MATH4512 – Fundamentals of Mathematical Finance

Topic 1 — Mean variance portfolio theory

1.1 Mean and variance of portfolio return

1.2 Markowitz mean-variance formulation

1.3 Two-fund Theorem

1.4 Inclusion of the risk free asset: One-fund Theorem

1.5 Addition of risk tolerance factor

Appendix: Degenerate case of singular covariance matrix
1.1 Mean and variance of portfolio return

Single-period investment model – Asset return

Suppose that you purchase an asset at time zero, and 1 year later you sell the asset. The total return on your investment is defined to be

\[
\text{total return} = \frac{\text{amount received}}{\text{amount invested}}.
\]

If \( X_0 \) and \( X_1 \) are, respectively, the amounts of money invested and received and \( R \) is the total return, then

\[
R = \frac{X_1}{X_0}.
\]

The rate of return is defined by

\[
r = \frac{\text{amount received} - \text{amount invested}}{\text{amount invested}} = \frac{X_1 - X_0}{X_0}.
\]

It is clear that

\[
R = 1 + r \quad \text{and} \quad X_1 = (1 + r)X_0.
\]
- Amount received = dividend received during the investment period + terminal asset value, both quantities are uncertain.

- We treat \( r \) as a random variable, characterized by its probability distribution. For example, in the discrete case, we have

<table>
<thead>
<tr>
<th>rate of return</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>( \cdots )</th>
<th>( r_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability of occurrence</td>
<td>( p_1 )</td>
<td>( p_2 )</td>
<td>( \cdots )</td>
<td>( p_n )</td>
</tr>
</tbody>
</table>

- Two important statistics (discrete random variable)

\[
\text{mean} \quad = \quad \bar{r} = \sum_{i=1}^{n} r_ip_i
\]

\[
\text{variance} \quad = \quad \sigma^2(r) = \sum_{i=1}^{n} (r_i - \bar{r})^2p_i.
\]
Statement of the problem

- A portfolio is defined by allocating fractions of initial wealth to individual assets. The fractions (or weights) must sum to one (some of these weights may be negative, corresponding to short selling).

- Return is quantified by portfolio’s expected rate of return; Risk is quantified by variance of portfolio’s rate of return.

Goal: Maximize return for a given level of risk; or minimize risk for a given level of return.

(i) How do we determine the optimal portfolio allocation?

(ii) The characterization of the set of optimal portfolios (minimum variance funds and efficient funds).
Limitations in the mean variance portfolio theory

• Only the mean and variance of rates of returns are taken into consideration in the portfolio analysis. The higher order moments (like the skewness) of the probability distribution of the rates of return are irrelevant in the formulation.

• Indeed, only the Gaussian (normal) distribution is fully specified by its mean and variance. Unfortunately, the rates of return of asset investment are not Gaussian in general.

• Calibration of parameters in the model is always challenging.
  – Sample mean: \( \hat{r} = \frac{1}{n} \sum_{t=1}^{n} r_t \).
  – Sample variance \( \hat{\sigma}^2 = \frac{1}{n-1} \sum_{t=1}^{n} (r_t - \hat{r})^2 \),
    where \( r_t \) is the historical rate of return observed at time \( t, t = 1, 2, \ldots , n \).
Short sales

- It is possible to sell an asset that you do not own through the process of **short selling**, or **shorting**, the asset. You then sell the borrowed asset to someone else, receiving an amount $X_0$. At a later date, you repay your loan by purchasing the asset for, say, $X_1$ and return the asset to your lender. Short selling is profitable if the asset price declines.

- When short selling a stock, you are essentially duplicating the role of the issuing corporation. You sell the stock to raise immediate capital. If the stock pays dividends during the period that you have borrowed it, you too must pay that same dividend to the person from whom you borrowed the stock.
Return associated with short selling

We receive $X_0$ initially and pay $X_1$ later, so the outlay is $-X_0$ and the final receipt is $-X_1$, and hence the total return is

$$R = \frac{-X_1}{-X_0} = \frac{X_1}{X_0}.$$ 

The minus signs cancel out, so we obtain the same expression as that for purchasing the asset. The return value $R$ applies algebraically to both purchases and short sales.

We can write

$$-X_1 = -X_0R = -X_0(1 + r)$$

to show how the final receipt is related to the initial outlay.
Example of short selling transaction

Suppose I short 100 shares of stock in company CBA. This stock is currently selling for $10 per share. I borrow 100 shares from my broker and sell these in the stock market, receiving $1,000. At the end of 1 year the price of CBA has dropped to $9 per share. I buy back 100 shares for $900 and give these shares to my broker to repay the original loan. Because the stock price fell, this has been a favorable transaction for me. I made a profit of $100.

The rate of return is clearly negative as \( r = -10\% \).

Shorting converts a negative rate of return into a profit because the original investment is also negative.
**Portfolio weights**

Suppose now that $n$ different assets are available. We form a **portfolio** of these $n$ assets. Suppose that this is done by apportioning an amount $X_0$ among the $n$ assets. We then select amounts $X_{0i}, i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{n} X_{0i} = X_0$, where $X_{0i}$ represents the amount invested in the $i^{th}$ asset. If we are allowed to sell an asset short, then some of the $X_{0i}$’s can be negative.

We write

$$X_{0i} = w_i X_0, \quad i = 1, 2, \ldots, n$$

where $w_i$ is the **weight** of asset $i$ in the portfolio. Clearly,

$$\sum_{i=1}^{n} w_i = 1$$

and some $w_i$’s may be negative if short selling is allowed.
Portfolio return

Let $R_i$ denote the total return of asset $i$. Then the amount of money generated at the end of the period by the $i^{th}$ asset is $R_i X_{0i} = R_i w_i X_0$.

The total amount received by this portfolio at the end of the period is therefore $\sum_{i=1}^{n} R_i w_i X_0$. The overall total return of the portfolio is

$$R_P = \frac{\sum_{i=1}^{n} R_i w_i X_0}{X_0} = \sum_{i=1}^{n} w_i R_i.$$ 

Since $\sum_{i=1}^{n} w_i = 1$, we have

$$r_P = R_P - 1 = \sum_{i=1}^{n} w_i (R_i - 1) = \sum_{i=1}^{n} w_i r_i.$$
**Covariance of a pair of random variables**

When considering two or more random variables, their mutual dependence can be summarized by their **covariance**.

Let $x_1$ and $x_2$ be a pair random variables with expected values $\bar{x}_1$ and $\bar{x}_2$, respectively. The covariance of this pair of random variables is defined to be the expectation of the product of deviations from the respective mean of $x_1$ and $x_2$:

$$\text{cov}(x_1, x_2) = E[(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)].$$

The covariance of two random variables $x$ and $y$ is denoted by $\sigma_{xy}$. We write $\text{cov}(x_1, x_2) = \sigma_{12}$. By symmetry, $\sigma_{12} = \sigma_{21}$, where

$$\sigma_{12} = E[x_1x_2 - \bar{x}_1x_2 - x_1\bar{x}_2 + \bar{x}_1\bar{x}_2] = E[x_1x_2] - \bar{x}_1\bar{x}_2.$$
Correlation

- If the two random variables $x_1$ and $x_2$ have the property that $\sigma_{12} = 0$, then they are said to be uncorrelated.

- If the two random variables are independent, then they are uncorrelated. When $x_1$ and $x_2$ are independent, $E[x_1x_2] = \bar{x}_1 \bar{x}_2$ so that $\text{cov}(x_1, x_2) = 0$.

- If $\sigma_{12} > 0$, then the two variables are said to be positively correlated. In this case, if one variable is above its mean, the other is likely to be above its mean as well.

- On the other hand, if $\sigma_{12} < 0$, the two variables are said to be negatively correlated.
When $x_1$ and $x_2$ are positively correlated, a positive deviation from mean of one random variable has a higher tendency to have a positive deviation from mean of the other random variable.
The correlation coefficient of a pair of random variables is defined as

\[ \rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}. \]

It can be shown that \( |\rho_{12}| \leq 1. \)

This would imply that the covariance of two random variables satisfies

\[ |\sigma_{12}| \leq \sigma_1 \sigma_2. \]

If \( \sigma_{12} = \sigma_1 \sigma_2 \), the variables are perfectly correlated. In this situation, the covariance is as large as possible for the given variance. If one random variable were a fixed positive multiple of the other, the two would be perfectly correlated.

Conversely, if \( \sigma_{12} = -\sigma_1 \sigma_2 \), the two variables exhibit perfect negative correlation.
Mean return of a portfolio

Suppose that there are $N$ assets with (random) rates of return $r_1, r_2, \ldots, r_N$, and their expected values $E[r_1] = \bar{r}_1, E[r_2] = \bar{r}_2, \ldots, E[r_N] = \bar{r}_N$. The rate of return of the portfolio in terms of the rate of return of the individual assets is

$$r_P = w_1r_1 + w_2r_2 + \cdots + w_Nr_N,$$

so that

$$E[r_P] = w_1E[r_1] + w_2E[r_2] + \cdots + w_NE[r_N].$$

The portfolio’s rate of return is simply the weighted average of the rates of return of the assets.
Variance of portfolio’s rate of return

We denote the variance of the return of asset \( i \) by \( \sigma_i^2 \), the variance of the return of the portfolio by \( \sigma_P^2 \), and the covariance of the return of asset \( i \) with that of asset \( j \) by \( \sigma_{ij} \). Portfolio variance is given by

\[
\sigma_P^2 = E[(r_P - \bar{r}_P)^2]
\]

\[
= E \left[ \left( \sum_{i=1}^{N} w_i r_i - \sum_{i=1}^{N} w_i \bar{r}_i \right)^2 \right]
\]

\[
= E \left[ \left( \sum_{i=1}^{N} w_i (r_i - \bar{r}_i) \right) \left( \sum_{j=1}^{N} w_j (r_j - \bar{r}_j) \right) \right]
\]

\[
= E \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j (r_i - \bar{r}_i)(r_j - \bar{r}_j) \right]
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}.
\]
Zero correlation

Suppose that a portfolio is constructed by taking equal portions of $N$ of these assets; that is, $w_i = \frac{1}{N}$ for each $i$. The overall rate of return of this portfolio is

$$r_P = \frac{1}{N} \sum_{i=1}^{N} r_i.$$ 

Let $\sigma_i^2$ be the variance of the rate of return of asset $i$. When the rates of return are uncorrelated, the corresponding variance is

$$\text{var}(r_P) = \frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^2 = \frac{\sigma_{\text{aver}}^2}{N}.$$ 

The variance decreases rapidly as $N$ increases.
When the rates of return of assets are uncorrelated, the variance of a portfolio can be made very small.
Non-zero correlation

We form a portfolio by taking equal portions of \( w_i = \frac{1}{N} \) of these assets. In this case,

\[
\text{var}(r_P) = E \left[ \sum_{i=1}^{N} \frac{1}{N} (r_i - \bar{r}) \right]^2
\]

\[
= \frac{1}{N^2} E \left\{ \left[ \sum_{i=1}^{N} (r_i - \bar{r}) \right] \left[ \sum_{j=1}^{N} (r_j - \bar{r}) \right] \right\}
\]

\[
= \frac{1}{N^2} \sum_{i,j} \sigma_{ij} = \frac{1}{N^2} \left\{ \sum_{i=j} \sigma_{ij} + \sum_{i \neq j} \sigma_{ij} \right\}
\]

\[
= \frac{1}{N^2} \{N(\sigma_i^2)_{\text{aver}} + (N^2 - N)(\sigma_{ij})_{\text{aver}}\}
\]

\[
= \frac{1}{N} \left[ (\sigma_i^2)_{\text{aver}} - (\sigma_{ij})_{\text{aver}} \right] + (\sigma_{ij})_{\text{aver}}.
\]

The covariance terms remain even when we take \( N \to \infty \).

Also, \( \text{var}(r_P) \) may be decreased by choosing assets that are negatively correlated.
If returns of assets are correlated, there is likely to be a lower limit to the portfolio variance that can be achieved.
1.2 Markowitz mean-variance formulation

We consider a single-period investment model. Suppose there are $N$ risky assets, whose rates of return are given by the random variables $r_1, \ldots, r_N$, where

$$r_n = \frac{S_n(1) - S_n(0)}{S_n(0)}, \quad n = 1, 2, \ldots, N.$$ 

Here $S_n(0)$ is known while $S_n(1)$ is random, $n = 1, 2, \ldots, N$. Let $w = (w_1 \cdots w_N)^T$, $w_n$ denotes the proportion of wealth invested in asset $n$, with $\sum_{n=1}^N w_n = 1$. The rate of return of the portfolio $r_P$ is

$$r_P = \sum_{n=1}^N w_n r_n.$$
**Assumption**

The two vectors $\mu = (\bar{r}_1 \  \bar{r}_2 \cdots \bar{r}_N)$ and $\mathbf{1} = (1 \ 1 \cdots 1)$ are linearly independent. If otherwise, the mean rates of return are equal and so the portfolio return can only be the common mean rate of return. Under this degenerate case, the portfolio choice problem becomes a simpler minimization problem.

