1. Suppose there are $n$ mutually uncorrelated assets. The rate of return on asset $i$ has variance $\sigma_i^2$. The expected rates of return are unspecified at this point. The dollar amount of asset $i$ in the market is $X_i$. We let $T = \sum_{i=1}^{n} X_i$ and then set $x_i = X_i/T$, for $i = 1, 2, \cdots, n$. We take the market portfolio in normalized form to be $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Find an expression for the beta $\beta_j$ in terms of the $x_i$’s and $\sigma_i$’s.

2. In Simpleland there are only two risky stocks, $A$ and $B$, whose details are listed in the following table.

<table>
<thead>
<tr>
<th>Stock</th>
<th>Number of shares outstanding</th>
<th>Price per share</th>
<th>Expected rate of return</th>
<th>Standard deviation of return</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>100</td>
<td>$1.50$</td>
<td>15%</td>
<td>15%</td>
</tr>
<tr>
<td>B</td>
<td>150</td>
<td>$2.00$</td>
<td>12%</td>
<td>9%</td>
</tr>
</tbody>
</table>

Furthermore, the correlation coefficient between the returns of stocks $A$ and $B$ is $\rho_{AB} = \frac{1}{3}$.

There is also a risk-free asset, and Simpleland satisfies the CAPM exactly.

(a) What is the expected rate of return of the market portfolio?
(b) What is the standard deviation of the market portfolio?
(c) What is the beta of stock $A$?
(d) What is the risk-free rate in Simpleland?

3. Some risky assets are not easily marketable. For example, human capital, an individual’s future lifetime income, cannot be sold. Some assets such as one’s house may not be sold for psychological reasons or inertia, so it may be considered non-marketable. When some assets are not marketable, the CAPM can be reformulated. Show that the equilibrium rate of return on asset $i$ is given by

$$ R_i = r + \beta_i^r (\bar{R}_m - r) $$

where

$$ \beta_i^r = \frac{\text{cov}(R_i, R_m) + (V_N/V_m)\text{cov}(R_i, R_N)}{\sigma_m^2 + (V_N/V_m)\text{cov}(R_m, R_N)} $$

$V_N$ = value of all non-marketable assets, $V_m$ = value of marketable assets, and $R_N$ = one-period rate of return on non-marketable assets.

**Hint:** Let $w_j$ be the weight of the $j^{th}$ risky asset within the universe of marketable assets and let $w_N = V_N/V_m$. The objective function to be minimized is

$$ \text{var} \left( \sum_{j=1}^{J} w_j R_j + w_N R_N \right) $$

Here, $w_1, \cdots, w_J$ are the control variables while $w_N$ is not a control variable. The model has close resemblance to the asset-liability model.
4. Let \( r_j \) denote the equilibrium rate of return of risky asset \( j \) as deduced from CAPM and \( S \) be its equilibrium price. Let \( P_0 \) be the market price of asset \( j \) and \( \hat{P}_e \) be the random return of the asset. Let \( r'_j \) be the rate of return as deduced from the market price of the asset. Let \( r_m \) denote the equilibrium rate of return of the market portfolio. Show that

\[
E[r'_j] - r_f = (1 + r_f) \left( \frac{S}{P_0} - 1 \right) + \frac{\text{cov}(\hat{P}_e/P_0, r_m)}{\sigma_m^2} (\mu_m - r_f),
\]

where \( \sigma_m^2 = \text{var}(r_m) \) and \( \mu_m = E[r_m] \), \( r_f \) is the risk free interest rate.

**Hint:** First, find the relation between \( E[r'_j] \) and \( E[r_j] \). Note that they are the same if and only if \( S = P_0 \).

5. Take a subset of \( N \) risky assets from the financial market and assume their beta values to be \( \beta_m = (\beta_{1m} \quad \beta_{2m} \quad \beta_{Nm})^T \). We would like to construct the market proxy \( \hat{m} \) from these \( N \) risky assets such that the beta of \( \hat{m} \) is one and \( \beta_{\hat{m}} = \beta_m \), that is, \( \beta_{jm} = \beta_{j\hat{m}} \) for all \( j \). Consider the following minimization problem

\[
\min \ w^T \Omega w
\]

subject to \( \beta_m^T w = 1 \).

where \( \Omega \) is the covariance matrix of the random returns of the \( N \) assets. The constraint indicates that the beta of the market proxy \( \hat{m} \) is one. The first order conditions give

\[
\Omega w - \lambda \beta_m = 0 \quad \text{and} \quad \beta_m^T w = 1.
\]

Show that

\[
\lambda = \frac{1}{\beta_m^T \Omega^{-1} \beta_m} \quad \text{and} \quad w^* = \frac{\Omega^{-1} \beta_m}{\beta_m^T \Omega^{-1} \beta_m}.
\]

We set the market proxy to be \( w^* \). Check whether \( \beta_{\hat{m}} = \beta_m \), where \( \beta_{\hat{m}} = (\beta_{1\hat{m}} \cdots \beta_{N\hat{m}})^T \), and recall \( \beta_{jm} = \frac{\text{cov}(r_j, r_{\hat{m}})}{\sigma_{\hat{m}}^2} \).

