1. If outcome \( j \) occurs, then the corresponding gain is given by

\[
G_j = \sum_{i=1}^{m} g_{ij} \alpha_i,
\]

where \( \alpha_i = \frac{1}{1 + d_i} \) and \( g_{ij} = \begin{cases} d_i & \text{if } j = i \\ -1 & \text{if } j \neq i \end{cases} \).

We then have

\[
G_j = g_{jj} \alpha_j - \sum_{i=1, i \neq j}^{m} \alpha_i
\]

\[
= (1 + g_{jj}) \alpha_j - \sum_{i=1}^{m} \alpha_i
\]

\[
= (1 + d_j) \alpha_j - \sum_{i=1}^{m} \alpha_i
\]

\[
= \frac{1}{1 - \sum_{i=1}^{m} \frac{1}{1 + d_i}} - \frac{\sum_{i=1}^{m} \frac{1}{1 + d_i}}{1 - \sum_{i=1}^{m} \frac{1}{1 + d_i}}
\]

\[
= 1, \quad \text{for } j = 1, 2, \ldots, m.
\]

Therefore, the betting game will always yield a gain of exactly 1.

2. Suppose we hold \((\alpha_1, \alpha_2, \alpha_3)\) units of the three securities, with \( \sum_{i=1}^{3} \alpha_i \leq 1, \alpha_i \geq 0 \). In this problem, we can set \( \sum_{i=1}^{3} \alpha_i = 1 \) since the random returns are greater than one under all states of world. Using the log-utility criterion, the growth factor is

\[
m = E[\ln R] = \frac{1}{2} \ln(4\alpha_1 + 2\alpha_2 + 3(1 - \alpha_1 - \alpha_2)) + \frac{1}{2} \ln(2\alpha_1 + 4\alpha_2 + 3(1 - \alpha_1 - \alpha_2))
\]

\[
= \frac{1}{2} \ln(3 + \alpha_1 - \alpha_2) + \frac{1}{2} \ln(3 - \alpha_1 + \alpha_2).
\]
Applying the first order condition, we obtain
\[
\frac{\partial m}{\partial \alpha_1} = \frac{1}{2} + \frac{1}{3 - \alpha_1 - \alpha_2} - \frac{1}{2} \frac{1}{3 - \alpha_1 + \alpha_2} = 0 \quad \Leftrightarrow \quad 3 - \alpha_1 + \alpha_2 = 3 + \alpha_1 - \alpha_2 \\
\Leftrightarrow \quad \alpha_1 = \alpha_2;
\]
\[
\frac{\partial m}{\partial \alpha_2} = \frac{1}{2} \frac{(-1)}{3 + \alpha_1 - \alpha_2} + \frac{1}{2} \frac{1}{3 - \alpha_1 + \alpha_2} = 0 \quad \Leftrightarrow \quad 3 + \alpha_1 - \alpha_2 = 3 - \alpha_1 + \alpha_2 \\
\Leftrightarrow \quad \alpha_1 = \alpha_2.
\]

Two possible optimal strategies are \((\frac{1}{2}, \frac{1}{2}, 0)\) and \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\).

In fact, for any portfolio choice with \(\alpha_2 = \alpha_1, \alpha_3 = 1 - 2\alpha_1, \alpha_1 \geq 0\), the portfolio’s return is either
\[
4\alpha_1 + 2\alpha_2 + 3\alpha_3 = 4\alpha_1 + 2\alpha_1 + 3(1 - 2\alpha_1) = 3 \text{ if the first state occurs}
\]
or
\[
2\alpha_1 + 4\alpha_2 + 3\alpha_3 = 2\alpha_1 + 4\alpha_2 + 3(1 - 2\alpha_1) = 3 \text{ if the second state occurs}
\]
The optimal strategies always yield a return of 3 for all values of \(\alpha_1\).

3. Recall that the class of the power utility functions includes the logarithm utility since
\[
\lim_{\gamma \to 0^+} \left[ \frac{1}{\gamma} x^\gamma - \frac{1}{\gamma} \right] = \ln x.
\]

This class of functions has the same recursive property as the log utility; that is, the structure is preserved from period to period. This is seen from
\[
E[U(X_k)] = \frac{1}{\gamma} E[(R_k R_{k-1} \cdots R_1 X_0)^\gamma] = \frac{1}{\gamma} E[R_k^\gamma R_{k-1}^\gamma \cdots R_1^\gamma X_0^\gamma] \\
= \frac{1}{\gamma} E[R_k^\gamma] E[R_{k-1}^\gamma] \cdots E[R_1^\gamma] X_0^\gamma
\]
where the last equality follows from the fact that the expected value of a product of independent random variables is equal to the product of their expected values. To maximize \(E[U(X_k)]\) with a fixed-proportions strategy it is only necessary to maximize \(E[(R_1 X_0)^\gamma]\). Therefore, again if one wants to maximize \(E[U(X_k)]\), one needs only to maximize \(E[U(X_1)]\).

