1. The market portfolio consists of $n$ uncorrelated assets with weight vector $(x_1 \cdots x_n)^T$. Since the assets are uncorrelated, we obtain

$$\sigma^2_M = \text{cov}(x_1 r_1 + \cdots + x_n r_n, x_1 r_1 + \cdots + x_n r_n) = \sum_{j=1}^{n} x_j^2 \sigma_j^2,$$

and

$$\sigma_i M = \text{cov}(x_i r_1, x_1 r_1 + \cdots + x_n r_n) = x_i \sigma_i^2.$$

We then have

$$\beta_i = \frac{\sigma_{iM}}{\sigma^2_M} = \frac{x_i \sigma_i^2}{\sum_{j=1}^{n} x_j^2 \sigma_j^2}.$$

2. The market consists of $150$ in shares of $A$ and $300$ in shares of $B$. Hence, the market rate of return is

$$r_M = \left( \frac{150}{450} \right) r_A + \left( \frac{300}{450} \right) r_B = \frac{1}{3} r_A + \frac{2}{3} r_B.$$

(a) $\bar{r}_M = \frac{1}{3} \times 0.15 + \frac{2}{3} \times 0.12 = 0.13$;

(b) $\sigma_M = \left[ \frac{1}{9} (0.15)^2 + \frac{4}{9 \times 3} (0.15)(0.09) + \frac{4}{9} (0.09)^2 \right]^{\frac{1}{2}} = 0.09$;

(c) $\sigma_{AM} = \frac{1}{3} \sigma_A^2 + \frac{2}{3} \rho_{AB} \sigma_A \sigma_B = \frac{1}{3} (0.15)^2 + \frac{2}{9} (0.15)(0.09) = 0.0105; \beta_A = \frac{\sigma_{AM}}{\sigma^2_M} = 1.2963$.

(d) Since Simpleland satisfies the CAPM exactly, stocks $A$ and $B$ lie on the security market line. Specifically,

$$\bar{r}_A - r_f = \beta_A (\bar{r}_M - r_f).$$

Hence, the risk-free rate is given by

$$r_f = \frac{\bar{r}_A - \beta_A \bar{r}_M}{1 - \beta_A} = 0.0625.$$

3. Assuming that there are $J$ risky assets and one risk-free asset in the set of marketable assets, with total dollar value $V_m$. Define the weight vector $w = (w_1 \cdots w_j)^T$, where $w_j$ represents the weight of the $j$th risky asset within the universe of marketable assets. That is, the dollar value $V_j$ of the $j$th risky asset is $w_j V_m$. Let $w_0$ denote the weight of the risk free asset. Using $V_m$ as the numeraire, the weight of the non-marketable asset, denoted by $w_N$, is equal to $V_N / V_m$. Note that $\sum_{j=0}^{J} w_j = 1$ and define the covariance matrix $\Omega$ such that the $(i,j)^{th}$ entry of $\Omega = \sigma_{ij} = \text{cov}(R_i, R_j)$, where $R_i$ is the rate of return of the $i$th risky asset. Let $\bar{r}_m$ denote the expected rate of return of the portfolio of the marketable assets.
assets, including the riskfree asset. Defining $\overline{R} = (\overline{R}_1 \cdots \overline{R}_J)^T$ and $w = (w_1 \cdots w_J)^T$. We then have

$$\overline{R}_m - r = \sum_{j=1}^{J} w_j(\overline{R}_j - r) = w^T(\overline{R} - r1),$$

since the expected rate of return of the portfolio of marketable assets above the riskfree rate $r$ is the weighted average of all expected rate of the return above $r$ of the risky assets. The control variables in the optimal portfolio selection problem are $x_1, x_2, \cdots, x_J$. The objective function to be minimized is

$$\text{var} \left( \sum_{j=1}^{J} w_jR_j + w_NR_N \right)$$

which is proportional to the variance of portfolio’s return. Note that the riskfree asset does not contribute to the portfolio risk. As a remark, the actual weight $\hat{w}_j$ of the $j^{th}$ risky asset in the overall portfolio is

$$\hat{w}_j = \frac{V_j}{V_{total}} = \frac{w_jV_m}{V_{total}},$$

where $V_{total}$ is the total value of all assets, including the marketable risky assets, non-marketable risky assets and the marketable riskfree assets. Expanding the objective function, we obtain

$$\text{var} \left( \sum_{j=1}^{J} w_jR_j + w_NR_N \right) = w^T\Omega w + 2w_N \sum_{j=1}^{J} w_j \text{cov}(R_j, R_N) + w_N^2 \text{var}(R_N).$$

