MATH4512 — Fundamentals of Mathematical Finance

Topic 4 — Single-period securities models

4.1 Dominant trading strategies and linear pricing measure

4.2 No-arbitrage theory and risk neutral probability measure

4.3 Valuation of contingent claims and risk neutral valuation principles
4.1 Dominant trading strategies and linear pricing measure

- The initial prices of $M$ risky securities, denoted by $S_1(0), \cdots, S_M(0)$, are positive scalars that are known at $t = 0$.

- Their values at $t = 1$ are random variables, which are defined with respect to a sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$ of $K$ possible outcomes (or states of the world).

- At $t = 0$, the investors know the list of all possible outcomes, but which outcome does occur is revealed only at the end of the investment period $t = 1$.

- A probability measure $P$ satisfying $P(\omega) > 0$, for all $\omega \in \Omega$, is defined on $\Omega$.

- We use $S$ to denote the price process $\{S(t) : t = 0, 1\}$, where $S(t)$ is the row vector $S(t) = (S_1(t) \ S_2(t) \cdots S_M(t))$. 
Consider 3 risky assets with time-0 price vector

\[ S(0) = (S_1(0) \quad S_2(0) \quad S_3(0)) = (1 \quad 2 \quad 3). \]

At time 1, there are 2 possible states of the world:

\[ \omega_1 = \text{Hang Seng index is at or above 22,000} \]
\[ \omega_2 = \text{Hang Seng index falls below 22,000}. \]

If \( \omega_1 \) occurs, then

\[ S(1; \omega_1) = (1.2 \quad 2.1 \quad 3.4); \]

otherwise, \( \omega_2 \) occurs and

\[ S(1; \omega_2) = (0.8 \quad 1.9 \quad 2.9). \]
• The possible values of the asset price process at $t = 1$ are listed in the following $K \times M$ matrix

$$S(1; \Omega) = \begin{pmatrix}
S_1(1; \omega_1) & S_2(1; \omega_1) & \cdots & S_M(1; \omega_1) \\
S_1(1; \omega_2) & S_2(1; \omega_2) & \cdots & S_M(1; \omega_2) \\
\vdots & \vdots & \ddots & \vdots \\
S_1(1; \omega_K) & S_2(1; \omega_K) & \cdots & S_M(1; \omega_K)
\end{pmatrix}.$$  

• Since the assets are limited liability securities, the entries in $S(1; \Omega)$ are non-negative scalars.

• Existence of a strictly positive riskless security or bank account, whose value is denoted by $S_0$. Without loss of generality, we take $S_0(0) = 1$ and the value at time 1 to be $S_0(1) = 1 + r$, where $r \geq 0$ is the deterministic interest rate over one period.
We define the discounted price process of the $i^{\text{th}}$ risky asset, $i = 1, 2, \ldots, M$, by

$$S_i^*(t) = S_i(t)/S_0(t), \quad t = 0, 1,$$

that is, we use the riskless security as the *numeraire* or *accounting unit*.

The payoff matrix of the discounted price processes of the $M$ risky assets and the riskless security can be expressed in the form

$$\hat{S}^*(1; \Omega) = \begin{pmatrix}
1 & S_1^*(1; \omega_1) & \cdots & S_M^*(1; \omega_1) \\
1 & S_1^*(1; \omega_2) & \cdots & S_M^*(1; \omega_2) \\
\vdots & \vdots & \ddots & \vdots \\
1 & S_1^*(1; \omega_K) & \cdots & S_M^*(1; \omega_K)
\end{pmatrix}.$$

The discounted payoff matrix $\hat{S}^*(1; \Omega)$ augmented with the riskfree asset (first column with entries all being one) and the initial (discounted) price vector $(S_0^*(0) \ S_1^*(0) \ \cdots \ S_M^*(0))$ are the two most relevant quantities in the single-period securities model.
Trading strategies

- An investor adopts a trading strategy by selecting a portfolio of the $M$ assets at time 0. A trading strategy is characterized by asset holding in the portfolio.

- The number of units of asset $m$ held in the portfolio from $t = 0$ to $t = 1$ is denoted by $h_m, m = 0, 1, \ldots, M$.