The first two moments of $r_P$ are

$$\mu_P = E[r_P] = \sum_{n=1}^{N} E[w_nr_n] = \sum_{n=1}^{N} w_n\mu_n,$$

where $\mu_n = \bar{r}_n$,

and

$$\sigma_P^2 = \text{var}(r_P) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_iw_j\text{cov}(r_i, r_j) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_iw_j\sigma_{ij}.$$
Covariance matrix

Let $\Omega$ denote the covariance matrix so that

$$\sigma_P^2 = w^T \Omega w,$$

where $\Omega$ is symmetric and $(\Omega)_{ij} = \sigma_{ij} = \text{cov}(r_i, r_j)$. For example, when $n = 2$, we have

$$(w_1 \quad w_2) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w_1^2 \sigma_1^2 + w_1 w_2 (\sigma_{12} + \sigma_{21}) + w_2^2 \sigma_2^2.$$ 

Since portfolio variance $\sigma_P^2$ must be non-negative, so the covariance matrix must be symmetric and semi-positive definite. The eigenvalues are all real non-negative.

- In our later discussion, we always assume $\Omega$ to be symmetric and positive definite (avoiding the unlikely event where one of the eigenvalues is zero) so that $\Omega^{-1}$ always exists. The degenerate case of singular covariance matrix is discussed in the Appendix.
Sensitivity of $\sigma_P^2$ with respect to $w_k$

By the product rule in differentiation

$$\frac{\partial \sigma_P^2}{\partial w_k} = \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{\partial w_i}{\partial w_k} w_j \sigma_{ij} + \sum_{i=1}^{N} \sum_{j=1}^{N} w_i \frac{\partial w_j}{\partial w_k} \sigma_{ij}$$

$$= \sum_{j=1}^{N} w_j \sigma_{kj} + \sum_{i=1}^{N} w_i \sigma_{ik}.$$  

Since $\sigma_{kj} = \sigma_{jk}$, we obtain

$$\frac{\partial \sigma_P^2}{\partial w_k} = 2 \sum_{j=1}^{N} w_j \sigma_{kj} = 2(\Omega w)_k,$$

where $(\Omega w)_k$ is the $k^{th}$ component of the vector $\Omega w$. Alternatively, we may write

$$\nabla \sigma_P^2 = 2\Omega w,$$

where $\nabla$ is the gradient operator. This partial derivative gives the sensitivity of the portfolio variance with respect to the weight of a particular asset.
Remark

1. The portfolio risk of return is quantified by $\sigma_P^2$. In the mean-variance analysis, only the first two moments are considered in the portfolio investment model. Earlier investment theory prior to Markowitz only considered the maximization of $\mu_P$ without $\sigma_P$.

2. The measure of risk by variance would place equal weight on the upside and downside deviations. In reality, positive deviations should be more welcomed.

3. The assets are characterized by their random rates of return, $r_i, i = 1, \ldots, N$. In the mean-variance model, it is assumed that their first and second order moments: $\mu_i, \sigma_i$ and $\sigma_{ij}$ are all known. In the Markowitz mean-variance formulation, we would like to determine the choice variables: $w_1, \ldots, w_N$ such that $\sigma_P^2$ is minimized for a given preset value of $\mu_P$. 
Two-asset portfolio

Consider a portfolio of two assets with known means $\bar{r}_1$ and $\bar{r}_2$, variances $\sigma_1^2$ and $\sigma_2^2$, of the rates of return $r_1$ and $r_2$, together with the correlation coefficient $\rho$, where $\text{cov}(r_1, r_2) = \rho \sigma_1 \sigma_2$.

Let $1 - \alpha$ and $\alpha$ be the weights of assets 1 and 2 in this two-asset portfolio.

Portfolio mean: $\bar{r}_P = (1 - \alpha)\bar{r}_1 + \alpha \bar{r}_2$,

Portfolio variance: $\sigma_P^2 = (1 - \alpha)^2 \sigma_1^2 + 2\rho\alpha(1 - \alpha)\sigma_1 \sigma_2 + \alpha^2 \sigma_2^2$. 
## assets’ mean and variance

<table>
<thead>
<tr>
<th></th>
<th>Asset A</th>
<th>Asset B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean return (%)</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Variance (%)</td>
<td>10</td>
<td>15</td>
</tr>
</tbody>
</table>

### Portfolio mean\(^a\) and variance\(^b\) for weights and asset correlations

<table>
<thead>
<tr>
<th>weight ( w_A )</th>
<th>( w_B = 1 - w_A )</th>
<th>( \rho = -1 )</th>
<th>( \rho = -0.5 )</th>
<th>( \rho = 0.5 )</th>
<th>( \rho = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Variance</td>
<td>Mean</td>
<td>Variance</td>
<td>Mean</td>
</tr>
<tr>
<td>1.0</td>
<td>10.0</td>
<td>10.00</td>
<td>10.0</td>
<td>10.00</td>
<td>10.0</td>
</tr>
<tr>
<td>0.8</td>
<td>12.0</td>
<td>3.08</td>
<td>12.0</td>
<td>5.04</td>
<td>12.0</td>
</tr>
<tr>
<td>0.5</td>
<td>15.0</td>
<td>0.13</td>
<td>15.0</td>
<td>3.19</td>
<td>15.0</td>
</tr>
<tr>
<td>0.2</td>
<td>18.0</td>
<td>6.08</td>
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<td>18.0</td>
</tr>
<tr>
<td>0.0</td>
<td>20.0</td>
<td>15.00</td>
<td>20.0</td>
<td>15.00</td>
<td>20.0</td>
</tr>
</tbody>
</table>

\(^a\) The mean is calculated as \( E(R) = w_A 10 + (1 - w_A) 20. \)

\(^b\) The variance is calculated as \( \sigma_p^2 = w_A^2 10 + (1 - w_A)^2 15 + 2 w_A (1 - w_A) \rho \sqrt{10} \sqrt{5} \)

where \( \rho \) is the assumed correlation coefficient and \( \sqrt{10} \) and \( \sqrt{5} \) are standard deviations of the returns of the two assets, respectively.

**Observation:** A lower variance is achieved for a given mean when the correlation of the pair of assets’ returns becomes more negative.
We represent the two assets in a mean-standard deviation diagram.

As \( \alpha \) varies, \((\sigma_P, \bar{r}_P)\) traces out a conic curve in the \(\sigma - \bar{r}\) plane. With \(\rho = -1\), it is possible to have \(\sigma_P = 0\) for some suitable choice of weight \(\alpha\).
Consider the special case where $\rho = 1$,

$$
\sigma_P(\alpha; \rho = 1) = \sqrt{(1 - \alpha)^2 \sigma_1^2 + 2\alpha(1 - \alpha)\sigma_1\sigma_2 + \alpha^2 \sigma_2^2} = (1 - \alpha)\sigma_1 + \alpha\sigma_2.
$$

Since $\bar{r}_P$ and $\sigma_P$ are linear in $\alpha$, and if we choose $0 \leq \alpha \leq 1$, then the portfolios are represented by the straight line joining $P_1(\sigma_1, \bar{r}_1)$ and $P_2(\sigma_2, \bar{r}_2)$.

When $\rho = -1$, we have

$$
\sigma_P(\alpha; \rho = -1) = \sqrt{[(1 - \alpha)\sigma_1 - \alpha\sigma_2]^2} = |(1 - \alpha)\sigma_1 - \alpha\sigma_2|.
$$

When $\alpha$ is small (close to zero), the corresponding point is close to $P_1(\sigma_1, \bar{r}_1)$. The line $AP_1$ corresponds to

$$
\sigma_P(\alpha; \rho = -1) = (1 - \alpha)\sigma_1 - \alpha\sigma_2.
$$

The point $A$ corresponds to $\alpha = \frac{\sigma_1}{\sigma_1 + \sigma_2}$. It is a point on the vertical axis which has zero value of $\sigma_P$. 
The quantity \((1 - \alpha)\sigma_1 - \alpha\sigma_2\) remains positive until \(\alpha = \frac{\sigma_1}{\sigma_1 + \sigma_2}\).

When \(\alpha > \frac{\sigma_1}{\sigma_1 + \sigma_2}\), the locus traces out the upper line \(AP_2\).

Suppose \(-1 < \rho < 1\), the minimum variance point on the curve that represents various portfolio combinations is determined by

\[
\frac{\partial \sigma_P^2}{\partial \alpha} = -2(1 - \alpha)\sigma_1^2 + 2\alpha\sigma_2^2 + 2(1 - 2\alpha)\rho\sigma_1\sigma_2 = 0
\]

↑ set

giving

\[
\alpha = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}.
\]
Mean-standard deviation diagram

$P_2(\sigma_2, \bar{r}_2)$

$P_1(\sigma_1, \bar{r}_1)$

Minimum-variance point

$x$

$\bar{r}$

$\bar{r}$

$\sigma$
Mathematical formulation of Markowitz's mean-variance analysis

minimize \( \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} \)

subject to \( \sum_{i=1}^{N} w_i \bar{r}_i = \mu_P \) and \( \sum_{i=1}^{N} w_i = 1 \). Given the target expected rate of return of portfolio \( \mu_P \), we find the optimal portfolio strategy that minimizes \( \sigma_P^2 \). The constraint: \( \sum_{i=1}^{N} w_i = 1 \) refers to the strategy of putting all wealth into investment of risky assets.

Solution

We form the Lagrangian

\[
L = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} - \lambda_1 \left( \sum_{i=1}^{N} w_i - 1 \right) - \lambda_2 \left( \sum_{i=1}^{N} w_i \bar{r}_i - \mu_P \right),
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the Lagrangian multipliers.
We differentiate $L$ with respect to $w_i$ and the Lagrangian multipliers, then set all the derivatives be zero.

\[
\frac{\partial L}{\partial w_i} = \sum_{j=1}^{N} \sigma_{ij} w_j - \lambda_1 - \lambda_2 \bar{r}_i = 0, \quad i = 1, 2, \ldots, N; \quad (1)
\]

\[
\frac{\partial L}{\partial \lambda_1} = \sum_{i=1}^{N} w_i - 1 = 0; \quad (2)
\]

\[
\frac{\partial L}{\partial \lambda_2} = \sum_{i=1}^{N} w_i \bar{r}_i - \mu_P = 0. \quad (3)
\]

From Eq. (1), we deduce that the optimal portfolio vector weight $w^*$ admits solution of the form

\[
\Omega w^* = \lambda_1 \mathbf{1} + \lambda_2 \mu \quad \text{or} \quad w^* = \Omega^{-1}(\lambda_1 \mathbf{1} + \lambda_2 \mu)
\]

where $\mathbf{1} = (1 \ 1 \cdots 1)^T$ and $\mu = (\bar{r}_1 \ \bar{r}_2 \cdots \bar{r}_N)^T$.  

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Degenerate case

Consider the case where all assets have the same expected rate of return, that is, \( \mu = h \mathbf{1} \) for some constant \( h \). In this case, the solution to Eqs. (2) and (3) gives \( \mu_P = h \). The assets are represented by points that all lie on the horizontal line: \( \bar{r} = h \).

In this case, the expected portfolio return cannot be arbitrarily prescribed. Actually, we have to take \( \mu_P = h \), so the constraint on the expected portfolio return becomes irrelevant.
Solution procedure

To determine $\lambda_1$ and $\lambda_2$, we apply the two constraints:

\[
1 = \mathbf{1}^T \Omega^{-1} \Omega w^* = \lambda_1 \mathbf{1}^T \Omega^{-1} \mathbf{1} + \lambda_2 \mathbf{1}^T \Omega^{-1} \mu
\]

\[
\mu_P = \mu^T \Omega^{-1} \Omega w^* = \lambda_1 \mu^T \Omega^{-1} \mathbf{1} + \lambda_2 \mu^T \Omega^{-1} \mu.
\]

Writing $a = \mathbf{1}^T \Omega^{-1} \mathbf{1}$, $b = \mathbf{1}^T \Omega^{-1} \mu$ and $c = \mu^T \Omega^{-1} \mu$, we have two equations for $\lambda_1$ and $\lambda_2$:

\[
1 = \lambda_1 a + \lambda_2 b \quad \text{and} \quad \mu_P = \lambda_1 b + \lambda_2 c.
\]

Solving for $\lambda_1$ and $\lambda_2$:

\[
\lambda_1 = \frac{c - b \mu_P}{\Delta} \quad \text{and} \quad \lambda_2 = \frac{a \mu_P - b}{\Delta},
\]

where $\Delta = ac - b^2$. Provided that $\mu \neq h \mathbf{1}$ for some scalar $h$, we then have $\Delta \neq 0$. 