4. Let \((x_1, y_1), (x_2, y_2), (x_3, y_3) \in B\).

Reflexivity: \(x_1 = x_1\) and \(y_1 \geq y_1\) so that \((x_1, y_1) \succeq (x_1, y_1)\)

Comparability: If \(x_1 > x_2\), then \((x_1, y_1) \succeq (x_2, y_2)\).

If \(x_2 > x_1\), then \((x_2, y_2) \succeq (x_1, y_1)\).

If \(x_1 = x_2\), then

if \(y_1 \geq y_2\), then \((x_1, y_1) \succeq (x_2, y_2)\)

if \(y_2 \geq y_1\), then \((x_2, y_2) \succeq (x_1, y_1)\).

Transitivity: Given \((x_1, y_1) \succeq (x_2, y_2), (x_2, y_2) \succeq (x_3, y_3)\). If \(x_1 = x_2 > x_3\), with \(y_1 \geq y_2\), then \(x_1 > x_3\) so that \((x_1, y_1) \succeq (x_3, y_3)\).
If \( x_1 > x_2 = x_3 \), with \( y_2 \geq y_3 \), then \( x_1 > x_3 \) so that \((x_1, y_1) \succeq (x_3, y_3)\).

If \( x_1 = x_2 = x_3 \), with \( y_1 \geq y_2 \geq y_3 \), then \( x_1 = x_3, y_1 \geq y_3 \), so that \((x_1, y_1) \succeq (x_3, y_3)\).

If \( x_1 > x_2 > x_3 \), then \( x_1 > x_3 \), so that \((x_1, y_1) \succeq (x_3, y_3)\).

5. Suppose \((x_1, y_1) \succeq (x_2, y_2)\).
   
   Case I: \( x_1 > x_2 \)
   
   1: \( \alpha(x_1, y_1) + (1 - \alpha)(x_2, y_2) = (1 + \alpha(x_1 - x_2), 1 + \alpha(y_1 - y_2)) \)
   
   2: \( \beta(x_1, y_1) + (1 - \beta)(x_2, y_2) = (1 + \beta(x_1 - x_2), 1 + \beta(y_1 - y_2)) \).
   
   \( \alpha > \beta \iff 1 + \alpha(x_1 - x_2) > 1 + \alpha(y_1 - y_2) \) and since \( x_1 - x_2 > 0 \), so
   
   \[ \alpha > \beta \iff \alpha(x_1, y_1) + (1 - \alpha)(x_2, y_2) \succeq \beta(x_1, y_1) + (1 - \beta)(x_2, y_2). \]

Case II: \( x_1 = x_2, y_1 \geq y_2 \)

\( \alpha > \beta \iff 1 + \alpha(x_1 - x_2) = 1 + \beta(x_1 - x_2) \). However, we have \( 1 + \alpha(x_1 - x_2) \geq 1 + \beta(x_1 - x_2) \) and since \( y_1 - y_2 \geq 0 \), so

\[ \alpha > \beta \iff \alpha(x_1, y_1) + (1 - \alpha)(x_2, y_2) \succeq \beta(x_1, y_1) + (1 - \beta)(x_2, y_2). \]

6. Let \( u(x, y) = \ln(x + y) \), and consider \((1, 0)\) and \((0, 1) \in B\), we have \((1, 0) \succ (0, 1) \). But \( u(1, 0) = \ln 1 = u(0, 1) \). Hence, \( u \) cannot be a utility function representing the Dictionary Order.

7. Consider the HARA class of utility functions

\[
U(x) = \frac{1 - \gamma}{\gamma} \left( \frac{ax}{1 - \gamma} + b \right) \gamma \\
= \left( \frac{1 - \gamma^\gamma}{\gamma} \right) \frac{a}{1 - \gamma} x + \left( \frac{1 - \gamma}{\gamma} \right) b \right)^\gamma \tag{1}
\]

(a) Let \( a = (1 - \gamma) \left( \frac{\gamma}{1 - \gamma} \right)^{\frac{1}{\gamma}} \), \( b \to 0^+ \). Then

\[ U(x) = x^\gamma \to x \text{ as } \gamma \to 1. \]

