1. (a) The portfolio variance $\sigma_P^2$ is given by

$$\sigma_P^2 = \alpha^2 \sigma_A^2 + (1 - \alpha)^2 \sigma_B^2 + 2\alpha(1 - \alpha)\rho \sigma_A \sigma_B.$$ 

Differentiating $\sigma_P^2$ with respect to $\alpha$, we have

$$\frac{d\sigma_P^2}{d\alpha} = 2\alpha \sigma_A^2 - 2(1 - \alpha) \sigma_B^2 + (2 - 4\alpha)\rho \sigma_A \sigma_B.$$ 

Setting $\frac{d\sigma_P^2}{d\alpha} = 0$, we obtain

$$\alpha = \frac{\sigma_B^2 - \rho \sigma_A \sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho \sigma_A \sigma_B} = 0.8261.$$ 

(b) Substituting $\alpha = 0.8261$ into $\sigma_P^2$, the portfolio variance of the optimal portfolio is

$$\sigma_P^2 = \alpha^2 \sigma_A^2 + (1 - \alpha)^2 \sigma_B^2 + 2\alpha(1 - \alpha)\rho \sigma_A \sigma_B = 0.01937$$

so that $\sigma_P = 0.1392$.

(c) The expected rate of return of the optimal portfolio:

$$\mu_P = \alpha \mu_A + (1 - \alpha) \mu_B = 0.1139.$$ 

2. (a) The expected rate of return is given by

$$E[r] = \frac{0.5 \times 3 	imes 10^6 + 0.5 \times u}{10^6 + 0.5u} - 1.$$ 

(b) It is seen that buying 3 million units of insurance eliminates all uncertainty regarding the return, resulting in zero variance. The corresponding expected rate of return is

$$E[r] = \frac{0.5 \times 3 \times 10^6 + 0.5 \times 3 \times 10^6}{10^6 + 0.5 \times 3 \times 10^6} - 1 = \frac{3}{2.5} - 1 = 0.2.$$ 

3. (a) 

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Diagram:

![Minimum Variance Point Diagram](image-url)
The expected portfolio rate of return always remains to be \( r \). The set of minimum variance portfolio (also called the efficient set) reduces to one portfolio, which is represented by the minimum variance point in the above \( \sigma_P - \mu_P \) diagram.

(b) The minimum variance point is the global minimum variance portfolio. Recall 

\[
\mathbf{w}_g = \frac{\Omega^{-1}\mathbf{1}}{1^T \Omega^{-1} \mathbf{1}}, \quad \text{where} \quad \Omega = \begin{pmatrix}
\sigma_1^2 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sigma_n^2
\end{pmatrix}.
\]

Note that

\[
\Omega^{-1} \mathbf{1} = \begin{pmatrix}
1/\sigma_1^2 \\
1/\sigma_2^2 \\
\vdots \\
1/\sigma_n^2
\end{pmatrix} \quad \text{and} \quad 1^T \Omega^{-1} \mathbf{1} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2}.
\]

The minimum variance is given by

\[
\sigma_P^2 = \frac{1}{1^T \Omega^{-1} \mathbf{1}} = \frac{1}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}} = \overline{\sigma}^2.
\]

Note that \( \overline{\sigma}^2 \) is the harmonic mean of \( \sigma_i^2, i = 1, 2, \cdots, n \). Hence, the minimum variance point is \((\overline{\sigma}, \overline{r})\).

4. (a) Solve for \( \mathbf{v}_g \) such that

\[
\Omega \mathbf{v}_g = \mathbf{1} \quad \text{or} \quad \begin{pmatrix}
2 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
v_1^g \\
v_2^g \\
v_3^g
\end{pmatrix} = \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}.
\]

We obtain

\[
\mathbf{v}_g = \begin{pmatrix}
0.5 \\
0.5 \\
0.5
\end{pmatrix}.
\]

It happens that the sum of components in \( \mathbf{v}_g \) is already equal to 1. So, the optimal weight vector corresponding to the global minimum variance portfolio is \( \mathbf{w}_g = (0.5 \ 0 \ 0.5)^T \).