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Solution to the minimum portfolio variance

• Both $\lambda_1$ and $\lambda_2$ have dependence on $\mu_P$, where $\mu_P$ is the target mean prescribed in the variance minimization problem.

• The minimum portfolio variance for a given value of $\mu_P$ is given by

$$\sigma_P^2 = w^{*T} \Omega w^* = w^{*T} (\lambda_1 \mathbf{1} + \lambda_2 \mu)$$

$$= \lambda_1 + \lambda_2 \mu_P = \frac{a \mu_P^2 - 2b \mu_P + c}{\Delta}.$$

• $\sigma_P^2 = w^T \Omega w \geq 0$, for all $w$, so $\Omega$ is guaranteed to be semi-positive definite. In our subsequent analysis, we assume $\Omega$ to be positive definite. Given that $\Omega$ is positive definite, we have $a > 0, c > 0$ and $\Omega^{-1}$ exists. By virtue of the Cauchy-Schwarz inequality, $\Delta > 0$. Since $a$ and $\Delta$ are both positive, the quantity $a \mu_P^2 - 2b \mu_P + c$ is guaranteed to be positive.
The set of minimum variance portfolios is represented by a parabolic curve in the $\sigma_P^2 - \mu_P$ plane. The parabolic curve is generated by varying the value of the parameter $\mu_P$. Note that $\frac{1}{a} > 0$ while $\frac{b}{a}$ may become negative under some extreme adverse cases of negative mean rates of return.

Non-optimal portfolios are represented by points which must fall on the right side of the parabolic curve.
Global minimum variance portfolio

Given $\mu_P$, we obtain $\lambda_1 = \frac{c - b\mu_P}{\Delta}$ and $\lambda_2 = \frac{a\mu_P - b}{\Delta}$, and the optimal weight $w^* = \Omega^{-1}(\lambda_1 \mathbf{1} + \lambda_2 \mu) = \frac{c - b\mu_P}{\Delta} \Omega^{-1} \mathbf{1} + \frac{a\mu_P - b}{\Delta} \Omega^{-1} \mu$.

To find the global minimum variance portfolio, we set
\[
\frac{d\sigma_P^2}{d\mu_P} = \frac{2a\mu_P - 2b}{\Delta} = 0
\]
so that $\mu_P = b/a$ and $\sigma_P^2 = 1/a$. Correspondingly, $\lambda_1 = 1/a$ and $\lambda_2 = 0$. The weight vector that gives the global minimum variance portfolio is found to be
\[
w_g = \lambda_1 \Omega^{-1} \mathbf{1} = \frac{\Omega^{-1} \mathbf{1}}{a} = \frac{\Omega^{-1} \mathbf{1}}{1^{T} \Omega^{-1} \mathbf{1}}.
\]

Note that $w_g$ is independent of $\mu$. Obviously, $w_g^{T} \mathbf{1} = 1$ due to the normalization factor $1^{T} \Omega^{-1} \mathbf{1}$ in the denominator. As a check, we have $\mu_g = \mu^{T} w_g = \frac{b}{a}$ and $\sigma_g^2 = w_g^{T} \Omega w_g = \frac{1}{a}$. 

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Example

Given the variance matrix

\[
\Omega = \begin{pmatrix}
2 & 0.5 & 0 \\
0.5 & 3 & 0.5 \\
0 & 0.5 & 2
\end{pmatrix},
\]

find \( w_g \). This can be obtained effectively by solving

\[
2v_1 + 0.5v_2 = 1 \\
0.5v_1 + 3v_2 + 0.5v_3 = 1 \\
0.5v_2 + 2v_3 = 1.
\]

Observing the symmetry between asset 1 and asset 3 since \( \sigma_1^2 = \sigma_3^2 \) and \( \sigma_{12} = \sigma_{32} \), etc.
We expect \( v_1 = v_3 \) so that the above system reduces to

\[
2v_1 + 0.5v_2 = 1 \\
v_1 + 3v_2 = 1
\]

giving \( v_1 = v_3 = \frac{5}{11} \) and \( v_2 = \frac{2}{11} \). Lastly, by normalization to sum of weights equals 1, we obtain

\[
w_g = \left( \frac{5}{12}, \frac{2}{12}, \frac{5}{12} \right)^T.
\]
Two-parameter \((\lambda_1 - \lambda_2)\) family of minimum variance portfolios

Recall \(w^* = \lambda_1 \Omega^{-1} 1 + \lambda_2 \Omega^{-1} \mu\), so the minimum variance portfolios (frontier funds) are seen to be generated by a linear combination of \(\Omega^{-1} 1 + \Omega^{-1} \mu\).

It is not surprising to see that \(\lambda_2 = 0\) corresponds to \(w_g^*\) since the constraint on the target mean vanishes when \(\lambda_2\) is taken to be zero. In this case, we minimize risk while paying no regard to the target mean, thus the global minimum variance portfolio is resulted.

Suppose we normalize \(\Omega^{-1} \mu\) by \(b\) and define

\[
w_d = \frac{\Omega^{-1} \mu}{b} = \Omega^{-1} \mu \frac{1^T \Omega^{-1} \mu}{1^T \Omega^{-1} \mu}.
\]

Obviously, \(w_d\) also lies on the frontier.
The corresponding expected rate of return $\mu_d$ and $\sigma_d^2$ are given by

$$\mu_d = \mu^T w_d = \frac{c}{b}$$

$$\sigma_d^2 = \left(\Omega^{-1}\mu\right)^T \Omega \left(\Omega^{-1}\mu\right) = \frac{\mu^T \Omega \mu}{b^2} = \frac{c}{b^2}.$$ 

Since $\Omega^{-1}1 = aw_g$ and $\Omega^{-1}\mu = bw_d$, the weight of any frontier fund (minimum variance fund) can be represented by

$$w^* = (\lambda_1 a)w_g + (\lambda_2 b)w_d = \frac{c - b\mu_P}{\Delta}a w_g + \frac{a\mu_P - b}{\Delta}b w_d.$$ 

This provides the motivation of the Two-Fund Theorem. The above representation indicates that the optimal portfolio weight $w^*$ depends on $\mu_P$ set by the investor.

- Any minimum variance fund can be generated by an appropriate combination of the two funds corresponding to $w_g$ and $w_d$ (see Sec. 1.3: Two-fund Theorem).
Feasible set

Given $N$ risky assets, we can form various portfolios from these $N$ assets. We plot the point $(\sigma_P, \bar{r}_P)$ that represents a particular portfolio in the $\sigma - \bar{r}$ diagram. The collection of these points constitutes the feasible set or feasible region.
Argument to show that the collection of the points representing $(\sigma_P, \bar{r}_P)$ of a 3-asset portfolio generates a solid region in the $\sigma$-$\bar{r}$ plane

- Consider a 3-asset portfolio, the various combinations of assets 2 and 3 sweep out a curve between them (the particular curve taken depends on the correlation coefficient $\rho_{23}$).

- A combination of assets 2 and 3 (labelled 4) can be combined with asset 1 to form a curve joining 1 and 4. As 4 moves between 2 and 3, the family of curves joining 1 and 4 sweep out a solid region.
Properties of the feasible regions

1. For a portfolio with at least 3 risky assets (not perfectly correlated and with different means), the feasible set is a solid two-dimensional region.

2. The feasible region is *convex to the left*. That is, given any two points in the region, the straight line connecting them does not cross the left boundary of the feasible region. This property must be observed since any combination of two portfolios also lies in the feasible region. Indeed, the left boundary of a feasible region is a hyperbola (as solved by the Markowitz constrained minimization model).
Locate the efficient and inefficient investment strategies

- Since investors prefer the lowest variance for the same expected return, they will focus on the set of portfolios with the smallest variance for a given mean, or the mean-variance frontier (collection of minimum variance portfolios).

- The mean-variance frontier can be divided into two parts: an efficient frontier and an inefficient frontier.

- The efficient part includes the portfolios with the highest mean for a given variance.
Minimum variance set and efficient funds

The left boundary of a feasible region is called the *minimum variance set*. The most left point on the minimum variance set is called the *global minimum variance point*. The portfolios in the minimum variance set are called the *frontier funds*.

For a given level of risk, only those portfolios on the *upper half* of the efficient frontier with a higher return are desired by investors. They are called the *efficient funds*.

A portfolio $\mathbf{w}^*$ is said to be mean-variance efficient if there exists no portfolio $\mathbf{w}$ with $\mu_P \geq \mu_P^*$ and $\sigma_P^2 \leq \sigma_P^*$, except itself. That is, you cannot find a portfolio that has a higher return and lower risk than those of an efficient portfolio. The funds on the inefficient frontier do not exhibit the above properties.
Example – Uncorrelated assets with short sales constraint

Suppose there are three uncorrelated assets. Each has variance 1, and the mean values are 1, 2 and 3 (in percentage points), respectively. We have $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$ and $\sigma_{12} = \sigma_{23} = \sigma_{13} = 0$.

The first order conditions give $\Omega w = \lambda_1 1 + \lambda_2 \mu$, $\mu^T w = \mu_P$ and $1^T w = 1$, so we obtain

$$
\begin{align*}
   w_1 - \lambda_2 - \lambda_1 & = 0 \\
   w_2 - 2\lambda_2 - \lambda_1 & = 0 \\
   w_3 - 3\lambda_2 - \lambda_1 & = 0 \\
   w_1 + 2w_2 + 3w_3 & = \mu_P \\
   w_1 + w_2 + w_3 & = 1.
\end{align*}
$$

By eliminating $w_1, w_2, w_3$, we obtain two equations for $\lambda_1$ and $\lambda_2$

$$
\begin{align*}
   14\lambda_2 + 6\lambda_1 & = \mu_P \\
   6\lambda_2 + 3\lambda_1 & = 1.
\end{align*}
$$

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These two equations can be solved to yield $\lambda_2 = \frac{\mu_P}{2} - 1$ and $\lambda_1 = 2\frac{1}{3} - \mu_P$. The portfolio weights are expressed in terms of $\mu_P$:

$$w_1 = \frac{4}{3} - \frac{\mu_P}{2}$$
$$w_2 = \frac{1}{3}$$
$$w_3 = \frac{\mu_P}{2} - \frac{2}{3}.$$

The standard deviation of $r_P$ at the solution is $\sqrt{w_1^2 + w_2^2 + w_3^2}$, which by direct substitution gives

$$\sigma_P = \sqrt{\frac{7}{3} - 2\mu_P + \frac{\mu_P^2}{2}}.$$ 

The minimum-variance point is, by symmetry, at $\mu_P = 2$, with $\sigma_P = \sqrt{3}/3 = 0.58$. When $\mu_P = 2$, we obtain

$$w_1 = w_2 = w_3 = \frac{1}{3}.$$
Short sales not allowed (adding the constraints: \( w_i \geq 0, i = 1, 2, 3 \))

Unlike the unrestricted case of allowing short sales, we now impose \( w_i \geq 0, i = 1, 2, 3 \). As a result, \( \mu_P \) can only lie between \( 1 \leq \mu_P \leq 3 \) [recall \( \mu_P = w_1 + 2w_2 + 3w_3 \)]. The lower bound is easily seen since \( \mu_P = (w_1 + w_2 + w_3) + (w_2 + 2w_3) = 1 + w_2 + 2w_3 \geq 1 \) since \( w_2 \geq 0 \) and \( w_3 \geq 0 \). Also, \( \mu_P \) cannot go above 3 as the maximum value of \( \mu_P \) can only be achieved by choosing \( w_3 = 1, w_1 = w_2 = 0 \). For certain range of \( \mu_P \), some of the optimal portfolio weights may become negative when there is no short sales constraint.
It is instructive to consider separately, the following 3 intervals for \( \mu_P \): \([1, \frac{4}{3}], [\frac{4}{3}, \frac{8}{3}] \) and \([\frac{8}{3}, 3]\).