(b) Let \( \gamma = 2 \). Then

\[ U(x) = -\frac{1}{2} (-ax + b)^2 = -\frac{b^2}{2} + abx - \frac{a^2}{2} x^2 \text{ which is equivalent with } V(x) = abx - \frac{a^2}{2} x^2 \text{ since they differ by only a constant. Now let } a^2 = c, b = 1/a, \]

\[ U(x) = x - \frac{1}{2} cx^2. \]

(c) Note that \( (1 + \frac{\alpha x}{n})^n \to e^{\alpha x} \text{ as } n \to \infty. \) Let \( b = \left( \frac{\gamma}{1 - \gamma} \right)^{\frac{1}{\gamma}} a = \frac{1 - \gamma}{\gamma} \left( \frac{\gamma}{1 - \gamma} \right)^{\frac{1}{\gamma}} (-\alpha). \)

Then

\[ U(x) = \left( \frac{-ax}{\gamma} + 1 \right)^\gamma \to e^{-\alpha x} \text{ as } \gamma \to \infty. \]
(d) Let $b \to 0^+$, $a = c^{1/\gamma}(1 - \gamma)\left(\frac{r}{1 - \gamma}\right)^{1/\gamma}$. Then

$$U(x) = (c^{1/\gamma}c)^\gamma = cx^\gamma.$$  

(e) Take $c = \frac{1}{\gamma}$ from part (d). $U(x)$ is equivalent to $\frac{x^\gamma - 1}{\gamma} \to \ln x$ as $\gamma \to 0^+$. 

8. By setting $U(c) = E[U(x)]$, where $c$ is the certainty equivalent, we obtain

$$U'(x)(c - \bar{x}) \approx \frac{U''(\bar{x})}{2} \text{var}(x)$$

so that 

$$c \approx \bar{x} + \frac{U''(\bar{x})}{2U'(\bar{x})}\text{var}(x).$$

9. Recall that $\widetilde{W} = W_0(1 + r_f) + a(\bar{r} - r_f)$ and

$$\eta = 1 + \left(\frac{\frac{da}{dW_0}}{a}\right) W_0 - a.$$ 

From the known result on $\frac{da}{dW_0}$, we have

$$\eta = 1 + \frac{W_0(1 + r_f)E[u''(\widetilde{W})(\bar{r} - r_f)] + aE[u''(\widetilde{W})(\bar{r} - r_f)^2]}{aE[-u''(\widetilde{W})(\bar{r} - r_f)^2]}$$

$$= 1 + \frac{E[u''(\widetilde{W})W_0(1 + r_f) + a(\bar{r} - r_f)](\bar{r} - r_f)}{aE[-u''(\widetilde{W})(\bar{r} - r_f)^2]}$$

$$= 1 + \frac{E[R_R(\widetilde{W})u'(\widetilde{W})(\bar{r} - r_f)]}{aE[u''(\widetilde{W})(\bar{r} - r_f)^2]}.$$ 

Since $u''(W) < 0$ for a concave utility function, we have

$$\text{sign} (\eta - 1) = -\text{sign} (E[R_R(\widetilde{W})u'(\widetilde{W})(\bar{r} - r_f)]).$$

Since $R_R'(W) > 0$, so $R_R(W)$ is an increasing function. We then have

$$R_R(\widetilde{W}) = R_R(W_0(1 + r_f) + a(\bar{r} - r_f))$$

$$\begin{cases} 
\geq R_R(W_0(1 + r_f)) & \text{when } \bar{r} \geq r_f \\
< R_R(W_0(1 + r_f)) & \text{when } \bar{r} < r_f.
\end{cases}$$

Recall that

$$E[R_R(\widetilde{W})u'(\widetilde{W})(\bar{r} - r_f)] = \sum_{s \in S} p_s R_R(\widetilde{W})u'(\widetilde{W})(\bar{r} - r_f).$$

For any typical term in the summation, we consider the two possible scenarios:
When $\bar{r} - r_f \geq 0$, 
\[ p_s R_R(\bar{W}) u'(\bar{W})(\bar{r} - r_f) \geq R_R(W_0(1 + r_f))(\bar{r} - r_f) \]

(ii) when $\bar{r} - r_f < 0$, 
\[ p_s R_R(\bar{W}) u'(\bar{W})(\bar{r} - r_f) < R_R(W_0(1 + r_f))(\bar{r} - r_f) \]

This gives 
\[ E[R_R(\bar{W}) u'(\bar{W})(r - r_f)] > R_R(W_0(1 + r_f))E[u'(\bar{W})(\bar{r} - r_f)] = 0 \]
so that $\eta < 1$.