(b) The other efficient portfolio is obtained by first solving for

\[
\Omega \mathbf{v}_d = \overline{r}
\]

and normalize the components so that the sum of components equals 1. Consider

\[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
v_1^d \\
v_2^d \\
v_3^d
\end{pmatrix} = \begin{pmatrix}
0.4 \\
0.8 \\
0.8
\end{pmatrix},
\]

we obtain

\[
\mathbf{v}_d = (0.1 \ 0.2 \ 0.3)^T.
\]

Upon normalization, we obtain the weight vector of another efficient portfolio to be

\[
\mathbf{w}_d = \left( \frac{1}{6} \ \frac{1}{3} \ \frac{1}{2} \right)^T.
\]
(c) With the inclusion of the riskfree asset, we solve for
\[ \Omega\mathbf{v} = \mathbf{r} - r_f \mathbf{1} \]
and normalize the components so that the condition on target expected rate of return of the portfolio is met. It is seen that
\[ \mathbf{v} = \mathbf{v}_g - r_f \mathbf{v}_d = (0.1 \ 0.2 \ 0.3)^T - 0.2(0.5 \ 0 \ 0.5)^T = (0 \ 0.2 \ 0.2)^T. \]
The optimal weight vector \( \mathbf{w}^* = \lambda \mathbf{v} \), where \( \lambda \) is determined by enforcing
\[ \lambda \sum_{j=1}^{3} (\mathbf{r}_j - r_f) v_j = \mu_P - r_f, \quad \text{where} \quad \mu_P = 0.4. \]
We then obtain
\[ \lambda(0.6 \times 0.2 + 0.6 \times 0.2) = 0.4 - 0.2 = 0.2 \]
so that \( \lambda = \frac{1}{1.7} \). The weights of the risky assets are
\[ w_1 = 0, \quad w_2 = \frac{2}{1.2} = \frac{1}{6} \quad \text{and} \quad w_3 = \frac{2}{1.2} = \frac{1}{6}. \]
The weight of the risk free asset is \( 1 - \frac{1}{6} - \frac{1}{6} = \frac{2}{3} \).

5. Consider the betting wheel which has \( n \) segments. Let \( Y \) be the random variable of the outcome, where \( Y = i \) if the outcome of the wheel is \( i \). The payoff of a $1 bet on the segment \( i \) is given by \( A_i I_{\{Y=i\}} \), where the indicator function \( I_{\{Y=i\}} = \begin{cases} 1, & \text{if} \ Y = i \\ 0, & \text{otherwise} \end{cases} \).

By using the strategy stated in the question, the payoff is
\[ \sum_{i=1}^{n} \frac{1}{A_i} A_i I_{\{Y=i\}} = 1, \]
which is independent of the outcome of the wheel. Following this strategy, the initial total amount betted is \( \sum_{i=1}^{n} \frac{1}{A_i} \) and the final payoff is always 1 (risk free). Therefore, the corresponding deterministic rate of return is given by
\[ \frac{1}{\sum_{i=1}^{n} \frac{1}{A_i}} - 1. \]
For example, suppose the wheel has 4 segments with \( A_1 = 3, A_2 = 4, A_3 = 5, A_6 = 6 \). The betting strategy is to bet \( \frac{1}{3} \) on segment 1, \( \frac{1}{4} \) on segment 2, \( \frac{1}{5} \) on segment 3, and \( \frac{1}{6} \) on segment 4. The riskfree return is
\[ \frac{1}{\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}} - 1 = \frac{1}{\frac{57}{60}} - 1 = \frac{3}{57}. \]
Consider the variance of the difference of \( r - r_M \)

\[
\text{var}(r - r_M) = \text{var}(r) + \text{var}(r_M) - 2\text{cov}(r, r_M)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \sigma_{ij} + \sigma_M^2 - 2 \sum_{i=1}^{n} \alpha_i \sigma_{iM}, \quad \text{where } \sigma_{iM} = \text{cov}(r_i, r_M).
\]

To minimize \( \text{var}(r - r_M) \) subject to \( \sum_{i=1}^{n} \alpha_i = 1 \), we set up the Lagrangian

\[
L = \frac{1}{2} \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \sigma_{ij} + \sigma_M^2 - 2 \sum_{i=1}^{n} \alpha_i \sigma_{iM} \right] - \lambda \left( \sum_{i=1}^{n} \alpha_i - 1 \right).
\]