- The scalars $h_m$ can be positive (long holding), negative (short selling) or zero (no holding).

- An investor is endowed with an initial endowment $V_0$ at time 0 to set up the trading portfolio. How does she choose the portfolio holding of the assets such that the expected portfolio value at time 1 is maximized?
**Portfolio value process**

- Let $V = \{V_t : t = 0, 1\}$ denote the value process that represents the total value of the portfolio over time. It is seen that

$$V_t = h_0S_0(t) + \sum_{m=1}^{M} h_mS_m(t), \quad t = 0, 1.$$ 

- Let $G$ be the random variable that denotes the total gain generated by investing in the portfolio. We then have

$$G = h_0r + \sum_{m=1}^{M} h_m\Delta S_m, \quad \Delta S_m = S_m(1) - S_m(0).$$
Account balancing

• If there is no withdrawal or addition of funds within the investment horizon, then

\[ V_1 = V_0 + G. \]

• Suppose we use the bank account as the numeraire, and define the discounted value process by \( V_t^* = V_t/S_0(t) \) and discounted gain by \( G^* = V_1^* - V_0^* \), we then have

\[
V_t^* = h_0 + \sum_{m=1}^M h_m S_m^*(t), \quad t = 0, 1;
\]

\[
G^* = V_1^* - V_0^* = \sum_{m=1}^M h_m \Delta S_m^*.
\]

Note that the riskfree asset does not contribute to the discounted gain.
Dominant trading strategies

A trading strategy \( \mathcal{H} \) is said to be *dominant* if there exists another trading strategy \( \mathcal{\tilde{H}} \) such that

\[
V_0 = \tilde{V}_0 \quad \text{and} \quad V_1(\omega) > \tilde{V}_1(\omega) \quad \text{for all} \ \omega \in \Omega.
\]

- Suppose \( \mathcal{H} \) dominates \( \mathcal{\tilde{H}} \), we define a new trading strategy \( \mathcal{\tilde{H}} = \mathcal{H} - \mathcal{\tilde{H}} \). Let \( \tilde{V}_0 \) and \( \tilde{V}_1 \) denote the portfolio value of \( \mathcal{\tilde{H}} \) at \( t = 0 \) and \( t = 1 \), respectively. We then have \( \tilde{V}_0 = 0 \) and \( \tilde{V}_1(\omega) > 0 \) for all \( \omega \in \Omega \).

- This trading strategy is dominant since it dominates the strategy which starts with zero value and does no investment at all.

- Equivalent definition: A dominant trading strategy exists if and only if there exists a trading strategy satisfying \( V_0 < 0 \) and \( V_1(\omega) \geq 0 \) for all \( \omega \in \Omega \).
Asset span

- Consider two risky securities whose discounted payoff vectors are
  \[ S_1^*(1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad S_2^*(1) = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}. \]

- The payoff vectors are used to form the discounted terminal payoff matrix
  \[ S^*(1) = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}. \]

- Let the current prices be represented by the row vector \( S^*(0) = (1 \quad 2). \)
• We write \( h \) as the column vector whose entries are the portfolio holding of the securities in the portfolio. The trading strategy is characterized by specifying \( h \). The current portfolio value and the discounted portfolio payoff are given by \( S^*(0)h \) and \( S^*(1)h \), respectively.

• The set of all portfolio payoffs at \( t = 1 \) via different holding of securities is called the asset span \( S \). The asset span is seen to be the column space of the payoff matrix \( S^*(1) \), which is a subspace in \( \mathbb{R}^K \) spanned by the columns of \( S^*(1) \).

\[
\text{asset span} = \text{column space of } S^*(1) = \text{span}(S_1^*(1) \ldots S_M^*(1))
\]
Recall that

\[
\text{column rank} = \text{dimension of column space} = \text{number of independent columns}.
\]

It is well known that number of independent columns = number of independent rows, so column rank = row rank = rank \( \leq \min(K, M) \).

- In the above numerical example, the asset span consists of all vectors of the form \( h_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + h_2 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \), where \( h_1 \) and \( h_2 \) are scalars.
Redundant security and complete model

- If the discounted terminal payoff vector of an added security lies inside $S$, then its payoff can be expressed as a linear combination of $S^*_1(1)$ and $S^*_2(1)$. In this case, it is said to be a redundant security. The added security is said to be replicable by some combination of existing securities.