<table>
<thead>
<tr>
<th>Interval</th>
<th>( w_1 )</th>
<th>( w_2 )</th>
<th>( w_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 \leq \mu_P \leq \frac{4}{3} )</td>
<td>( 2 - \mu_P )</td>
<td>( \mu_P - 1 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \frac{4}{3} \leq \mu_P \leq \frac{8}{3} )</td>
<td>( \frac{4}{3} - \frac{\mu_P}{2} )</td>
<td>( \frac{1}{3} )</td>
<td>( 3 - \mu_P )</td>
</tr>
<tr>
<td>( \frac{8}{3} \leq \mu_P \leq 3 )</td>
<td>( \frac{\mu_P}{2} - \frac{2}{3} )</td>
<td>( \mu_P - 2 )</td>
<td></td>
</tr>
</tbody>
</table>

\[
\sigma_P = \begin{cases} 
\sqrt{2\mu_P^2 - 6\mu_P + 5} & \text{if } 1 \leq \mu_P \leq \frac{4}{3} \\
\sqrt{\frac{7}{3} - 2\mu_P + \frac{\mu_P^2}{2}} & \text{if } \frac{4}{3} \leq \mu_P \leq \frac{8}{3} \\
\sqrt{2\mu_P^2 - 10\mu_P + 13} & \text{if } \frac{8}{3} \leq \mu_P \leq 3 
\end{cases}
\]
• From \( w_3 = \frac{\mu_P}{2} - \frac{2}{3} \), we deduce that when \( 1 \leq \mu_P < \frac{4}{3} \), \( w_3 \) becomes negative in the minimum variance portfolio when short sales are allowed. This is truly an inferior investment choice as the investor sets \( \mu_P \) to be too low while asset 3 has the highest mean rate of return. When short sales are not allowed, we expect to have “\( w_3 = 0 \)” in the minimum variance portfolio. The problem reduces to two-asset portfolio model and the corresponding optimal weights \( w_1 \) and \( w_2 \) can be easily obtained by solving

\[
\begin{align*}
    w_1 + 2w_2 &= \mu_P \\
    w_1 + w_2 &= 1.
\end{align*}
\]

• Similarly, when \( \frac{8}{3} \leq \mu_P \leq 3 \), it is optimal to choose \( w_1 = 0 \). In this case, the investor sets \( \mu_P \) to be too high while asset 1 has the lowest mean rate of return.

• When \( \frac{4}{3} \leq \mu_P \leq \frac{8}{3} \), we have the same solution as the case without the short sales constraint. This is because the solutions to the weights happen to be non-negative under the unconstrained case. The short sales constraint becomes redundant.
1.3 Two-fund Theorem

Take any two frontier funds (portfolios), then any combination of these two frontier funds remains to be a frontier fund. Indeed, any frontier portfolio can be duplicated, in terms of mean and variance, as a combination of these two frontier funds. In other words, all investors seeking frontier portfolios need only invest in various combinations of these two funds.

- This property can be extended to a combination of efficient funds (frontier funds that lie on the upper portion of the efficient frontier)?
Proof of the Two-fund Theorem

Let \( w^1 = (w^1_1 \cdots w^1_n), \lambda^1_1, \lambda^1_2 \) and \( w^2 = (w^2_1 \cdots w^2_n)^T, \lambda^2_1, \lambda^2_2 \) be two known solutions to the minimum variance formulation with expected rates of return \( \mu^1_P \) and \( \mu^2_P \), respectively. By setting \( \mu_P \) equal \( \mu^1_P \) and \( \mu^2_P \) successively, both solutions satisfy

\[
\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda_1 - \lambda_2 \bar{r}_i = 0, \quad i = 1, 2, \ldots, n
\]  

\[
\sum_{i=1}^{n} w_i \bar{r}_i = \mu_P
\]  

\[
\sum_{i=1}^{n} w_i = 1.
\]  

We would like to show that \( \alpha w_1 + (1-\alpha)w_2 \) is a solution corresponds to the expected rate of return \( \alpha \mu^1_P + (1-\alpha)\mu^2_P \).
1. The new weight vector $\alpha w^1 + (1 - \alpha) w^2$ is a legitimate portfolio with weights that sum to one.

2. Check the condition on the expected rate of return

$$\sum_{i=1}^{n} \left[ \alpha w^1_i + (1 - \alpha) w^2_i \right] \bar{r}_i$$

$$= \alpha \sum_{i=1}^{n} w^1_i \bar{r}_i + (1 - \alpha) \sum_{i=1}^{n} w^2_i \bar{r}_i$$

$$= \alpha \mu_P^1 + (1 - \alpha) \mu_P^2.$$

3. Eq. (1) is satisfied by $\alpha w^1 + (1 - \alpha) w^2$ since the system of equations is linear. The corresponding $\lambda_1$ and $\lambda_2$ are given by

$$\lambda_1 = \alpha \lambda_1^1 + (1 - \alpha) \lambda_1^2$$

and

$$\lambda_2 = \alpha \lambda_2^1 + (1 - \alpha) \lambda_2^2.$$

4. Given $\mu_P$, the appropriate portion $\alpha$ is determined by

$$\mu_P = \alpha \mu_P^1 + (1 - \alpha) \mu_P^2.$$
Global minimum variance portfolio $w_g$ and the counterpart $w_d$

For convenience, we choose the two frontier funds to be $w_g$ and $w_d$. To obtain the optimal weight $w^*$ for a given $\mu_P$, we solve for $\alpha$ using $\alpha \mu_g + (1 - \alpha) \mu_d = \mu_P$ and $w^*$ is then given by $\alpha w_g^* + (1 - \alpha) w_d^*$. Recall $\mu_g = b/a$ and $\mu_d = c/b$, so $\alpha = \frac{(c - b \mu_P) a}{\Delta}$.

**Proposition**

Any minimum variance portfolio with the target mean $\mu_P$ can be uniquely decomposed into the sum of two portfolios

$$w^*_P = \alpha w_g + (1 - \alpha) w_d$$

where $\alpha = \frac{c - b \mu_P}{\Delta} a$. 
Indeed, any two minimum variance portfolios $w_u$ and $w_v$ can be used to substitute for $w_g$ and $w_d$. Suppose

$$w_u = (1 - u)w_g + uw_d$$
$$w_v = (1 - v)w_g + vw_d$$

we then solve for $w_g$ and $w_d$ in terms of $w_u$ and $w_v$. Recall

$$w^*_P = \lambda_1 \Omega^{-1} \mathbf{1} + \lambda_2 \Omega^{-1} \mu$$

so that

$$w^*_P = \lambda_1 aw_g + (1 - \lambda_1 a)w_d$$
$$= \frac{\lambda_1 a + v - 1}{v - u} w_u + \frac{1 - u - \lambda_1 a}{v - u} w_v,$$

whose sum of coefficients remains to be 1 and $\lambda_1 = \frac{c - b \mu_P}{\Delta}$. 

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Convex combination of efficient portfolios

Any convex combination (that is, weights are non-negative) of efficient portfolios is an efficient portfolio.

Proof

Let \( w_i \geq 0 \) be the weight of the efficient fund \( i \) whose random rate of return is \( r^i_e \). Recall that \( \frac{b}{a} \) is the expected rate of return of the global minimum variance portfolio.

It suffices to show that such convex combination has an expected rate of return greater than \( \frac{b}{a} \) in order that the combination of funds remains to be efficient.

Since \( E \left[ r^i_e \right] \geq \frac{b}{a} \) for all \( i \) as all these funds are efficient and \( w_i \geq 0 \), \( i = 1, 2, \ldots, n \), we have

\[
\sum_{i=1}^{n} w_i E \left[ r^i_e \right] \geq \sum_{i=1}^{n} w_i \frac{b}{a} = \frac{b}{a}.
\]
Example

Mean, variances, and covariances of the rates of return of 5 risky assets are listed:

<table>
<thead>
<tr>
<th>Security</th>
<th>covariance, $\sigma_{ij}$</th>
<th>mean, $\bar{r}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.30 0.93 0.62 0.74 -0.23</td>
<td>15.1</td>
</tr>
<tr>
<td>2</td>
<td>0.93 1.40 0.22 0.56 0.26</td>
<td>12.5</td>
</tr>
<tr>
<td>3</td>
<td>0.62 0.22 1.80 0.78 -0.27</td>
<td>14.7</td>
</tr>
<tr>
<td>4</td>
<td>0.74 0.56 0.78 3.40 -0.56</td>
<td>9.02</td>
</tr>
<tr>
<td>5</td>
<td>-0.23 0.26 -0.27 -0.56 2.60</td>
<td>17.68</td>
</tr>
</tbody>
</table>

Recall that $w^*$ has the following closed form solution

$$w^* = \frac{c - b\mu_P}{\Delta} \Omega^{-1} \mathbf{1} + \frac{a\mu_P - b}{\Delta} \Omega^{-1} \mathbf{\mu}$$

$$= \alpha \mathbf{w}_g + (1 - \alpha) \mathbf{w}_d,$$

where $\alpha = (c - b\mu_P) \frac{a}{\Delta}$. Here, $\alpha$ satisfies

$$\mu_P = \alpha \mu_g + (1 - \alpha) \mu_d = \alpha \left( \frac{b}{a} \right) + (1 - \alpha) \frac{c}{b}.$$
We compute $w^*_g$ and $w^*_d$ through finding $\Omega^{-1}1$ and $\Omega^{-1}\mu$, then normalize by enforcing the condition that their weights are summed to one.

1. To find $v^1 = \Omega^{-1}1$, we solve the system of equations

$$\sum_{j=1}^{5} \sigma_{ij}v^1_j = 1, \quad i = 1, 2, \ldots, 5.$$ 

Normalize the component $v^1_i$'s so that they sum to one

$$w_i^1 = \frac{v_i^1}{\sum_{j=1}^{5} v_j^1}.$$ 

After normalization, this gives the solution to $w_g$. Why?
We first solve for \( v^1 = \Omega^{-1} \mathbf{1} \) and later divide \( v^1 \) by some constant \( k \) such that \( \mathbf{1}^T v^1 / k = 1 \). We see that \( k \) must be equal to \( a \), where \[ a = \mathbf{1}^T \Omega^{-1} \mathbf{1} \]. Actually, \[ a = \mathbf{1}^T \Omega^{-1} \mathbf{1} = \sum_{j=1}^{N} v_j^1. \]

2. To find \( v^2 = \Omega^{-1} \mu \), we solve the system of equations:

\[
\sum_{j=1}^{5} \sigma_{ij} v_j^2 = \bar{r}_i, \quad i = 1, 2, \ldots, 5.
\]

Normalize \( v_i^2 \)'s to obtain \( w_i^2 \). After normalization, this gives the solution to \( w_d \). Also, \[ b = \mathbf{1}^T \Omega^{-1} \mu = \sum_{j=1}^{N} v_j^2 \] and \[ c = \mu^T \Omega^{-1} \mu = \sum_{j=1}^{N} \bar{r}_j v_j^2 \].
<table>
<thead>
<tr>
<th>security</th>
<th>$v^1$</th>
<th>$v^2$</th>
<th>$w_g$</th>
<th>$w_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.141</td>
<td>3.652</td>
<td>0.088</td>
<td>0.158</td>
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<td>2</td>
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<td>3.583</td>
<td>0.251</td>
<td>0.155</td>
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<td>0.282</td>
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<tr>
<td>4</td>
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<td>variance</td>
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<tr>
<td>standard deviation</td>
<td></td>
<td></td>
<td>0.791</td>
<td>0.812</td>
</tr>
</tbody>
</table>

Recall $v^1 = \Omega^{-1}1$ and $v^2 = \Omega^{-1}\mu$ so that

- sum of components in $v^1 = 1^T \Omega^{-1}1 = a$
- sum of components in $v^2 = 1^T \Omega^{-1}\mu = b$.

Note that $w_g = v^1/a$ and $w_d = v^2/b$. 
Relation between $w_g$ and $w_d$

Both $w_g$ and $w_d$ are frontier funds with

$$
\mu_g = \frac{\mu^T \Omega^{-1} \mathbf{1}}{a} = \frac{b}{a} \quad \text{and} \quad \mu_d = \frac{\mu^T \Omega^{-1} \mu}{b} = \frac{c}{b}.
$$

Their variances are

$$
\sigma^2_g = w_g^T \Omega w_g = \frac{(\Omega^{-1} \mathbf{1})^T \Omega (\Omega^{-1} \mathbf{1})}{a^2} = \frac{1}{a},
$$

$$
\sigma^2_d = w_d^T \Omega w_d = \frac{(\Omega^{-1} \mu)^T \Omega (\Omega^{-1} \mu)}{b^2} = \frac{c}{b^2}.
$$

Difference in expected returns $= \mu_d - \mu_g = \frac{c}{b} - \frac{b}{a} = \frac{\Delta}{ab}$. Note that $\mu_d > \mu_g$ if and only if $b > 0$.