Differentiating \( L \) with respect to \( \alpha_i \) and \( \lambda \), we obtain

\[
\sum_{j=1}^{n} \alpha_j \sigma_{ij} - \sigma_{iM} - \lambda = 0, \quad i = 1, 2, \ldots, n,
\]

\[
\sum_{i=1}^{n} \alpha_i = 1.
\]

In matrix form:

\[
\Omega \alpha - \sigma_M - \lambda \mathbf{1} = 0
\]

\[
\mathbf{1}^T \alpha = 1,
\]

where \( \sigma_M = (\sigma_{1M}, \sigma_{2M}, \ldots, \sigma_{nM})^T \). Assuming \( \Omega^{-1} \) exists, we have

\[
\alpha = \Omega^{-1} \sigma_M + \lambda \Omega^{-1} \mathbf{1}.
\]

Applying the constraint: \( \mathbf{1}^T \alpha = 1 \), we obtain

\[
\mathbf{1}^T \Omega^{-1} \sigma_M + \lambda \mathbf{1}^T \Omega^{-1} \mathbf{1} = 1
\]

so that

\[
\lambda = \frac{1 - \mathbf{1}^T \Omega^{-1} \sigma_M}{\mathbf{1}^T \Omega^{-1} \mathbf{1}}.
\]

Finally, we obtain

\[
\alpha = \Omega^{-1} \sigma_M + \frac{1 - \mathbf{1}^T \Omega^{-1} \sigma_M}{\mathbf{1}^T \Omega^{-1} \mathbf{1}} \Omega^{-1} \mathbf{1}.
\]

(b) The modified Lagrangian is given by

\[
L = \frac{1}{2} \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \sigma_{ij} - 2 \sum_{i=1}^{n} \alpha_i \sigma_{iM} + \sigma_M^2 \right] - \lambda_1 \left( \sum_{i=1}^{n} \alpha_i - 1 \right) - \lambda_2 \left( \sum_{i=1}^{n} \alpha_i \bar{r}_i - m \right),
\]

where \( m \) is the target mean. Differentiating \( L \) with respect to \( \alpha_i, \lambda_1, \lambda_2 \), we obtain

\[
\sum_{j=1}^{n} \alpha_j \sigma_{ij} - \sigma_{iM} - \lambda_1 - \lambda_2 \bar{r}_i = 0, \quad i = 1, 2, \ldots, n,
\]

\[
\sum_{i=1}^{n} \alpha_i = 1
\]

\[
\sum_{i=1}^{n} \alpha_i \bar{r}_i = m.
\]
In matrix form:

$$\Omega \alpha - \sigma_M - \lambda_1 \mathbf{1} - \lambda_2 \mathbf{r} = 0 \quad (i)$$

$$\mathbf{1}^T \alpha = 1 \quad (ii)$$

$$\mathbf{r}^T \alpha = m \quad (iii)$$

We write

$$a = \mathbf{1}^T \Omega^{-1} \mathbf{1}, b = \mathbf{1}^T \Omega^{-1} \mathbf{r}, c = \mathbf{r}^T \Omega^{-1} \mathbf{r}, s_1 = \mathbf{1}^T \Omega^{-1} \sigma_M, s_2 = \mathbf{r}^T \Omega^{-1} \sigma_M.$$  

Assuming \( \Omega^{-1} \) exists, eq. (i) can be expressed as

$$\alpha = \Omega^{-1} \sigma_M + \lambda_1 \Omega^{-1} \mathbf{1} + \lambda_2 \Omega^{-1} \mathbf{r}. \quad (iv)$$

Invoking conditions (ii) and (iii), we obtain the following pair of algebraic equations for \( \lambda_1 \) and \( \lambda_2 \):

$$1 = s_1 + \lambda_1 a + \lambda_2 b \quad m = s_2 + \lambda_1 b + \lambda_2 c.$$

Solving for \( \lambda_1 \) & \( \lambda_2 \):

$$\lambda_1 = \frac{1 - s_1}{m - s_2} \begin{vmatrix} a & b \\ b & c \end{vmatrix} = \frac{c(1 - s_1) - b(m - s_2)}{ac - b^2},$$

$$\lambda_2 = \frac{1 - s_1}{m - s_2} \begin{vmatrix} a & b \\ b & c \end{vmatrix} = \frac{a(m - s_2) - b(1 - s_1)}{ac - b^2}.$$

Both \( \lambda_1 \) and \( \lambda_2 \) are linear functions of \( m \). We are able to express \( \alpha \) in terms of \( m \) [see eq. (iv)].