- A securities model is said to be complete if every payoff vector lies inside the asset span. That is, all new securities can be replicated by existing securities. This occurs if and only if the dimension of the asset span equals the number of possible states, that is, the asset span $[\text{column space of } S^*(1; \Omega)]$ becomes the whole $\mathbb{R}^K$. 

Given the securities model with 4 risky securities and 3 possible states of world:

\[ S^*(1; \Omega) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 8 & 11 \end{pmatrix}, \quad S^*(0) = (1 \ 2 \ 4 \ 7). \]

Here, asset span = \( \text{span}(S^*_1(1), S^*_2(1)) \), which has dimension = 2 < 3 = number of possible states. Hence, the securities model is not complete! For example,

\[ S^*_\beta(1; \Omega) = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \]

does not lie in the asset span of the securities model. There is no solution to

\[ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 8 & 11 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}. \]
Pricing problem

Given a new security that is replicable by existing securities, its price with reference to a given securities model is given by the cost of setting up the replicating portfolio.

Consider a new security with discounted payoff at $t = 1$ as given by

$$S_\alpha^*(1; \Omega) = \begin{pmatrix} 5 \\ 8 \\ 13 \end{pmatrix},$$

which is seen to be

$$S_\alpha^*(1; \Omega) = S_2^*(1; \Omega) + S_3^*(1; \Omega) = S_1^*(1; \Omega) + 2S_2^*(1; \Omega).$$

This new security is redundant. As there are two distinct replicating portfolios, the price of this security can be either

$$S_2^*(0) + S_3^*(0) = 6 \quad \text{or} \quad S_1^*(0) + 2S_2^*(0) = 5.$$

There are two possible prices, so the law of one price does NOT hold.
Question

How do we modify the initial prices in the initial price vector \( S^*(0) \) such that the law of one price hold?

Note that \( S^*_3(1; \Omega) = S^*_1(1; \Omega) + S^*_2(1; \Omega) \) and \( S^*_4(1; \Omega) = S^*_1(1; \Omega) + S^*_3(1; \Omega) \), both the third and fourth security are redundant securities. To achieve the law of one price, we modify \( S^*_3(0) \) and \( S^*_4(0) \) to ensure that

\[
S^*_3(0) = S^*_1(0) + S^*_2(0) = 3 \quad \text{and} \quad S^*_4(0) = 2S^*_1(0) + S^*_2(0) = 4.
\]

Conjecture

If there are no redundant securities, then the law of one price holds for securities that lie in the asset span. Mathematically, non-existence of redundant securities means \( S^*(1; \Omega) \) has full column rank. That is, column rank = number of columns. This gives a sufficient condition for “law of one price”.
Law of one price (pricing of securities that lie in the asset span)

1. The law of one price states that all portfolios with the same terminal payoff have the same initial price.

2. Consider two portfolios with different portfolio weights $h$ and $h'$. Suppose these two portfolios have the same discounted payoff, that is, $S^*(1)h = S^*(1)h'$, the law of one price infers that $S^*(0)h = S^*(0)h'$.

3. The trading strategy $h$ is obtained by solving

$$S^*(1)h = S^*_\alpha(1).$$

Solution exists if $S^*_\alpha(1)$ lies in the asset span. Uniqueness of solution is equivalent to null space of $S^*(1)$ having zero dimension. There is only one trading strategy that replicates the security with discounted terminal payoff $S^*_\alpha(1)$. In this case, the law of one price always holds.
Law of one price and dominant trading strategy

If the law of one price fails, then it is possible to have two trading strategies $h$ and $h'$ such that $S^*(1)h = S^*(1)h'$ but $S^*(0)h > S^*(0)h'$.

Let $G^*(\omega)$ and $G^{*'}(\omega)$ denote the respective discounted gain corresponding to the trading strategies $h$ and $h'$. We then have $G^{*'}(\omega) > G^*(\omega)$ for all $\omega \in \Omega$, so there exists a dominant trading strategy. The corresponding dominant trading strategy is $h' - h$ so that $V_0 < 0$ but $V_1^*(\omega) = 0$ for all $\omega \in \Omega$.