Since $w_N$ is fixed, so it is not one of the control variables. We form the following Lagrangian:

$$L = \frac{w^T\Omega w}{2} + w_N \sum_{j=1}^{J} w_j \text{cov}(R_j, R_N) + \lambda \left[ \overline{R}_m - r - \sum_{j=1}^{J} w_j(\overline{R}_j - r) \right].$$

The first order conditions are

$$\frac{\partial L}{\partial w_i} = \sum_{j=1}^{J} w_j \sigma_{ij} + w_N \text{cov}(R_i, R_N) - \lambda(\overline{R}_i - r) = 0, \quad i = 1, 2, \cdots, J, \quad (1a)$$

$$\overline{R}_m - r = \sum_{j=1}^{J} w_j(\overline{R}_j - r). \quad (1b)$$

The first $J$ equations can be rewritten as

$$\lambda(\overline{R}_i - r) = \sum_{j=1}^{J} w_j \sigma_{ij} + w_N \text{cov}(R_i, R_N)$$

$$= \text{cov} \left( \sum_{j=1}^{J} w_jR_j + w_NR_N, R_i \right)$$

$$= \sum_{j=1}^{J} w_j \sigma_{ij} + w_N \text{cov}(R_i, R_N), \quad i = 1, 2, \cdots, J. \quad (2)$$
Combining Eqs. (1b) and (2), we obtain
\[
\sum_{i=1}^{J} w_i \left( \sum_{j=1}^{J} \sigma_{ij} w_j + w_N \text{cov}(R_i, R_N) \right) = \lambda w^T (\overline{R} - r) = \lambda (\overline{R_m} - r). \tag{3}
\]

Eliminating \( \lambda \) in Eqs. (2) and (3), we obtain
\[
\frac{\overline{R}_i - r}{\overline{R}_m - r} = \frac{\text{cov} \left( \sum_{j=1}^{J} w_j R_j + w_N R_N, R_i \right)}{\sum_{i=1}^{J} w_i \left[ \sum_{j=1}^{J} \sigma_{ij} w_j + w_N \text{cov}(R_i, R_N) \right]}
= \frac{\text{cov}(R_i, R_m) + \frac{w_N}{\overline{R}_m} \text{cov}(R_i, R_N)}{\sigma_m^2 + \frac{w_N}{\overline{R}_m} \text{cov}(R_i, R_N)},
\]
where
\[
\sigma_m^2 = \sum_{i=1}^{J} \sum_{j=1}^{J} w_i w_j \sigma_{ij}
\]
\[
\text{cov}(R_i, R_m) = \text{cov} \left( R_i, \sum_{j=1}^{J} w_j R_j \right).
\]

4. Consider
\[
E[r'_j] = E \left[ \frac{\overline{P}_e - P_0}{P_0} \right] = E \left[ \frac{\overline{P}_e}{S} \right] \frac{S}{P_0} - 1
= (E[r_j] + 1) \frac{S}{P_0} - 1
= [1 + r_f + \beta'_{jm}(\mu_m - r_f)] \frac{S}{P_0} - 1
\]
so that
\[
E[r'_j] - r_f = (r_f + 1) \left( \frac{S}{P_0} - 1 \right) + \frac{\text{cov}(r_j, r_m)(\mu_m - r_f)}{\sigma_m^2} \frac{S}{P_0}
= (r_f + 1) \left( \frac{S}{P_0} - 1 \right) + \frac{\text{cov}(\overline{P}_e/P_0, r_m)(\mu_m - r_f)}{\sigma_m^2}
= \alpha_j + \beta'_{jm}(\mu_m - r_f)
\]
where
\[
\alpha_j = (1 + r_f) \left( \frac{S}{P_0} - 1 \right) \quad \text{and} \quad \beta'_{jm} = \text{cov}(\overline{P}_e/P_0, r_m)/\sigma_m^2.
\]
Here, \( \beta'_{jm} \) is the beta as deduced from the market price.