7. (a) Recall \( \mathbf{w}_0 = \frac{\Omega^{-1} \mathbf{1}}{\mathbf{1}^T \Omega^{-1} \mathbf{1}}, \sigma_0^2 = \mathbf{w}_0^T \Omega \mathbf{w}_0 = \frac{1}{\mathbf{1}^T \Omega^{-1} \mathbf{1}}, \) so that

$$\text{cov}(r_0, r_1) = \mathbf{w}_0^T \Omega \mathbf{w}_1 = \frac{1}{\mathbf{1}^T \Omega^{-1} \mathbf{1}} \frac{1}{\mathbf{1}^T \Omega^{-1} \mathbf{1}} = \sigma_0^2$$

giving \( A = 1 \). Consider the variance \( \sigma_0^2 \)

$$\sigma_0^2 = \text{cov}((1 - \alpha)r_0 + \alpha r_1, (1 - \alpha)r_0 + \alpha r_1)$$

$$= (1 - \alpha)^2 \sigma_0^2 + 2(1 - \alpha)\text{cov}(r_0, r_1) + \alpha^2 \sigma_1^2$$

$$= (1 - \alpha)^2 \sigma_0^2 + 2(1 - \alpha)\sigma_0^2 + \alpha^2 \sigma_1^2$$

$$= \sigma_0^2 + \alpha^2(\sigma_1^2 - \sigma_0^2).$$

The result agrees with the intuition that variations of the variance of the given portfolio around \( \alpha = 0 \) should be second order in \( \alpha \).

(b) Writing \( \mathbf{r} = (r_1 \cdots r_N)^T \), consider

$$\text{cov}(r_1, r_2) = \text{cov}(\mathbf{w}_1^T \mathbf{r}, [(1 - \alpha)\mathbf{w}_0 + \alpha \mathbf{w}_1]^T \mathbf{r})$$

$$= \text{cov}(\mathbf{w}_1^T \mathbf{r}, (1 - \alpha)\mathbf{w}_0^T \mathbf{r} + \alpha \mathbf{w}_1^T \mathbf{r})$$

$$= \text{cov}(\mathbf{w}_1^T \mathbf{r}, (1 - \alpha)\mathbf{w}_0^T \mathbf{r}) + \text{cov}(\mathbf{w}_1^T \mathbf{r}, \alpha \mathbf{w}_1^T \mathbf{r})$$

$$= (1 - \alpha)\sigma_0^2 + \alpha \sigma_1^2.$$
Setting \( \text{cov}(r_1, r_z) = 0 \), we obtain

\[
0 = (1 - \alpha)\sigma_0^2 + \alpha\sigma_1^2
\]

giving

\[
\alpha = -\frac{\sigma_0^2}{\sigma_1^2 - \sigma_0^2} < 0.
\]

(c) We have

\[
\bar{r}_z = (1 - \alpha)\bar{r}_0 + \alpha\bar{r}_1 = \bar{r}_0 + \alpha(\bar{r}_1 - \bar{r}_0)
\]

so that \( \bar{r}_z < \bar{r}_0 \) (since \( \alpha < 0 \) and \( \bar{r}_1 - \bar{r}_0 > 0 \)). Now,

\[
\sigma_z^2 = \text{var}(r_z) = (1 - \alpha)^2\sigma_0^2 + \alpha^2\sigma_1^2 + 2\alpha(1 - \alpha)\text{cov}(r_0, r_1)
\]

\[
= (1 - \alpha)^2\sigma_0^2 + 2\alpha(1 - \alpha)\sigma_0^2 + \alpha^2\sigma_1^2 = \frac{\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2}.
\]

Note that Portfolio \( z \) is a minimum variance portfolio but it is not efficient.