Hence, the non-existence of a dominant trading strategy implies the law of one price. However, the converse statement does not hold.

[See later numerical example.]
Pricing functional

• Given a discounted portfolio payoff $x$ that lies inside the asset span, the payoff can be generated by some linear combination of the securities in the securities model. We have $x = S^*(1)h$ for some $h \in \mathbb{R}^M$. Existence of the solution $h$ is guaranteed since $x$ lies in the asset span, or equivalently, $x$ lies in the column space of $S^*(1)$.

• The current value of the portfolio is $S^*(0)h$, where $S^*(0)$ is the initial price vector.

• We may consider $S^*(0)h$ as a pricing functional $F(x)$ on the payoff $x$. If the law of one price holds, then the pricing functional is single-valued. Furthermore, it is a linear functional, that is,

$$F(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F(x_1) + \alpha_2 F(x_2)$$

for any scalars $\alpha_1$ and $\alpha_2$ and payoffs $x_1$ and $x_2$. 
Arrow security and state price

• Let $e_k$ denote the $k^{th}$ coordinate vector in the vector space $\mathbb{R}^K$, where $e_k$ assumes the value 1 in the $k^{th}$ entry and zero in all other entries. The vector $e_k$ can be considered as the discounted payoff vector of a security, and it is called the Arrow security of state $k$. This Arrow security has unit discounted payoff when state $k$ occurs and zero payoff otherwise.

• Suppose the securities model is complete (all Arrow securities lie in the asset span) and the law of one price holds, then the pricing functional $F$ assigns unique value to each Arrow security. We write $s_k = F(e_k)$, which is called the state price of state $k$. We expect the state prices to be non-negative. Take

$$S^*_\alpha(1) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{pmatrix} = \sum_{k=1}^K \alpha_k e_k,$$

then its unique initial price is

$$S^*_\alpha(0) = F(S^*_\alpha(1)) = F\left( \sum_{k=1}^K \alpha_k e_k \right) = \sum_{k=1}^K \alpha_k F(e_k) = \sum_{k=1}^K \alpha_k s_k.$$
Example – State prices

Given $F\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = 7$ and $F\left(\begin{pmatrix} 4 \\ 2 \end{pmatrix}\right) = 9$, find $F\left(\begin{pmatrix} 5 \\ 3 \end{pmatrix}\right)$.

By the linear property of pricing functional, we deduce that

$$F\left(\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 9 - 7 = 2$$

so that $s_1 = 2$;

$$F\left(\frac{1}{2} \left[ \begin{pmatrix} 3 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \right) = \frac{1}{2} \left[ F\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) - 3F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right] = \frac{1}{2} (7 - 3 \times 2) = \frac{1}{2}$$

so that $s_2 = \frac{1}{2}$. 
The fair price of \( \begin{pmatrix} 5 \\ 3 \end{pmatrix} \) is given by

\[
F \left( \begin{pmatrix} 5 \\ 3 \end{pmatrix} \right) = 5F \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + 3F \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 5s_1 + 3s_2 = \frac{23}{2}.
\]

The actual probabilities of occurrence of the two states are irrelevant in the pricing of the new contingent claim \( \begin{pmatrix} 5 \\ 3 \end{pmatrix} \). Lastly, we observe that

\[
\begin{pmatrix} 2 & \frac{1}{2} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 9 \end{pmatrix}.
\]
Summary

Given a securities model endowed with $S^*(1; \Omega)$ and $S^*(0)$, can we find a trading strategy to form a portfolio that replicates a new security $S^*_\alpha(1; \Omega)$ (also called a contingent claim) that is outside the universe of the $M$ available risky securities in the securities model?

Replication means the terminal payoff of the replicating portfolio matches with that of the contingent claim under all scenarios of occurrence of the state of the world at $t = 1$.

1. Formation of the replicating portfolio is possible if we have existence of solution $h$ to the following system

$$S^*(1; \Omega)h = S^*_\alpha(1; \Omega). \quad \text{(R)}$$
This is equivalent to the fact that \( S^*_\alpha(1; \Omega) \) lies in the asset span (column space) of \( S^*(1; \Omega) \). The solution \( h \) is the corresponding trading strategy. Note that \( h \) may not be unique.