5. Form the Lagrangian: \( L = \frac{1}{2} w^T \Omega w + \lambda(1 - \beta_m^T w) \). The first order conditions are
\[
\Omega w - \lambda \beta_m = 0 \quad \text{and} \quad \beta_m^T w = 1.
\]
We then obtain
\[
\lambda = \frac{1}{\beta_m^T \Omega^{-1} \beta_m} \quad \text{and} \quad w^* = \frac{\Omega^{-1} \beta_m}{\beta_m^T \Omega^{-1} \beta_m}.
\]
Consider 
\[ \beta_{jm} = \frac{\text{cov}(r_j, r_m)}{\sigma_m^2} = \frac{e_j^T \Omega w^*}{w^T \Omega w^*}, \] 
where \( e_j = (0 \cdots 1 \cdots 0)^T \),

so that 
\[ \beta_m = \frac{\Omega w^*}{w^T \Omega w^*} = \frac{\Omega (\lambda \Omega^{-1} \beta_m)}{\lambda^2 (\Omega^{-1} \beta_m)^T \Omega (\Omega^{-1} \beta_m)} = \frac{\beta_m}{\lambda^2 \beta_m^T \Omega^{-1} \beta_m} = \beta_m. \]

6. According to the APT, the expected rate of return \( \overline{R}_t \) of the \( i^{th} \) asset is given by 
\[ \overline{R}_t = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}. \]

We obtain
\[
\begin{align*}
12 - \lambda_0 &= \lambda_1 + 0.5 \lambda_2 \\
13.4 - \lambda_0 &= 3 \lambda_1 + 0.2 \lambda_2 \\
12 - \lambda_0 &= 3 \lambda_1 - 0.5 \lambda_2
\end{align*}
\]

\[
\begin{array}{cccc}
1 & 0.5 & 1 & \lambda_1 \\
3 & 0.2 & 1 & \lambda_2 \\
3 & -0.5 & 1 & \lambda_0
\end{array} = \begin{array}{c} 12 \\
13.4 \\
12 \end{array}.
\]

Solving the algebraic system, we have \( \lambda_0 = 10, \lambda = 1, \lambda_2 = 2 \). Hence, \( \overline{R}_i = 10 + b_{i1} + 2b_{i2} \).

7. (a) The beta of the portfolio is a weighted average of the individual betas:
\[ \beta = 0.2 \times 1.1 + 0.5 \times 0.8 + 0.3 \times 1 = 0.92. \]

Hence, by applying the CAPM to the portfolio, we find 
\[ \overline{R}_P = 0.05 + 0.92 (0.12 - 0.05) = 11.44\%. \]

(b) Using the single-factor model, we have
\[ \sigma_e^2 = \sum_{i=A}^{C} w_i^2 \sigma_{e_i}^2 = 0.2^2 \times 0.07^2 + 0.5^2 \times 0.023^2 + 0.3^2 \times 0.01^2 \]
\[ = 0.00033725; \]
\[ \sigma^2 = \beta^2 \sigma_M^2 + \sigma_e^2 = 0.92^2 \times 0.18^2 + 0.00033725 = 0.2776; \]
\[ \sigma = 16.7\%. \]

8. By the APT, we have \( \lambda_0 = r_f = 10\% \). Also, we deduce the following pair of equations for \( \lambda_1 \) and \( \lambda_2 \):
\[ 0.15 = 0.10 + 2 \lambda_1 + \lambda_2 \]
\[ 0.20 = 0.10 + 3 \lambda_1 + 4 \lambda_2. \]

This yields \( \lambda_1 = 0.02 \) and \( \lambda_2 = 0.01 \).

9. (a) \( a_{it} \) is the constant rate of return expected for a security whose \( b_{ik} \) values are all zero; \( b_{iN} \) is the sensitivity of stock \( i \) to percentage changes in non-durable consumer purchases (the average value across all stocks is \( 1.0 \)); \( \tilde{N}_t \) is the uncertain percentage change in non-durable goods purchases in period \( t \).
\[ E(R_1) = 4.0\% + 1.0(2.0\%) + 1.5(3.0\%) - 0.5(1.5\%) = 9.75\%. \]
(c) Given that they expect the common factor $D$ to have a larger payoff than that expected by other investors, their portfolio should have a sensitivity to factor $D$ larger than normal. Usually this means the portfolio will have a factor sensitivity in excess of 1.0.

(d) The term $e_{it}$ is the stock specific return during period $t$. To minimize the impacts of stock specific returns on portfolio returns, the portfolio should be broadly diversified.

(e) Probably the most difficult aspect of using the APT as a tool for active management is the identification of the common factors. If this can be done, then there are difficulties of creating a model that will successfully predict the common factor outcomes. For example, how should David and Sue develop their opinion that factor $D$ would be greater than what is expected by others?