10. (a) 

<table>
<thead>
<tr>
<th>Outcomes (%)</th>
<th>$F_A$</th>
<th>$\int F_A$</th>
<th>$F_B$</th>
<th>$\int F_B$</th>
<th>$F_C$</th>
<th>$\int F_C$</th>
</tr>
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<tbody>
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<td>4</td>
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<td>0.7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
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<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
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<td>0.8</td>
<td>2.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
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<td>0.3</td>
<td>0.7</td>
<td>0.2</td>
<td>0.4</td>
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<tr>
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<td>0.4</td>
<td>1.1</td>
<td>0.4</td>
<td>0.8</td>
</tr>
<tr>
<td>9</td>
<td>1.0</td>
<td>5.0</td>
<td>0.6</td>
<td>1.7</td>
<td>0.6</td>
<td>1.4</td>
</tr>
<tr>
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<td>1.0</td>
<td>6.0</td>
<td>1.0</td>
<td>2.7</td>
<td>0.6</td>
<td>2.0</td>
</tr>
<tr>
<td>11</td>
<td>1.0</td>
<td>7.0</td>
<td>1.0</td>
<td>3.7</td>
<td>1.0</td>
<td>3.0</td>
</tr>
</tbody>
</table>

$F_A > F_B > F_C$, for all outcomes; $\int F_A \geq \int F_B \geq \int F_C$, for all outcomes.

(b) Consider the geometric average of the 3 investments:
\[
\bar{X}_{geo}(A) = 3^{0.4} 4^{0.3} 6^{0.1} 9^{0.1} = 4.2581 \\
\bar{X}_{geo}(B) = 5^{0.1} 6^{0.2} 8^{0.1} 10^{0.1} = 8.0665 \\
\bar{X}_{geo}(C) = 5^{0.1} 1^{0.1} 2^{0.2} 9^{0.2} 11^{0.4} = 8.7585.
\]

Hence, $C$ is preferred to $B$ and $A$, and $B$ is preferred to $A$, according to the geometric mean criterion.

11. We would like to show that $F(x)$ dominates $G(x)$ by the 3rd order stochastic dominance (TSD) if
\[
(i) \quad \int_a^x \int_a^t F(y) dy dt \leq \int_a^x \int_a^t G(y) dy dt \quad \text{for all } x \in [a, b]
\]
and \quad \text{(ii) } \int_a^b F(t) dt \leq \int_a^b G(t) dt.

According to the above definition, $F(x)$ dominates $G(x)$ in TSD if and only if
\[
\int_c u(x) dF(x) \geq \int_c u(x) dG(x)
\]
for all utility functions with $u'(x) > 0$, $u''(x) < 0$ and $u'''(x) > 0$ for all $x \in C$, where $C$ is the set of all possible outcomes.
Let $a$ and $b$ be the smallest and largest values $F$ and $G$ can take on, where $F(a) = G(a) = 0$, $F(b) = G(b) = 1$. Consider

$$\int_a^b u(x)d(F(x) - G(x)) = u(x)[F(x) - G(x)]|_a^b - \int_a^b u'(x)[F(x) - G(x)] \, dx$$

$$= - \int_a^b u'(x)[F(x) - G(x)] \, dx$$

$$= - u'(x) \int_a^x [F(y) - G(y)] \, dy \bigg|_a^b + \int_a^b u''(x) \int_a^x [F(y) - G(y)] \, dy \, dx$$

$$= - u'(b) \int_a^b [F(y) - G(y)] \, dy + \int_a^b u''(x) \int_a^x [F(y) - G(y)] \, dy \, dx.$$  

By parts integration, we obtain

$$\int_a^b u''(x) \int_a^x [F(y) - G(y)] \, dy \, dx = u''(x) \int_a^b \int_a^t [F(y) - G(y)] \, dy \, dt \bigg|_a^b$$

$$- \int_a^b u''(x) \int_a^y \int_a^t [F(y) - G(y)] \, dy \, dt \, dx.$$  

By property (i), we have

$$\int_a^b u''(x) \int_a^y [F(y) - G(y)] \, dy \, dx \geq 0.$$  

Here, we assume that both

$$\int_a^x \int_a^t F(y) \, dy \, dt \quad \text{and} \quad \int_a^x \int_a^t G(y) \, dy \, dt$$

are continuous function of $x$, otherwise (*) holds only at discontinuous point, since $u''(x) < 0$ and $u'''(x) > 0$. Also, by (ii) and $u'(x) > 0$, we obtain

$$-u'(b) \int_a^b [F(y) - G(y)] \, dy \geq 0.$$  

Hence, the combination of properties (i) and (ii) implies TSD.