Completeness of securities model

If all contingent claims are replicable, then the securities model is said to be complete. This is equivalent to

\[
\dim(\text{asset span}) = K = \text{number of possible states},
\]

that is, asset span = \( \mathbb{R}^K \). In this case, solution \( h \) always exists.
2. Uniqueness of trading strategy

If \( h \) is unique, then there is only one trading strategy that generates the replicating portfolio. This occurs when the columns of \( S^*(1; \Omega) \) are independent. Equivalently, column rank = \( M \) and all securities are non-redundant. Mathematically, this is equivalent to observe that the homogeneous system

\[
S^*(1; \Omega)h = 0
\]

admits only the trivial zero solution. In other words, the dimension of the null space of \( S^*(1; \Omega) \) is zero.

When we always have unique solution \( h \) conditional on existence of solution \( h \) to eq.(R), the initial cost of setting up the replicating portfolio (price at time 0) as given by \( S^*(0)h \) is unique. In this case, law of one price holds for the given securities model.
Matrix properties of $S^*(1)$ that are related to financial economics concepts

The securities model is endowed with

(i) discounted terminal payoff matrix $= \begin{pmatrix} S_1^*(1) & \cdots & S_M^*(1) \end{pmatrix}$, and
(ii) initial price vector; $S^*(0) = (S_1^*(0) \cdots S_M^*(0))$.

Recall that

$$\text{column rank} \leq \min(K, M)$$

where $K =$ number of possible states, $M =$ number of risky securities.

List of terms: redundant securities, complete model, replicating portfolio, asset holding, asset span, law of one price, dominant trading strategy, Arrow securities, state prices
Given a risky security with the discounted terminal payoff $S^*_\alpha(1)$, we are interested to explore the existence and uniqueness of solution to

$$S^*(1)h = S^*_\alpha(1).$$

Here, $h$ is the asset holding of the portfolio that replicates $S^*_\alpha(1)$.

(i) column rank = $K$

asset span = $\mathbb{R}^K$, so the securities model is complete. Any risky securities is replicable. In this case, solution $h$ always exists.

(ii) column rank = $M$ (all columns of $S^*(1)$ are independent)

All securities are non-redundant. Note that $h$ may or may not exist. However, if $h$ exists, then it must be unique. The price of any replicable security is unique.
(iii) column rank < $K$

Solution $h$ exists if and only if $S^*_\alpha(1)$ lies in the asset span. However, there is no guarantee for the uniqueness of solution.

(iv) column rank < $M$

Existence of redundant securities, so the law of one price may fail.

To explore “law of one price” and “existence of dominant trading strategies”, one has to consider the nature of the solution to the linear system of equations

$$S^*(0) = \pi S^*(1).$$
Law of one price (necessary and sufficient condition)

Law of one price holds if and only if solution to

\[ \pi S^*(1) = S(0) \]  \hspace{1cm} (A)

exists. Note that it is not necessary to include the riskfree security in the securities model.

1. Suppose solution to (A) exists, let \( h \) and \( h' \) be two trading strategies such that their respective discounted terminal payoff are the same. That is,

\[ S^*(1)h = S^*(1)h'. \]

Since \( \pi \) exists, we then have

\[ \pi S^*(1)(h - h') = 0. \]

Noting that \( \pi S^*(1) = S(0) \), we obtain

\[ S(0)(h - h') = 0 \quad \text{so that} \quad V_0 = V_0'. \]
2. Suppose solution to $A$ does not exist for the given $S(0)$, this implies that $S(0)$ that does not lie in the row space of $S^*(1)$. The row space of $S^*(1)$ does not span the whole $\mathbb{R}^M$. Therefore, \(\dim(\text{row space of } S^*(1)) = \text{rank}(S^*(1)) < M\), where $M$ is the number of securities $=$ number of columns in $S^*(1)$.

