var} \left( \frac{de}{e} \right) - 2 \frac{D}{1 + i} \text{cov}(di, \frac{de}{e}).$$

- The covariance between changes in interest rates and modifications in the exchange rates is usually very low, and the bulk of the variance of changes in the foreign bond’s value stems mainly from the variance of the exchange rate.

- Empirical studies show that the share of the exchange rate variance is easily two-thirds of the total variance.
For finite changes in $e$ and $B$, the correct formula should be

$$\frac{\Delta V}{V} = \frac{(B + \Delta B)(e + \Delta e) - Be}{Be} = \frac{\Delta B}{B} + \frac{\Delta e}{e} + \frac{\Delta B \Delta e}{Be}.$$ 

**Numerical example**

Suppose that the loss on the Jakarta stock market was 60% in a given period and that the rupee lost 60% of its value against the dollar in the same period.

- The rate of change in the investment's value in American dollars cannot be $(-60\%) + (-60\%) = (-120\%)$.

- It is more proper to use

$$\frac{\Delta V}{V} = (-60\%) + (-60\%) + (-60\%)(-60\%)$$

$$= -120\% + 36\% = -84\%.$$
Cash matching problem – Linear programming with constraints

- A known sequence of future monetary obligations over \( n \) periods.

\[
y = (y_1 \ldots y_n)
\]

- Purchase bonds of various maturities and use the coupon payments and redemption values to meet the obligations.

Suppose there are \( m \) bonds, and the cash stream on dates \( 1, 2, \ldots, n \) associated with one unit of bond \( j \) is \( c_j = (c_{1j} \ldots c_{nj}) \).

\[
p_j = \text{price of bond } j
\]

\[
x_j = \text{amount of bond } j \text{ held in the portfolio}
\]

Minimize

\[
\sum_{j=1}^{m} p_j x_j
\]

subject to

\[
\sum_{j=1}^{m} c_{ij} x_j \geq y_i \quad i = 1, 2, \ldots, n
\]

\[
x_j \geq 0 \quad j = 1, 2, \ldots, m.
\]
Numerical example – Six-year match

To match the cash obligations over a 6-year period using 10 bonds.

<table>
<thead>
<tr>
<th>Yr</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Req’d</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>5</td>
<td>10</td>
<td>8</td>
<td>7</td>
<td>100</td>
<td>100</td>
<td>171.74</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>5</td>
<td>10</td>
<td>8</td>
<td>107</td>
<td>200</td>
<td>200.00</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>5</td>
<td>110</td>
<td>108</td>
<td></td>
<td></td>
<td>800</td>
<td>800.00</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>105</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>100</td>
<td>119.34</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>106</td>
<td>107</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>800</td>
<td>800.00</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
<td>107</td>
<td>108</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1,200</td>
<td>1,200.00</td>
</tr>
<tr>
<td>p</td>
<td>109</td>
<td>94.8</td>
<td>99.5</td>
<td>93.1</td>
<td>97.2</td>
<td>92.9</td>
<td>110</td>
<td>104</td>
<td>102</td>
<td>95.2</td>
<td>2,381.14</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>0</td>
<td>11.2</td>
<td>0</td>
<td>6.81</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6.3</td>
<td>0.28</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In two of the 6 years, some extra cash is generated beyond what is required.
Difficulties and weaknesses

- There are high requirements in some years so a larger number of bonds must be purchased that mature on those dates. These bonds generate coupon payments in earlier years and only a portion of these payments is needed to meet obligations in these early years. Such problem is alleviated with a smoother set of cash requirements.

- At the end of Year 6, the cash flow required is $1,200, so either Bond 1, Bond 2 and / or Bond 3 must be purchased. Bond 2 is chosen since it has lower coupon payments in earlier years so that less extra cash is generated in those earlier years.

- Liabilities are being met by coupon payments or maturing bonds. Bonds are not sold to meet cash flows. The only risk is default risk. Adverse interest rate changes do not affect the ability to meet liabilities.
Given typical liability schedules and bonds available for cash flow matching, perfect matching is unlikely.