6. Assume that the following two-index model describes the rate of return of asset \( i \) as

\[ R_i = b_{1i} I_1 + b_{2i} I_2 + e_i. \]

Assume that the following three portfolios are observed.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Expected Return</th>
<th>( b_{1i} )</th>
<th>( b_{2i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>12.0</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>B</td>
<td>13.4</td>
<td>3</td>
<td>0.2</td>
</tr>
<tr>
<td>C</td>
<td>12.0</td>
<td>3</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

According to the APT, the expected rate of return \( \overline{R}_i \) is given by

\[ \overline{R}_i = \lambda_0 + \lambda_1 b_{1i} + \lambda_2 b_{2i}. \]

Determine \( \lambda_0, \lambda_1, \) and \( \lambda_2 \).

7. Someone who believes that the collection of all stocks satisfies a single-factor model with the market portfolio serving as the factor gives you information on three stocks which make up a portfolio. (See Table.) In addition, you know that the market portfolio has an expected rate of return of 12% and a standard deviation of 18%. The riskfree rate is 5%.
(a) What is the portfolio’s expected rate of return?

(b) Assuming the factor model is accurate, what is the standard deviation of this rate of return?

### Simple Portfolio

<table>
<thead>
<tr>
<th>Stock</th>
<th>Beta</th>
<th>Standard deviation of random error term</th>
<th>Weight in portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.10</td>
<td>7.0%</td>
<td>20%</td>
</tr>
<tr>
<td>B</td>
<td>0.80</td>
<td>2.3%</td>
<td>50%</td>
</tr>
<tr>
<td>C</td>
<td>1.00</td>
<td>1.0%</td>
<td>30%</td>
</tr>
</tbody>
</table>

8. Two stocks are believed to satisfy the two-factor model

\[
\begin{align*}
    r_1 &= a_1 + 2f_1 + f_2 \\
    r_2 &= a_2 + 3f_1 + 4f_2.
\end{align*}
\]

In addition, there is a riskfree asset with a rate of return of 10%. It is known that \( r_1 = 15\% \), \( r_2 = 20\% \). What are the values of \( \lambda_0 \), \( \lambda_1 \) and \( \lambda_2 \) for this model?

9. David and Sue are portfolio managers for a defined benefit pension fund that has \$20 billion dollars invested in U.S. stocks. A large portion of these funds are passively invested to match the returns of the S & P 500 index. Recently, top administrators of the pension plan have asked David and Sue to develop an active investment strategy that will initially be used on a small portion of the \$20 billion. If the active strategy they develop is successful, it will be used on a larger percentage of the \$20 billion. David and Sue have decided to use the concept of arbitrage pricing theory (APT) in developing their active investment strategy.

(a) They decide their initial APT model will include the following three common factors:

- \( N \): percentage change in consumer non-durable good purchases
- \( D \): percentage change in consumer durable good purchases
- \( I \): percentage change in consumer price inflation

Following the standard way of symbolically expressing the APT, they write the following equation for stock \( i \):

\[
\bar{R}_it = a_{0t} + b_{1N}(\bar{N}_t) + b_{1D}(\bar{D}_t) + b_{1I}(\bar{I}_t) + e_{it}.
\]

Define what the following terms mean: \( a_{0t}, b_{iN}, \bar{N}_t \).

(b) After considerable statistical analysis, they develop the following factor estimates for stock 1:

\[
\begin{align*}
    b_{1N} &= 1.0; \\
    b_{1D} &= 1.5; \\
    b_{1I} &= -0.5.
\end{align*}
\]

They believe \( a_{0t} \) should be the one-year risk-free rate at 4%. In addition, they believe the security markets expect \( N, D, \) and \( I \) to be the following during the next year.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Expected value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>2.0%</td>
</tr>
<tr>
<td>( D )</td>
<td>3.0%</td>
</tr>
<tr>
<td>( I )</td>
<td>1.5%</td>
</tr>
</tbody>
</table>

If these assessments are correct, what expected return does the market expect on stock 1 during the next year?
(c) David and Sue agree with the expected value of $N$ and $I$. But they believe the percentage growth of durable consumer purchases (common factor $D$) during the next year will be 5.0%. How should they use this opinion in developing their active management strategy?

(d) What is the role of the $e_{it}$ term in the equation?

(e) What types of difficulties do you see in using the APT to develop an active management strategy?