Recall that the nullspace is the collection of all the solutions to the homogeneous system: $S^*(1)h = 0$. The nullspace is a subspace of $\mathbb{R}^M$ and

$$\dim(\text{null space of } S^*(1)) + \text{rank}(S^*(1)) = M.$$ 

Hence, we deduce that \(\dim(\text{null space of } S^*(1)) > 0\). That is, there exists non-zero solution $h$ to

$$S^*(1)h = 0.$$
Furthermore, since row space = orthogonal complement of null space, any of these non-zero solution $h$ in the nullspace of $S^*(1)$ cannot be orthogonal to $S(0)$. That is, $S(0)h \neq 0$. Otherwise, this leads to contradiction as $S(0)$ does not lie in the row space.

Let $h = h_1 - h_2$, where $h_1 \neq h_2$, then there exist two distinct trading strategies such that

$$S^*(1)h_1 = S^*(1)h_2.$$  

The two strategies have the same discounted terminal payoff under all states of the world. However, by virtue of the property: $S(0)h \neq 0$, their initial prices are unequal since

$$S(0)h_1 \neq S(0)h_2.$$  

Hence, the law of one price does not hold.
Linear pricing measure

We consider securities models with the inclusion of the riskfree security. A non-negative row vector \( q = (q(\omega_1) \cdots q(\omega_K)) \) is said to be a linear pricing measure if for every trading strategy we have

\[
V_0^* = \sum_{k=1}^{K} q(\omega_k)V_1^*(\omega_k).
\]

Remark

Note that \( q \) is not required to be unique. Here, the same initial price \( V_0^* \) is always resulted as there is no dependence of \( V_0^* \) on the asset holding of the portfolio. Two portfolios with the same terminal payoff for all states of the world would have the the same price. Implicitly, this implies that the law of one price holds. The rigorous justification of the statement “existence of a linear pricing measure implying law of one price” will be presented later.
1. Suppose we take the holding amount of every risky security to be zero, thereby \( h_1 = h_2 = \cdots = h_M = 0 \), then

\[
V_0^* = h_0 = \sum_{k=1}^{K} q(\omega_k) h_0
\]

so that

\[
\sum_{k=1}^{K} q(\omega_k) = 1.
\]

2. Provided that the securities model is complete so all Arrow securities exist, by taking the portfolio to have the same terminal payoff as that of the \( k^{th} \) Arrow security, we obtain

\[
s_k = q(\omega_k), \quad k = 1, 2, \ldots, K.
\]

That is, the state price of the \( k^{th} \) state is simply \( q(\omega_k) \), so sum of state prices equals one. This is not surprising when we compare

\[
V_0^* = \sum_{k=1}^{K} q(\omega_k) V_1^*(\omega_k) \quad \text{and} \quad S_\alpha^*(0) = \sum_{k=1}^{K} \alpha_k s_k.
\]
• Since we have taken \( q(\omega_k) \geq 0, k = 1, \cdots, K \), and their sum is one, we may interpret \( q(\omega_k) \) as a probability measure on the sample space \( \Omega \). Instead of being strictly positive, \( q(\omega_k) \) is non-negative for all \( \omega_k \).

• Note that \( q(\omega_k) \) is not related to the actual probability of occurrence \( P(\omega_k) \) of the state \( k \), though the current security price is given by the expectation of the security payoff one period later under the linear pricing measure.

• By taking the portfolio weights to be zero except for the \( m^{\text{th}} \) security, we generate the following linear system of equations for \( q(\omega_k) \):

\[
S^*_m(0) = \sum_{k=1}^{K} q(\omega_k) S^*_m(1; \omega_k), \quad m = 1, \cdots, M.
\]

In matrix form, letting \( q = (q(\omega_1) q(\omega_2) \cdots q(\omega_K)) \), we have

\[
\hat{S}^*(0) = q \hat{S}^*(1; \Omega), \quad q \geq 0.
\]
Numerical example: State prices calculations

Take $\hat{S}^*(1) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\hat{S}^*(0) = (1 \quad 1\frac{1}{3})$, then

$$q = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

is a linear pricing measure. The linear pricing measure is not unique! Actually, we have $q(\omega_1) = \frac{1}{3}$ and $q_2(\omega_2) + q(\omega_3) = \frac{2}{3}$.

- The securities model is not complete. Though $e_1$ is replicable and its initial price is $\frac{1}{3}$, but $e_2$ and $e_3$ are not replicable so the two state prices of $\omega_2$ and $\omega_3$ do not exist.