Also, difference in variances $= \sigma^2_d - \sigma^2_g = \frac{c}{b^2} - \frac{1}{a} = \frac{\Delta}{ab^2} > 0$. 
Covariance of the portfolio returns for any two minimum variance portfolios

The random rates of return of \( u \)-portfolio and \( v \)-portfolio are given by

\[
r^u_P = w^T_u r \quad \text{and} \quad r^v_P = w^T_v r
\]

where \( r = (r_1 \ldots r_N)^T \) is the random rate of return vector. First, for the two special frontier funds, \( w_g \) and \( w_d \), their covariance is given by

\[
\sigma_{gd} = \text{cov}(r^g_P, r^d_P) = \text{cov} \left( \sum_{i=1}^{N} w^g_i r_i, \sum_{j=1}^{N} w^d_j r_j \right)
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} w^g_i w^d_j \text{cov}(r_i, r_j) \quad \text{(bilinear property of covariance)}
\]

\[
= w^T_g \Omega w_d = \left( \frac{\Omega^{-1} 1}{a} \right)^T \Omega \left( \frac{\Omega^{-1} \mu}{b} \right)
\]

\[
= \frac{1^T \Omega^{-1} \mu}{ab} = \frac{1}{a} \quad \text{since} \quad b = 1^T \Omega^{-1} \mu.
\]
In general, consider the two portfolios parametrized by $u$ and $v$:

$$w_u = (1 - u)w_g + uw_d \quad \text{and} \quad w_v = (1 - v)w_g + vw_d$$

so that the covariance of their portfolio returns is given by

$$\text{cov}(r_u, r_v) = (1 - u)(1 - v)\sigma_g^2 + uv\sigma_d^2 + [u(1 - v) + v(1 - u)]\sigma_{gd}$$

\[= \frac{(1 - u)(1 - v)}{a} + \frac{uvc}{b^2} + \frac{u + v - 2uv}{a}\]

\[= \frac{1}{a} + \frac{uv\Delta}{ab^2}.\]

For any portfolio $w_P$, we always have

$$\text{cov}(r_g, r_P) = w_P^T\Omega w_P = \frac{1^T\Omega^{-1}\Omega w_P}{a} = \frac{1}{a} = \text{var}(r_g).$$
Minimum variance portfolio and its uncorrelated counterpart

For any frontier portfolio $u$, we can find another frontier portfolio $v$ such that these two portfolios are uncorrelated. This can be done by setting

$$\frac{1}{a} + \frac{uv\Delta}{ab^2} = 0,$$

and solve for $v$, provided that $u \neq 0$. Portfolio $v$ is the uncorrelated counterpart of portfolio $u$.

The case $u = 0$ corresponds to $w_g$. We cannot solve for $v$ when $u = 0$, indicating that the uncorrelated counterpart of the global minimum variance portfolio does not exist. This observation is consistent with the result that $\text{cov}(r_g, r_P) = \text{var}(r_g) = 1/a \neq 0$. 
1.4 Inclusion of the risk free asset: One-fund Theorem

Consider a portfolio with weight $\alpha$ for the risk free asset and $1 - \alpha$ for a risky asset. The risk free asset has the deterministic rate of return $r_f$. The expected rate of portfolio return is

$$\bar{r}_P = \alpha \bar{r}_f + (1 - \alpha) \bar{r}_j \quad \text{(note that } r_f = \bar{r}_f).$$

The covariance $\sigma_{fj}$ between the risk free asset and any risky asset $j$ is zero since

$$E[(r_j - \bar{r}_j)(r_f - \bar{r}_f)|\{z\}] = 0.$$

Therefore, the variance of portfolio return $\sigma_P^2$ is

$$\sigma_P^2 = \alpha^2 \sigma_j^2 + (1 - \alpha)^2 \sigma_j^2 + 2\alpha(1 - \alpha) \sigma_{fj}$$

so that

$$\sigma_P = |1 - \alpha|\sigma_j.$$
Since both $\bar{r}_P$ and $\sigma_P$ are linear functions of $\alpha$, so $(\sigma_P, \bar{r}_P)$ lies on a pair of line segments in the $\sigma$-$\bar{r}$ diagram.

1. For $0 < \alpha < 1$, the points representing $(\sigma_P, \bar{r}_P)$ for varying values of $\alpha$ lie on the straight line segment joining $(0, r_f)$ and $(\sigma_j, \bar{r}_j)$.
2. If borrowing of the risk free asset is allowed, then $\alpha$ can be negative. In this case, the line extends beyond the right side of $(\sigma_j, \bar{r}_j)$ (possibly up to infinity).

3. When $\alpha > 1$, this corresponds to short selling of the risky asset. In this case, the portfolios are represented by a line with slope negative to that of the line segment joining $(0, r_f)$ and $(\sigma_j, \bar{r}_j)$ (see the lower dotted-dashed line).

- The lower dotted-dashed line can be seen as the mirror image with respect to the vertical $\bar{r}$-axis of the upper solid line segment that would have been extended beyond the left side of $(0, r_f)$. This is due to the swapping in sign in $|1 - \alpha| \sigma_j$ when $\alpha > 1$.

- The holder bears the same risk, like long holding of the risky asset, while $\mu_P$ falls below $r_f$. This is highly undesirable for the investor.
Consider a portfolio that starts with $N$ risky assets originally, what is the impact of the inclusion of a risk free asset on the feasible region?

*Lending and borrowing of the risk free asset is allowed*

For each portfolio formed using the $N$ risky assets, the new combinations with the inclusion of the risk free asset trace out the pair of symmetric half-lines originating from the risk free point and passing through the point representing the original portfolio.

The totality of these lines forms an infinite triangular feasible region bounded by a pair of symmetric half-lines through the risk free point, one line is tangent to the original feasible region while the other line is the mirror image about the horizontal line: $\bar{r} = r_f$. The infinite triangular wedge contains the original feasible region.
We consider the more realistic case where $r_f < \mu_g$ (a risky portfolio should demand an expected rate of return high than $r_f$). For $r_f < \frac{b}{a}$, the upper line of the symmetric double line pair touches the original feasible region.

The new efficient set is the single straight line on the top of the new triangular feasible region. This tangent line touches the original feasible region at a point $F$, where $F$ lies on the efficient frontier of the original feasible set.
No shorting of the risk free asset \((r_f < \mu_g)\)

The line originating from the risk free point cannot be extended beyond the points in the original feasible region (otherwise entails borrowing of the risk free asset). The upper half line is extended up to the tangency point only while the lower half line can be extended to infinity.
One-fund Theorem

Any efficient portfolio (represented by a point on the upper tangent line) can be expressed as a combination of the risk free asset and the portfolio (or fund) represented by $M$.

“There is a single fund $M$ of risky assets such that any efficient portfolio can be constructed as a combination of the fund $M$ and the risk free asset.”

The One-fund Theorem is based on the assumptions that

- every investor is a mean-variance optimizer
- they all agree on the probabilistic structure of asset returns
- a unique risk free asset exists.

Then everyone purchases a single fund, which is then called the market portfolio.
The proportion of wealth invested in the risk free asset is \( 1 - \sum_{i=1}^{N} w_i \).
Write \( r \) as the constant rate of return of the risk free asset.

**Modified Lagrangian formulation**

minimize \( \frac{\sigma_P^2}{2} = \frac{1}{2} w^T \Omega w \)

subject to \( \mu^T w + (1 - 1^T w) r = \mu_P. \)

Define the Lagrangian: \( L = \frac{1}{2} w^T \Omega w + \lambda [\mu_P - r - (\mu - r 1)^T w] \)

\[
\frac{\partial L}{\partial w_i} = \sum_{j=1}^{N} \sigma_{ij} w_j - \lambda (\mu_i - r) = 0, \quad i = 1, 2, \cdots, N \tag{1}
\]

\[
\frac{\partial L}{\partial \lambda} = 0 \quad \text{giving} \quad (\mu - r 1)^T w = \mu_P - r. \tag{2}
\]

\((\mu - r 1)^T w\) is interpreted as the weighted sum of the expected excess rate of return above the risk free rate \( r \).
Remark

In the earlier mean-variance model without the risk free asset, we have

\[
\sum_{j=1}^{N} w_j \overline{r}_j = \mu P.
\]

However, with the inclusion of the risk free asset, the corresponding relation is modified to become

\[
\sum_{j=1}^{N} w_j (\overline{r}_j - r) = \mu P - r.
\]

In the new formulation, we now consider \(\overline{r}_j - r\), which is the excess expected rate of return of asset \(j\) above the riskfree rate of return \(r\). This is more convenient since the contribution of the riskfree asset to this excess expected rate of return is zero so that the weight of the riskfree asset becomes immaterial in the new formulation.
Solution to the constrained optimization model

Comparing to the earlier Markowitz model without the riskfree asset, the new formulation considers the expected rate of return above the riskfree rate of the risky assets. There is no constraint on "sum of weights" equals one. We have

\[(1): \Omega w^* = \lambda(\mu - r1) \quad \text{and} \quad (2): w^T(\mu - r1) = \mu_P - r.\]

Solving (1): \(w^* = \lambda\Omega^{-1}(\mu - r1)\). Substituting into (2)

\[
\mu_P - r = \lambda(\mu - r1)^T\Omega^{-1}(\mu - r1) = \lambda(c - 2br + ar^2).
\]

By eliminating \(\lambda\), the relation between \(\mu_P\) and \(\sigma_P\) is given by the following pair of half lines ending at the risk free asset point \((0, r)\):

\[
\sigma_P^2 = w^{*T}\Omega w^* = \lambda(w^{*T}\mu - rw^{*T}1) = \lambda(\mu_P - r) = (\mu_P - r)^2/(c - 2br + ar^2).
\]
With the inclusion of the risk free asset, the set of minimum variance portfolios are represented by portfolios on the two half lines

\[ L_{up} : \mu_P - r = \sigma_P \sqrt{ar^2 - 2br + c} \]  \hspace{1cm} (3a)  

\[ L_{low} : \mu_P - r = -\sigma_P \sqrt{ar^2 - 2br + c}. \]  \hspace{1cm} (3b)  

Recall that \( ar^2 - 2br + c > 0 \) for all values of \( r \) since \( \Delta = ac - b^2 > 0. \)

The pair of half lines give the frontier boundary of the feasible region of the risky assets plus the risk free asset?

The minimum variance portfolios without the risk free asset lie on the hyperbola

\[ \sigma_P^2 = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}. \]
When \( r < \mu_g = \frac{b}{a} \), one can show geometrically that the upper half line is a tangent to the hyperbola. The tangency portfolio is the tangent point to the efficient frontier (upper part of the hyperbolic curve) through the point \((0, r)\).

\[
\mu_p = r + \sigma_p \sqrt{c - 2rb + r^2a}
\]
Though we normally expect \( r < \mu_g = \frac{b}{a} \), what happens when \( r > \frac{b}{a} \)? The lower half line touches the feasible region with risky assets only.

- Any portfolio on the upper half line involves short selling of the tangency portfolio and investing the proceeds in the risk free asset. It makes good sense to short sell the tangency portfolio since it has an expected rate of return that is lower than the risk free asset.
Solution of the tangency portfolio when $r < \mu_g$

The tangency portfolio $M$ is represented by the point $(\sigma_{P,M}, \mu_P^M)$, and the solution to $\sigma_{P,M}$ and $\mu_P^M$ are obtained by solving simultaneously

\[
\sigma_P^2 = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}, \quad \mu_P = r + \sigma_P\sqrt{ar^2 - 2br + c}.
\]

From the first order conditions that are obtained by differentiating the Lagrangian by the control variables $w$, we obtain

\[
w^* = \lambda \Omega^{-1}(\mu - r1), \quad (a)
\]

where $\lambda$ is then determined by the constraint condition:

\[
\mu_P - r = (\mu - r1)^T w. \quad (b)
\]
Recall that the tangency portfolio lies in the feasible region that corresponds to the absence of the riskfree asset, so \( \mathbf{1}^T \mathbf{w}_M = 1 \). Note that \( \mathbf{w}_M \) should satisfy eq. (a) while pays no reference to eq. (b) since \( \mu_p^M \) is not yet known (to be determined as part of the solution). We then deduce that

\[
1 = \lambda_M [ \mathbf{1}^T \Omega^{-1} \mu - r \mathbf{1}^T \Omega^{-1} \mathbf{1} ]
\]

so that \( \lambda_M = \frac{1}{b-ar} \) (note that \( r < \frac{b}{a} \) has been assumed). The corresponding \( \mu_p^M \) and \( \sigma_{P,M}^2 \) can be determined as follows:

\[
\mu_p^M = \mu^T \mathbf{w}^*_M = \frac{1}{b-ar} (\mu^T \Omega^{-1} \mu - r \mu^T \Omega^{-1} \mathbf{1}) = \frac{c - br}{b-ar},
\]

\[
\sigma_{P,M}^2 = \mathbf{w}^*_M \Omega \mathbf{w}^*_M = \frac{1}{(b-ar)^2} (\mu - r \mathbf{1})^T \Omega^{-1} (\mu - r \mathbf{1})
\]

\[
= \frac{ar^2 - 2br + c}{(b-ar)^2}.
\]
Recall $\mu_g = \frac{b}{a}$. When $r < \frac{b}{a}$, we can establish $\mu_P^M > \mu_g$ as follows:

\[
\left( \mu_P^M - \frac{b}{a} \right) \left( \frac{b}{a} - r \right) = \left( \frac{c - br}{b - ar} - \frac{b}{a} \right) \frac{b - ar}{a} \\
= \frac{c - br}{a} - \frac{b^2}{a^2} + \frac{br}{a} \\
= \frac{ac - b^2}{a^2} = \frac{\Delta}{a^2} > 0,
\]

so we deduce that $\mu_P^M > \frac{b}{a} > r$.