8. The Lagrangian is given by

\[
L = \tau(w^T\mu + w_0r) - \frac{w^T\Omega w}{2} + \lambda(w^T1 + w_0 - 1),
\]

and the first order conditions:

\[
\tau\mu - \Omega w + \lambda 1 = 0
\]

\[
\tau\tau + \lambda = 0
\]

\[
w^T1 + w_0 = 1.
\]
Using $\lambda = -\tau r$, we obtain
\[ w = \tau \Omega^{-1}(\mu - r 1) \]
\[ w_0 = 1 - \tau (1^T \Omega^{-1} \mu) + \tau r (1^T \Omega^{-1} 1) = 1 - b r + a r. \]

The portfolio’s expected rate of return is
\[ \mu_P = r w_0 + \mu^T w \]
\[ = \tau [r^2 a - br - r (1^T \Omega^{-1} \mu) + \mu^T \Omega^{-1} \mu] + r = \tau (ar^2 - 2br + c) + r. \]

Finally, the relation between $\mu_P$ and $\sigma_P$ is found to be
\[ \sigma_P^2 = w^T \Omega w \]
\[ = \tau^2 [r^2 (1^T \Omega^{-1} 1) - 2r (1^T \Omega^{-1} \mu) + \mu^T \Omega^{-1} \mu] = \tau^2 (ar^2 - 2br + c) \]
\[ = \tau (\mu_P - r). \]

9. In the asset-liability model, we want to maximize
\[ \tau E[r_S] - \frac{\text{var}(r_S)}{2} \]
subject to $\sum_{i=1}^{N} w_i = 1$, where $r_S = r w - \frac{1}{f_0} r_L$. Since $r_L$ is independent of $w$, it is equivalent to maximize
\[ \tau \mu^T w - \frac{w^T \Omega w}{2} + \text{cov}(r w, \frac{r_L}{f_0}). \]

Now, $\text{cov}(r w, r_L) = \sum_{i=1}^{N} w^T \text{cov}(r_i, r_L) = \gamma^T w$, with $\gamma_i = \frac{1}{f_0} \text{cov}(r_i, r_L)$. Define the Lagrangian
\[ L = \tau \mu^T w - \frac{w^T \Omega w}{2} + \gamma^T w - \lambda (1^T w - 1) \]
\[ \frac{\partial L}{\partial w_i} = \tau \mu_i - \sum_{j=1}^{N} w_j \sigma_{ij} + \gamma_i - \lambda = 0, \quad i = 1, \ldots, N; \quad (1) \]
\[ \frac{\partial L}{\partial \lambda} = 1^T w - 1 = 0. \quad (2) \]

From Eq. (1), we obtain
\[ \tau \mu - \Omega w + \gamma - \lambda 1 = 0 \quad \Rightarrow \quad w = \tau \Omega^{-1} \mu + \Omega^{-1} \gamma - \lambda \Omega^{-1} 1. \]

From Eq. (2), we obtain
\[ 1 = 1^T w = \tau 1^T \Omega^{-1} \mu + 1^T \Omega^{-1} \gamma - \lambda 1^T \Omega^{-1} 1 \]
so that
\[ \lambda = \frac{\tau 1^T \Omega^{-1} \mu + 1^T \Omega^{-1} \gamma - 1}{1^T \Omega^{-1} 1}. \]
Finally, we obtain

\[ w = \tau \Omega^{-1} \mu + \Omega^{-1} \gamma - \frac{\tau^T \Omega^{-1} \mu}{\Omega^{-1} 1} \Omega^{-1} 1 - \frac{\tau^T \Omega^{-1} \gamma}{\Omega^{-1} 1} \Omega^{-1} 1 + \frac{1}{\Omega^{-1} 1} \Omega^{-1} 1 \]

\[ = \frac{1}{\Omega^{-1} 1} \Omega^{-1} 1 + \Omega^{-1} \gamma - \frac{\tau^T \Omega^{-1} \gamma}{\Omega^{-1} 1} \Omega^{-1} 1 + \tau \left[ \Omega^{-1} \mu - \frac{\Omega^{-1} \mu}{\Omega^{-1} 1} \Omega^{-1} 1 \right] \]

\[ = x^{\min} + z^L + \tau z^\star, \]

where

\[ x^{\min} = \frac{1}{\Omega^{-1} 1} \Omega^{-1} 1, \]

\[ z^L = \Omega^{-1} \gamma - \frac{\Omega^{-1} \gamma}{\Omega^{-1} 1} \Omega^{-1} 1, \]

\[ z^\star = \Omega^{-1} \mu - \frac{\Omega^{-1} \mu}{\Omega^{-1} 1} \Omega^{-1} 1. \]