Strike the tradeoff between (i) avoidance of the risk of not satisfying the liability stream (ii) lower cost.

How to combine immunization with cash matching? More precisely, how one strikes the balance between lower cost of constructing the bond portfolio against less difference in duration (more ideal to have higher convexity as well).

The extra surpluses should be reinvested so this creates reinvestment risks. This requires the estimation of future interest rate movements.
Allowing cash carryforward

If cash can be carried forward, then there are two possible sources of funds: cash flows from the bond investment (coupons and principles) and cash carryover from the prior period.

- Let $F_t$ represent the amount of short-term investment and $r$ be the one period interest rate.

$$\text{minimize } \sum_{j=1}^{m} p_j x_j$$

subject to

$$\sum_{j=1}^{m} c_{ij} x_j + F_{i-1}(1 + r) \geq y_i + F_i, \quad i = 1, 2, \ldots, n,$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, m,$$

$$F_i \geq 0 \quad \text{for } i = 1, 2, \ldots, n; \quad \text{and } F_{-1} = 0.$$

Note that $r$ should be set to approximate current expectations about future short term rates.
Rebalancing an immunized portfolio

Since the market yield will fluctuate over the investment horizon, how often should the portfolio be rebalanced in order to adjust its duration?

- Immunization involves minimizing the initial portfolio cost subject to the constraint of having sufficient cash to satisfy the liabilities.

- Transaction cost must be included in the optimization framework such that a tradeoff between transaction costs and risk minimization is considered.
The above immunization works perfectly only when the yield curve is flat and any changes in the yield curve are parallel changes. Duration matched portfolios are not unique.

**Goal**  How to construct an immunized portfolio that has the lowest risk of not realizing the target yield?

**Rules of thumb**

- Immunization risk involves reinvestment risk of cash flows received prior to liabilities payment.

- When the cash flows are concentrated around the horizon date, the portfolio is subject to less reinvestment risk.
**Immunization risk measure (simple view)**

Immunization risk measure

\[
\text{Immunization risk measure} = \sum \text{PVCF} \times (\text{payment time} - \text{horizon date})^2
\]

where \( \text{PVCF} \) is the present value of cash flow.

The cash flows represented on the left side have a lower immunization risk measure. Zero immunization risk portfolio is a portfolio consisting of zero-coupon bonds maturing exactly on the horizon dates.
Bankruptcy of Orange County, California (see Qn 8 in HW 1)

“A prime example of the interest rate risk incurred when the duration of asset investments is not equal to the duration of fund needs.”

Orange County (like most municipal governments) maintained an operating account of cash from which operating expenses were paid. During the 1980’s and early 1990s, interest rates in US had been falling.

Seeing the larger returns being earned on long-term securities, the treasurer of Orange County decided to invest in long-term fixed income securities.

- The County has $7.5 billion and borrowed $12.5 billion from Wall Street brokerages. Leverage killed.
• Between 1991 and 1993, the County enjoyed more than a 8.5% return on investments.

• Started in February 1994, the Federal Reserve Board raised the interest rate in order to cool an expanding economy. All through the year, paper losses on the fund led to margin calls from Wall Street brokers that had provided short-term financing.

• In December 1994, as news of the loss spread, brokers tried to pull out their money. Finally, as the fund defaulted on collateral payments, brokers started to liquidate their collateral.

• Bankruptcy caused the County to have difficulties to meet payrolls, 40% cut in health and welfare benefits and school employees were laid off.
• County officials blamed the county treasurer, Bob Citron, for undertaking risky investments. He claimed that there was no risk in the portfolio since he was holding to maturity.

• Since government accounting standards do not require municipal investment pools to report “paper” gains or losses, Citron did not report the market value of the portfolio.

• Investors, in touch with monthly fluctuations in values, also may have refrained from the “run on the bank” (擠提) that happened in December 1994. The sudden loss in value in the long-term bonds due to increased interest rate can be compensated by the higher interest rate earned in the remaining life of the long-term bonds.