Similarly, when $r > \frac{b}{a}$, we have $\mu_P^M < \frac{b}{a} < r$.

Also, we can deduce that $\sigma_P, M > \sigma_g$ as expected. This is because both Portfolio $M$ and Portfolio $g$ are portfolios generated by the universe of risky assets (with no inclusion of the riskfree asset), and $g$ is the global minimum variance portfolio.
Example (5 risky assets and one riskfree asset)

Data of the 5 risky assets are given in the earlier example, and \( r = 10\% \).

The system of linear equations to be solved is

\[
\sum_{j=1}^{5} \sigma_{ij} v_j = \bar{r}_i - r = 1 \times \bar{r}_i - r \times 1, \quad i = 1, 2, \ldots, 5.
\]

Recall that \( v^1 \) and \( v^2 \) in the earlier example are solutions to

\[
\sum_{j=1}^{5} \sigma_{ij} v^1_j = 1 \quad \text{and} \quad \sum_{j=1}^{5} \sigma_{ij} v^2_j = \bar{r}_i, \quad i = 1, 2, \ldots, 5.
\]

Hence, \( v_j = v^2_j - r v^1_j, j = 1, 2, \ldots, 5 \) (numerically, we take \( r = 10\% \)).

In matrix representation, we have

\[
v_1 = \Omega^{-1} \mathbf{1} \quad \text{and} \quad v_2 = \Omega^{-1} \mu.
\]
Now, we have obtained $v$ where

$$v = \Omega^{-1}(\mu - r\mathbf{1}) = v_1 - rv_2$$

Note that the optimal weight vector for the 5 risky assets satisfies

$$w = \lambda v$$

for some scalar $\lambda$.

We determine $\lambda$ by enforcing $(\mu - r\mathbf{1})^T w = \mu_P - r$, or equivalently,

$$\lambda(\mu - r\mathbf{1})^T v = \lambda(c - 2br + ar^2) = \mu_P - r,$$

where $\mu_P$ is the target rate of return of the portfolio.

Recall $a = \mathbf{1}^T \Omega^{-1} \mathbf{1} = \sum_{j=1}^{5} v_j^1$, $b = \mathbf{1}^T \Omega^{-1} \mu = \sum_{j=1}^{5} v_j^2$, and

$$c = \mu^T \Omega^{-1} \mu = \sum_{j=1}^{5} \bar{r}_j v_j^2.$$

The weight of the risk free asset is then given by $1 - \sum_{j=1}^{5} w_j$. 
Properties of the minimum variance portfolios for $r < b/a$

1. **Efficient portfolios**

Any portfolio on the upper half line

$$\mu_P = r + \sigma_P \sqrt{ar^2 - 2br + c}$$

within the segment $FM$ joining the two points $F(0, r)$ and $M$ involves long holding of the market portfolio $M$ and the risk free asset $F$, while those outside $FM$ involves short selling of the risk free asset and long holding of the market portfolio.

2. Any portfolio on the lower half line

$$\mu_P = r - \sigma_P \sqrt{ar^2 - 2br + c}$$

involves short selling of the market portfolio and investing the proceeds in the risk free asset. This represents a non-optimal investment strategy since the investor faces risk but gains no extra expected return above $r$. 
Furthermore, note that $\mu^M_P - r = \frac{ar^2 - 2br + c}{b-ar}$ and $\sigma_{P,M} = \sqrt{\frac{ar^2 - 2br + c}{|b-ar|}}$. One can show that

(i) when $r < \frac{b}{a}$, we obtain

$$\mu^M_P - r = \sigma_{P,M} \sqrt{ar^2 - 2br - c}$$

(equation of $L_{up}$);

(ii) when $r > \frac{b}{a}$, we obtain

$$\mu^M_P - r = -\sigma_{P,M} \sqrt{ar^2 - 2br + c}$$

(equation of $L_{low}$).

Interestingly, the flip of sign in $b - ar$ with respect to $r < \mu_g$ or $r > \mu_g$ would indicate whether $(\sigma_{P,M}, \mu^M_P)$ lies in the upper or lower half line, respectively.
Degenerate case occurs when $\mu_g = \frac{b}{a} = r$

- What happens when $r = b/a$? The pair of half lines become

$$\mu_P = r \pm \sigma_P \sqrt{c - 2 \left( \frac{b}{a} \right) b + \frac{b^2}{a}} = r \pm \sigma_P \sqrt{\Delta},$$

which correspond to the asymptotes of the hyperbolic left boundary of the feasible region with risky assets only. The tangency portfolio does not exist, consistent with the mathematical result that $\lambda_M = \frac{1}{b - ar}$ is not defined when $r = \frac{b}{a}$.

- Under the scenario: $r = \frac{b}{a}$, efficient funds still lie on the upper half line, though the tangency portfolio does not exist. Recall that

$$w^* = \lambda \Omega^{-1}(\mu - r \mathbf{1})$$

so that

$$\mathbf{1}^T w^* = \lambda (\mathbf{1}^T \Omega^{-1} \mu - r \mathbf{1}^T \Omega^{-1} \mathbf{1}) = \lambda(b - ra).$$
When $r = b/a$, sum of weights of risky assets $= \mathbf{1}^T \mathbf{w}^* = 0$ as $\lambda$ is finite. Any minimum variance portfolio involves investing everything in the riskfree asset and holding a portfolio of risky assets whose weights are summed to zero.

- The optimal weight vector $\mathbf{w}^*$ equals $\lambda \Omega^{-1}(\mu - r \mathbf{1})$. Suppose we specify $\mu_P$ to be the target expected rate of return of the efficient portfolio, then the multiplier $\lambda$ is determined by (see p.75)

\[
\lambda = \frac{\mu_P - r}{c - 2br + ar^2} \bigg|_{r=b/a} = \frac{\mu_P - r}{c - 2 \left(\frac{b}{a}\right) b + \frac{b^2}{a}} = \frac{a(\mu_P - r)}{\Delta}.
\]
Financial interpretation

Given the target expected rate of portfolio return $\mu_P$, the corresponding optimal portfolio is to hold 100% on the riskfree asset and $w_j$ on the $j^{th}$ risky asset, $j = 1, 2, \ldots, N$, where $w_j$ is given by the $j^{th}$ component of $\frac{a(\mu_P - r)}{\Delta} \Omega^{-1}(\mu - r \mathbf{1})$.

One should check whether the expected rate of return of the whole portfolio equals $\mu_P$.

The expected rate of return from all the risky assets is

$$\frac{a(\mu_P - r)}{\Delta} \mu^T [\Omega^{-1}(\mu - r \mathbf{1})] = \frac{a(\mu_P - r)}{ac - b^2} \left( \frac{c - b^2}{a} \right) = \mu_P - r.$$  

The overall expected rate of return of the portfolio is

$$w_0 r + \sum_{j=1}^{N} w_j \bar{r}_j = r + (\mu_P - r) = \mu_P,$$  

where $w_0 = 1$. 

**One-fund Theorem under** \( r = \mu_g = \frac{b}{a} \)

In this degenerate case, \( r = \frac{b}{a} \), the tangency fund does not exist. At its global minimum variance portfolio \( g \), the universe of risky assets just provide an expected rate of return that is the same as the riskfree return \( r \).

The optimal portfolio is to invest 100% on the riskfree asset and a scalar multiple of the fund \( z \) whose weight vector is

\[
w_z = \Omega^{-1}(\mu - r1).
\]

The scalar \( \lambda \) is determined by the investor’s target expected rate of return \( \mu_P \). The sum of weights in this portfolio \( z \) is zero.

Since the global minimum variance portfolio of risky assets \( g \) has the same expected rate of return as that of the riskfree asset, a sensible investor would place 100% weight on the riskfree asset to generate the level of expected rate of return equals \( r \). This is consistent with the observation that \( \lambda = 0 \) when \( \mu_P = r \).
Nature of the portfolio $z$: $w_z = \Omega^{-1}(\mu - r1)$, where $r = b/a$

1. Recall $w_d = \Omega^{-1}\mu/b$ and $w_g = \Omega^{-1}1/a$, so

$$w_z = bw_d - \frac{b}{a}(aw_g) = b(w_d - w_g).$$

Its sum of weights is seen to be zero since it longs $b$ units of $w_d$ and short the same number of units of $w_g$.

2. Location of the portfolio $z$ in the mean-variance plot

$$\sigma_z^2 = w_z^T \Omega w_z = b^2 (w_d - w_g)^T \Omega (w_d - w_g) = b^2 (\sigma_{d}^2 - 2\sigma_{gd} + \sigma_{g}^2) = b^2 \frac{\Delta}{ab^2} = \frac{\Delta}{a};$$

$$\mu_z = \mu^T w_z = b(\mu_d - \mu_g) = b \left(\frac{c}{b} - \frac{b}{a}\right) = \frac{\Delta}{a}.$$  

We have $\sigma_z = \sqrt{\frac{\Delta}{a}}$ and $\mu_z = \frac{\Delta}{a}$; so any scalar multiple of $z$ lies on the line: $\mu = \sqrt{\frac{\Delta}{a}} \sigma$. 

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3. **Location of efficient portfolios in the mean-variance diagram**

Suppose the investor specifies her target rate of return to be $\mu_P$, then the scalar $\lambda$ is determined by setting

$$\mu_P = r + \lambda \mu_z = r + \lambda \frac{\Delta}{a}$$
giving

$$\lambda = \frac{a (\mu_P - r)}{\Delta}.$$  

Also, the standard deviation of the optimal portfolio’s return is $\sigma_P = \lambda \sigma_z = \lambda \sqrt{\frac{\Delta}{a}}$. The efficient portfolio lies on the line:

$$\mu_P - r = \sqrt{\frac{\Delta}{a} \sigma_P}.$$  

The target expected rate of portfolio return above $r$ is produced by appropriate long and short positions on the risky assets with sum of weights equals zero.
The upper line represents the set of frontier funds. The point \( \left( \sqrt{\frac{\Delta}{a}}, \frac{\Delta}{a} \right) \) on the lower line represents the \( z \)-fund, \( w_z \).
**Market Portfolio** is a portfolio consisting of a weighted sum of every asset in the market, with weights in the proportions that they exist in the market (under the assumption that these assets are infinitely divisible). This is also called a *value-weighted portfolio*.

- In the market portfolio, HSBC has a larger weight than Bank of East Asia due to her larger market capitalization.

- The Hang Seng index may be considered as a proxy of the market portfolio of the Hong Kong stock market.
Tangency portfolio under One-fund Theorem and market portfolio

- The One-fund Theorem states that everyone purchases a single fund (tangency portfolio) of risky assets and borrow or lend at the riskfree rate.

- If everyone purchases the same tangency portfolio (same weights of all risky assets in the market), what must that fund be? This fund is simply the market portfolio. In other words, if everyone buys just one fund, and their purchases add up to the market, then the proportional weights in the tangency fund must be the same as those of the market portfolio.

- In the situation where everyone follows the mean-variance methodology with the same estimates of parameters, the efficient fund of risky assets will be the market portfolio.
How can this happen? The answer is based on the equilibrium argument.

- If everyone else (or at least a large number of people) solves the problem, we do not need to. The return on an asset depends on both its initial price and its final price. The other investors solve the mean-variance portfolio problem using their common estimates, and they place orders in the market to acquire their portfolios.

- If orders placed do not match with what is available, the prices must change. The prices of the assets under heavy demand will increase while the prices of the assets under light demand will decrease. These price changes affect the estimates of asset returns directly, and hence investors will recalculate their optimal portfolio.
This process continues until demand exactly matches supply, that is, it continues until an equilibrium prevails.

Summary

• In the idealized world, where every investor is a mean-variance investor and all have the same estimates, everyone buys the same portfolio and that must be equal to the market portfolio.

• Prices adjust to drive the market to efficiency. Then after other people have made the adjustments, we can be sure that the single efficient portfolio is the market portfolio.
1.5 Addition of a risk tolerance factor

The universe of assets include $N$ risky assets but no riskfree asset.

Maximize $\tau \mu_P - \frac{\sigma_P^2}{2}$, with $\tau \geq 0$, where $\tau$ is the risk tolerance.

Optimization problem: $\max_{\mathbf{w} \in \mathbb{R}^N} \tau \mu_P - \frac{\sigma_P^2}{2}$ subject to $\mathbf{1}^T \mathbf{w} = 1$.

- Instead of only minimizing risk as in the mean variance models, the new objective function represents the tradeoff between return and risk with weighted factor $2\tau$. When $\tau$ is high, the investor is more interested in expected return and has a high tolerance on risk.

- Note that $\mu_P$ in the objective function is not preset. The tolerance factor $\tau$ is chosen by the investor and will be fixed in the formulation. The choice variables are the portfolio weights $w_i, i = 1, 2, \ldots, N$. 
**Quadratic optimization problem**

\[
\max_{\mathbf{w} \in \mathbb{R}^N} \left[ \tau \mathbf{\mu}^T \mathbf{w} - \frac{\mathbf{w}^T \Omega \mathbf{w}}{2} \right] \quad \text{subject to } \mathbf{w}^T \mathbf{1} = 1.
\]

The Lagrangian formulation becomes

\[
L(\mathbf{w}; \lambda) = \tau \mathbf{\mu}^T \mathbf{w} - \frac{\mathbf{w}^T \Omega \mathbf{w}}{2} + \lambda (\mathbf{w}^T \mathbf{1} - 1).
\]

The first order conditions are

\[
\begin{cases} 
\tau \mathbf{\mu} - \Omega \mathbf{w}^* + \lambda \mathbf{1} = 0 \\
\mathbf{w}^* \mathbf{1} = 1
\end{cases}
\]

When \( \tau \) is taken to be zero, the problem reduces to the minimization of portfolio return variance without regard to expected portfolio return. This gives the global minimum variance portfolio \( \mathbf{w}_g^* \).
One can show that the optimal solution $w^*$ can be expressed as $w_g + \tau z^*, \tau \geq 0$.

1. When $\tau = 0$, the two first order conditions become

$$\Omega w = \lambda_0 \mathbf{1} \quad \text{and} \quad \mathbf{1}^T w_g = 1.$$ 

Solving

$$w_g = \lambda_0 \Omega^{-1} \mathbf{1} \quad \text{and} \quad 1 = \mathbf{1}^T w_g = \lambda_0 \mathbf{1}^T \Omega^{-1} \mathbf{1}$$

hence

$$w_g = \frac{\Omega^{-1} \mathbf{1}}{\mathbf{1}^T \Omega^{-1} \mathbf{1}} = \frac{\Omega^{-1} \mathbf{1}}{a} \quad \text{(independent of $\mu$)}.$$ 

The formulation does not depend on $\mu$ when $\tau$ is taken to be zero.
2. When $\tau > 0$, we obtain $w^* = \tau \Omega^{-1} \mu + \lambda \Omega^{-1} \mathbf{1}$. To determine $\lambda$, we apply

$$1 = \mathbf{1}^T w = \tau \mathbf{1}^T \Omega^{-1} \mu + \lambda \mathbf{1}^T \Omega^{-1} \mathbf{1}$$

so that $\lambda = \frac{1 - \tau b}{a}$.

$$w^* = \tau \Omega^{-1} \mu + \frac{1 - \tau b}{\Omega^{-1} \mathbf{1}}$$

$$= \tau \left( \Omega^{-1} \mu - \frac{b}{a} \Omega^{-1} \mathbf{1} \right) + w_g.$$

We obtain $w^* = w_g + \tau z^*$, where

$$z^* = \Omega^{-1} \mu - \frac{b}{a} \Omega^{-1} \mathbf{1} = b(w_d - w_g)$$

and $\mathbf{1}^T z^* = 0$.

Write $r w_g$ as the random rate of return of portfolio $g$, and a similar notation for $z^*$. Observe that $\text{cov}(r w_g, r z^*) = z^* \Omega w_g = 0$,

$$\mu z^* = \mu^T z^* = c - \frac{b^2}{a} = \frac{\Delta}{a} > 0 \quad \text{and} \quad \sigma_{z^*}^2 = \frac{\Delta}{a} > 0.$$
Financial interpretation

The zero tolerance solution $w_g$ leads to the global minimum risk position. This position is modified by investing in the portfolio $z^*$ [note that $1^T z^* = 0$] so as to maximize $\tau \mu^T w - \frac{w^T \Omega w}{2}$.

Set of optimal portfolios

For a given value of $\tau$, we have solved for $w^*$ (with dependence on $\tau$). We then compute $\mu_P$ and $\sigma^2_P$ corresponding to the optimal weight $w^*$.

$$
\mu_P = \mu^T (w_g + \tau z^*) = \mu_g + \tau \mu_z^* \\
\sigma^2_P = \sigma^2_g + 2\tau \text{COV}(r w_g, r z^*) + \tau^2 \sigma^2_{z^*}.
$$

By eliminating $\tau$, we obtain

$$
\sigma^2_P = \sigma^2_g + \left(\frac{\mu_P - \mu_g}{\mu_z^*}\right)^2 \sigma^2_{z^*} = \frac{\Delta}{a}, \quad \mu_z^* = \sigma^2_{z^*} = \frac{\Delta}{a}.
$$

This is an equation of a hyperbola in the $\sigma_P-\mu_P$ diagram.
\[ \mu_P = \frac{b}{a} + \frac{\Delta}{a} \tau \]
\[ \sigma_P^2 = \frac{1}{a} + \frac{\Delta}{a} \tau^2 \]

The points representing these optimal portfolios in the \( \sigma_P-\mu_P \) diagram lie on the upper half of the hyperbola. We expect that for a higher value of \( \tau \) chosen by the investor, the optimal portfolio has higher \( \mu_P \) and \( \sigma_P \).
How to reconcile the mean-variance model and risk-tolerance model?

Recall that the left boundary of the feasible region of the risky assets is given by

$$\sigma_P^2 = \frac{a \mu_P^2 - 2b \mu_P + c}{\Delta}, \quad \Delta = ac - b^2. \quad (1)$$

The parabolic curve that traces all optimal portfolios of the risk-tolerance model in the $\sigma_P^2-\mu_P$ diagram is

$$\mu_P = \frac{b}{a} + \frac{\Delta}{a} \tau \quad \text{and} \quad \sigma_P^2 = \frac{1}{a} + \frac{\Delta}{a} \tau^2. \quad (2)$$

It can be shown that the solutions to $\mu_P$ and $\sigma_P^2$ in Eq. (2) satisfy the parabolic equation (1) since

$$a \left( \frac{b}{a} + \frac{\Delta}{a} \tau \right)^2 - 2b \left( \frac{b}{a} + \frac{\Delta}{a} \tau \right) + c = \frac{1}{a} + \frac{\Delta}{a} \tau^2.$$
With a given tolerance \( \tau \), the objective function line

\[
\tau \mu_p - \frac{\sigma_p^2}{2} = \text{constant}
\]

is pushed upwards until it becomes the tangent line to the parabolic curve.
• The objective function line: \( \tau \mu_P - \frac{\sigma_P^2}{2} = \text{constant} \) in the \( \sigma_P^2 - \mu_P \) diagram is pushed upwards as much as possible in the maximization procedure.

• However, the optimal portfolio must lie in the feasible region of risky assets. Recall that the feasible region is bounded on the left by the parabolic curve:
  \[ \sigma_P^2 = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta} \]
  The objective function \( \tau \mu_P - \frac{\sigma_P^2}{2} \) is maximized when the objective function line touches the left boundary of the feasible region.

• In the degenerate case where \( \tau = 0 \), the slope of the line:
  \( \tau \mu_P - \frac{\sigma_P^2}{2} = \text{constant} \) becomes infinite. When we maximize \( -\frac{\sigma_P^2}{2} = \text{constant} \) by pushing the vertical line to the far left, the corresponding optimal portfolio obtained is Portfolio \( g \).
Another version of the Two-fund Theorem

Given $\mu_P$, the efficient fund under the mean-variance model is given by

$$w^* = w_g + \frac{a}{\Delta} \left( \mu_P - \frac{b}{a} \right) z^*, \quad \mu_P > \mu_g = \frac{b}{a},$$

where $w_g = \frac{\Omega^{-1} 1}{a}$, $z^* = \Omega^{-1} \mu - \frac{b}{a} \Omega^{-1} 1$.

This implies that any efficient fund can be generated by the two funds: global minimum variance fund $w_g$ and the fund $z^*$. 
Proof

1. Note that $w^*$ is of the form $\lambda_1 \Omega^{-1}1 + \lambda_2 \Omega^{-1} \mu$.

2. Consider the expected portfolio return:

$$\mu^T w^* = \mu_g + \frac{a}{\Delta} \left( \mu_P - \frac{b}{a} \right) \mu z^* = \frac{b}{a} + \frac{a}{\Delta} \left( \mu_P - \frac{b}{a} \right) \Delta \frac{a}{a} = \mu_P.$$

3. Consider the sum of weights:

$$1^T w^* = 1^T w_g + \frac{a}{\Delta} \left( \mu_P - \frac{b}{a} \right) 1^T z^* = 1.$$
Comparing the first order conditions of the mean-variance model and risk-tolerance model

1. \( \Omega w^* = \lambda_1 \mathbf{1} + \lambda_2 \mu \)
   \( \mathbf{1}^T w^* = 1 \)
   \( \mathbf{1}^T \mu = \mu_P \)

2. \( \Omega w^* = \lambda \mathbf{1} + \tau \mu \)
   \( \mathbf{1}^T w^* = 1 \)

We observe that

\[
\lambda_1 = \lambda = \frac{c - b \mu_P}{\Delta} \\
\lambda_2 = \frac{a \mu_P - b}{\Delta} \text{ is simply } \tau.
\]

The specification of the risk tolerance factor \( \tau \) is somewhat equivalent to the specification of \( \mu_P \).
Summary

1. The objective function $\tau \mu^T w - \frac{w^T \Omega w}{2}$ represents a balance of maximizing return $\tau \mu^T w$ against risk $\frac{w^T \Omega w}{2}$.

2. The optimal solution takes the form

$$w^* = w_g + \tau z^*$$

where $w_g$ is the portfolio weight of the global minimum variance portfolio and the weights in $z^*$ are summed to zero.
Note that
\[ z^* = \Omega^{-1} \mu - \frac{b}{a} \Omega^{-1} 1 = b(w_d - w_g). \]
and \( \tau \) and \( \mu_P \) are related by
\[ \tau = \frac{a}{\Delta} \left( \mu_P - \frac{b}{a} \right). \]
Putting the results together, we have
\[ w^* = w_g + \frac{ab}{\Delta} \left( \mu_P - \frac{b}{a} \right) (w_d - w_g) \]
\[ = \frac{ab}{\Delta} \left( \frac{c}{b} - \mu_P \right) w_g + \frac{ab}{\Delta} \left( \mu_P - \frac{b}{a} \right) w_d. \]
3. The additional variance above $\sigma_g^2$ is given by

$$\tau^2 \sigma_{z*}^2 = \tau^2 \frac{\Delta}{a}, \quad \Delta = ac - b^2.$$  

Also, $\text{cov}(r_{wg}, r_{z*}) = 0$, that is, $r_{wg}$ and $r_{z*}$ are uncorrelated.

4. The efficient frontier of the mean-variance model coincides with the set of optimal portfolios of the risk-tolerance model. The risk tolerance $\tau$ and expected portfolio return $\mu_P$ are related by

$$\frac{\mu_P - \mu_g}{\mu_{z*}} = \frac{\mu_P - \mu_g}{\sigma_{z*}^2} = \tau.$$  

5. Another version of the Two-fund Theorem can be established where any efficient fund can be generated by the two funds: $w_g$ and $z^*$. 

**Asset-liability model**

Liabilities of a pension fund = future benefits – future contributions

Market value can hardly be determined since liabilities are not readily marketable, unlike tradeable assets. Assume that some specific accounting rules are used to calculate an initial value $L_0$. If the same rule is applied one period later, a value for $L_1$ results. Note that $L_1$ is random.

Rate of growth of the liabilities = $r_L = \frac{L_1 - L_0}{L_0}$, where $r_L$ is expected to depend on the changes of interest rate structure, mortality and other stochastic factors.

Let $A_0$ be the initial value of assets. The investment strategy of the pension fund is characterized by the portfolio choice $w$. Let $rw$ denote the rate of growth of the asset portfolio.
Surplus optimization

Depending on the portfolio choice $w$, the surplus gain after one period

$$S_1 - S_0 = [A_0(1 + rw) - L_0(1 + r_L)] - (A_0 - L_0) = A_0rw - L_0r_L.$$ 

The rate of return on the surplus relative to asset value is defined by

$$r_S = \frac{S_1 - S_0}{A_0} = rw - \frac{1}{f_0}r_L$$

where $f_0 = A_0/L_0$ is the initial funding ratio.

Here, $r_S$ is the difference of the two random variables: $rw$ and $\frac{1}{f_0}r_L$, one of them is dependent on the choice variable $w$ while the other is not.
Maximization formulation

\[
\max_{\mathbf{w} \in \mathbb{R}^N} \left\{ \tau E \left[ r\mathbf{w} - \frac{1}{f_0} r_L \right] - \frac{1}{2} \text{var} \left( r\mathbf{w} - \frac{1}{f_0} r_L \right) \right\}
\]

subject to \( \sum_{i=1}^N w_i = 1 \). Since \( E \left[ \frac{1}{f_0} r_L \right] \) and \( \text{var}(r_L) \) are independent of \( \mathbf{w} \) so that they can be omitted from the objective function.

We rewrite the quadratic maximization formulation as

\[
\max_{\mathbf{w} \in \mathbb{R}^N} \left\{ \tau E[r\mathbf{w}] - \frac{\text{var}(r\mathbf{w})}{2} + \frac{1}{f_0} \text{cov}(r\mathbf{w}, r_L) \right\}
\]

subject to \( \sum_{i=1}^N w_i = 1 \). Recall that

\[
\text{cov}(r\mathbf{w}, r_L) = \text{cov} \left( \sum_{i=1}^N w_i r_i, r_L \right) = \sum_{i=1}^N w_i \text{cov}(r_i, r_L).
\]
Modified maximization formulation

\[
\max_{w \in \mathbb{R}^N} \left\{ \tau \mu^T w + \gamma^T w - \frac{w^T \Omega w}{2} \right\} \text{ subject to } 1^T w = 1,
\]

where \( \gamma^T = (\gamma_1 \cdots \gamma_N) \) with \( \gamma_i = \frac{1}{f_0} \text{cov}(r_i, r_L) \), \( \mu^T = (\mu_1 \cdots \mu_N) \) with \( \mu_i = E[r_i], \sigma_{ij} = \text{cov}(r_i, r_j) \).

1. The additional term \( \gamma^T w \) in the objective function arises from the correlation \( \text{cov}(r_i, r_L) \) multiplied by the factor \( L_0/A_0 \).

2. Compared to the earlier risk tolerance model, we just need to replace \( \mu \) by \( \mu + \frac{1}{\tau} \gamma \). The efficient portfolios are of the form

\[
w^* = w_g + z^L + \tau z^*, \quad \tau \geq 0,
\]

where \( z^L = \Omega^{-1} \gamma - \frac{1^T \Omega^{-1} \gamma}{1^T \Omega^{-1} 1} \Omega^{-1} 1 \) with \( \sum_{i=1}^N z^L_i = 0 \).
Appendix: Mathematical properties of covariance matrix $\Omega$

1. All eigenvalues of $\Omega$ are non-negative

If otherwise, suppose $\lambda$ is a negative eigenvalue of $\Omega$ and $x$ is the corresponding eigenvector. We have

$$\Omega x = \lambda x, \quad x \neq 0,$$

so that

$$x^T \Omega x = \lambda x^T x < 0,$$

a contradiction to the semi-positive definite property of $\Omega$.

2. $\Omega$ is non-singular ($\Omega^{-1}$ exists) if and only if all eigenvalues are positive

Recall

$$\det \Omega = \text{product of eigenvalues}$$

and

$$\Omega \text{ is non-singular } \iff \det \Omega \neq 0.$$
3. Decomposition of $\Omega$ and representation of $\Omega^{-1}$ when $\Omega$ is non-singular

Let $\lambda_i, i = 1, 2, \ldots, n,$ be the eigenvalues of $\Omega$ (allowing multiplicities) and $x_i$ be the corresponding eigenvector of eigenvalue $\lambda_i$. Since $\Omega$ is symmetric, it has a full set of eigenvectors. We then have

$$\Omega S = S \Lambda,$$

where $\Lambda$ is the diagonal matrix whose entries are the eigenvalues of $\Omega$ and $S$ is the matrix whose columns are the eigenvectors of $\Omega$ (arranged in the corresponding sequential order). The eigenvectors are orthogonal to each other since $\Omega$ is symmetric and we can always normalize the eigenvectors to be unit length.

That is, $S$ can be constructed to be an orthonormal matrix so that $S^{-1} = S^T$. We then have

$$\Omega = S \Lambda S^{-1} = S \Lambda S^T.$$
Provided that all eigenvalues of $\Omega$ are non-zero so that $\Lambda^{-1}$ exists, we then have

$$\Omega^{-1} = (S\Lambda S^T)^{-1} = (S^T)^{-1}\Lambda^{-1}S^{-1} = S\Lambda^{-1}S^T.$$  

4. $\Delta = ac - b^2 > 0$, where $\mu \neq h1$

Note that

$$a = 1^T \Omega^{-1} 1 = (1^T S\Lambda^{-1/2})(\Lambda^{-1/2}S^T 1)$$
$$b = \mu^T \Omega^{-1} 1 = (\mu^T S\Lambda^{-1/2})(\Lambda^{-1/2}S^T 1)$$
$$c = \mu^T \Omega^{-1} \mu = (\mu^T S\Lambda^{-1/2})(\Lambda^{-1/2}S^T \mu).$$

We write $x = \Lambda^{-1/2}S^T 1$ and $y = \Lambda^{-1/2}S^T \mu$ so that $a = x^T x$, $b = y^T x$ and $c = y^T y$.

The Cauchy-Schwarz inequality gives

$$|y^T x|^2 \leq (x^T x)(y^T y).$$

For $\mu \neq h1$, $x$ and $y$ are then linearly independent, we have

$$\Delta = ac - b^2 > 0.$$
Appendix: Degenerate case of singular covariance matrix

All the formulas derived earlier are based on the assumption that the covariance matrix $\Omega$ is non-singular. In this Appendix, we investigate how to determine the feasible region and the corresponding efficient funds when $\Omega^{-1}$ does not exist. Recall

$\Omega$ is singular $\Leftrightarrow$ the set of eigenvalues of $\Omega$ contains “zero”.

That is, there exists non-zero vector $w_0$ such that

$$\Omega w_0 = 0.$$ 

We seek $w_0$ such that $w_0^T \mathbf{1} = 1$. Write $r w_0$ as the random rate of return of the portfolio with the weight vector $w_0$. Note that

$$\text{var}(r w_0) = w_0^T \Omega w_0 = 0.$$
Since \( \text{var}(r_w) = 0 \), the corresponding portfolio can be considered as a “proxy” for the riskfree asset (zero variance), whose riskfree rate of return \( r_0 \) is given by

\[
r_0 = w_0^T \mu.
\]

**Example**

Consider the two-asset portfolio with \( \rho = -1 \), the corresponding covariance matrix is

\[
\Omega = \begin{pmatrix}
\sigma_1^2 & -\sigma_1 \sigma_2 \\
-\sigma_1 \sigma_2 & \sigma_2^2
\end{pmatrix}.
\]

Obviously, \( \Omega \) is singular since the columns are dependent. Accordingly, we obtain

\[
w_0 = \begin{pmatrix}
\frac{\sigma_2}{\sigma_1 + \sigma_2} \\
\frac{\sigma_1}{\sigma_1 + \sigma_2} \\
\frac{\sigma_2}{\sigma_1 + \sigma_2}
\end{pmatrix},
\]

where \( \Omega w_0 = 0 \) and \( w_0^T \mathbf{1} = 1 \). See p. 28 for the pictorial representation of the zero-variance portfolio in the mean-variance diagram.
Let $E_0$ denote the eigenspace corresponding to the eigenvalue 0. It is seen that $E_0$ is simply the null space of $Ω$.

First, we assume that there exists unique (up to a scalar multiple) eigenvector $w_0$ corresponding to the eigenvalue 0. We have

$$\text{rank}(Ω) = \text{dim}(\text{column space of } Ω) = N - 1,$$

where $Ω$ is a $N \times N$ matrix. The global minimum variance portfolio is the “proxy” riskfree asset generated by $w_0$, whose riskfree rate of return is $r_0 = w_0^T \mu$.

The feasible region and efficient frontier would be similar to the portfolio model with the inclusion of the riskfree asset. Since any combination of any portfolio and $w_0$ lies on the line joining the portfolio and this “proxy” riskfree asset, the feasible region is an infinite triangular wedge with its tip at $(0, r_0)$ in the $\sigma_P - r_P$ diagram.
We would like to find the equation of the upper half line that bounds the triangular wedge region. Similar to the portfolio model with the inclusion of the riskfree asset, we impose the constraint:

$$\mu_P - r_0 = \sum_{j=1}^{N} w_j (\bar{r}_j - r_0).$$

Note that $\sum_{j=1}^{N} w_j \neq 1$ in general. The weight vector $w$ of an efficient fund is obtained by solving

$$\Omega w = \lambda(\mu - r_0 \mathbf{1}) = \lambda(\mu - \mathbf{1} w_0^T \mu) = \lambda(I - \mathbf{1} w_0^T) \mu, \quad (1)$$

where the matrix $\mathbf{1} w_0^T$ is obtained by taking the outer product of $\mathbf{1}$ and $w_0$. Similarly, the parameter $\lambda$ is determined by specifying the investor’s target expected rate of return of the portfolio.
Several mathematical issues:

1. Since $\Omega^{-1}$ does not exist, does solution to eq. (1) always exist?

2. Suppose solution to eq. (1) exists, it is not unique since adding any scalar multiple of $w_0$ is also a solution (since $\Omega w_0 = 0$).

3. What is the interpretation of $\lambda$ in eq. (1) and how to determine its value?

Though $\Omega$ is singular, recall that solution $w$ exists in eq. (1) provided that the column vector $(I - 1w_0^T)\mu$ lies in the column space of $\Omega$. 
Mathematical proof that solution to eq. (1) always exists

Since \( \Omega w_0 = 0 \), the eigenvector \( w_0 \) is orthogonal to all the rows in \( \Omega \). Since \( \Omega \) is symmetric, so the column space of \( \Omega \) is the orthogonal complement of the eigenspace \( E_0 \). To establish the existence of solution, it suffices to show that \( \mu - r_0 \mathbf{1} \) lies in the orthogonal complement of \( E_0 \) (which is the column space of \( \Omega \)). This is easily seen since

\[
w_0^T (\mu - r_0 \mathbf{1}) = w_0^T \mu - r_0 = 0.
\]
Example

Given the singular matrix

$$
\Omega = \begin{pmatrix}
\sigma_1^2 & -\sigma_1 \sigma_2 \\
-\sigma_1 \sigma_2 & \sigma_2^2
\end{pmatrix}
$$

and $w_0 = \frac{1}{\sigma_1 + \sigma_2} \begin{pmatrix} \sigma_2 \\ \sigma_1 \end{pmatrix}$ in $E_0$, we have $\Omega w_0 = 0$. The “proxy” riskfree rate of return is given by

$$
r_0 = w_0^T \mu = \frac{1}{\sigma_1 + \sigma_2} \begin{pmatrix} \sigma_2 \\ \sigma_1 \end{pmatrix} \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \end{pmatrix} = \frac{\sigma_2 \bar{r}_1 + \sigma_1 \bar{r}_2}{\sigma_1 + \sigma_2}.
$$

As a check, it is readily seen that solution exists for

$$
\Omega w = \lambda \left[ \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \end{pmatrix} - \frac{\sigma_2 \bar{r}_1 + \sigma_1 \bar{r}_2}{\sigma_1 + \sigma_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = \lambda \left( \frac{\sigma_1 (\bar{r}_1 - \bar{r}_2)}{\sigma_1 + \sigma_2} \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} \frac{\sigma_2 (\bar{r}_2 - \bar{r}_1)}{\sigma_1 + \sigma_2} \right) = \frac{\lambda (\bar{r}_1 - \bar{r}_2)}{\sigma_1 + \sigma_2} \begin{pmatrix} \sigma_1 \\ -\sigma_2 \end{pmatrix}^T
$$

since the column vector $\frac{\lambda (\bar{r}_1 - \bar{r}_2)}{\sigma_1 + \sigma_2} \begin{pmatrix} \sigma_1 \\ -\sigma_2 \end{pmatrix}$ lies in the column space of $\Omega$. 

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Other mathematical issues

1. What would happen when \( \text{dim}(E_0) > 1 \)? In this case, there are multiple “proxy” riskfree assets. However, such scenario is ruled out; otherwise, this leads to arbitrage opportunities where the riskfree rate of return of one portfolio is higher than the riskfree rate of return of another portfolio.

2. Suppose \( \mu \) happens to lie in \( E_0 \), the projection of \( \mu \) onto the orthogonal complement of \( E_0 \) is the zero vector. Now, solution to eq. (1) is \( w_0 \) since the right-hand side vector is the zero vector. In this case, the investor puts 100% of her wealth into the “proxy” riskfree asset.