- The existence of a linear pricing measure does not require completeness of the securities model. However, if the market is complete, then the linear pricing probability values are the state prices.
Suppose we add a new risky security with discounted terminal payoff 
\[
\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}
\] 
and initial price \( \frac{2}{3} \) into the securities model, then the securities model becomes complete. We have the following state prices: \( s_1 = \frac{1}{3}, \ s_2 = -\frac{1}{3}, \ s_3 = 1 \); and 

\[
V_0^* = \begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix} \begin{pmatrix} V_1^*(\omega_1) \\ V_1^*(\omega_2) \\ V_1^*(\omega_3) \end{pmatrix}.
\]

In this case, law of one price holds but dominant trading strategy exists. For example, we may choose 3 units of the first Arrow security, 6 units of the second Arrow security and one unit of the third Arrow security in the portfolio. This gives 

\[
V_1^*(\omega) = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} > 0, \quad V_0^* = 3s_1 + 6s_2 + s_3 = 0.
\]
Numerical example: Linear pricing measure and dominant trading strategy

Consider a securities model with 2 risky securities and the riskfree security, and there are 3 possible states. The current discounted price vector \( \widehat{S}^*(0) \) is \((1 \ 4 \ 2)\) and the discounted payoff matrix at \( t = 1 \) is \( \widehat{S}^*(1) = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 4 \end{pmatrix} \). Here, the law of one price holds since the only solution to \( \widehat{S}^*(1)h = 0 \) is \( h = 0 \). This is because the columns of \( \widehat{S}^*(1) \) are independent so that the dimension of the nullspace of \( \widehat{S}^*(1) \) is zero.
Non-existence of linear pricing measure

The linear pricing probabilities $q(\omega_1), q(\omega_2)$ and $q(\omega_3)$, if exist, should satisfy the following equations:

$$
1 = q(\omega_1) + q(\omega_2) + q(\omega_3) \\
4 = 4q(\omega_1) + 3q(\omega_2) + 2q(\omega_3) \\
2 = 3q(\omega_1) + 2q(\omega_2) + 4q(\omega_3).
$$

Solving the above equations, we obtain $q(\omega_1) = q(\omega_2) = 2/3$ and $q(\omega_3) = -1/3$.

- Since not all the pricing probabilities are non-negative, the linear pricing measure does not exist for this securities model.
Existence of dominant trading strategies

- For convenience in the illustration, we consider the scenario where the portfolio contains only the risky securities. That is, we take $h_0 = 0$. Can we find a trading strategy $(h_1, h_2)$ such that $V^*_0 = 4h_1 + 2h_2 = 0$ but $V^*_1(\omega_k) > 0, k = 1, 2, 3$? This is equivalent to ask whether there exist $h_1$ and $h_2$ such that $4h_1 + 2h_2 = 0$ and

\[
4h_1 + 3h_2 > 0 \\
3h_1 + 2h_2 > 0 \\
2h_1 + 4h_2 > 0.
\]  \hspace{1cm} (A)

- The region that satisfies all the inequalities in (A) is found to be lying on the top right sides above the two bold lines: (i) $3h_1 + 2h_2 = 0, h_1 < 0$ and (ii) $2h_1 + 4h_2 = 0, h_1 > 0$. It is seen that all the points on the dotted half line: $4h_1 + 2h_2 = 0, h_1 < 0$ represent dominant trading strategies that start with zero wealth but end with positive wealth with certainty.
The region above the two bold lines represents trading strategies that satisfy inequalities (A). The trading strategies that lie on the dotted line: \(4h_1 + 2h_2 = 0, h_1 < 0\) are dominant trading strategies.
• Suppose the initial discounted price vector is changed from
(4 2) to (3 3), the new set of linear pricing probabilities will
be determined by

\[
\begin{align*}
1 &= q(\omega_1) + q(\omega_2) + q(\omega_3) \\
3 &= 4q(\omega_1) + 3q(\omega_2) + 2q(\omega_3) \\
3 &= 3q(\omega_1) + 2q(\omega_2) + 4q(\omega_3),
\end{align*}
\]

which is seen to have the solution: \( q(\omega_1) = q(\omega_2) = q(\omega_3) = 1/3 \). Now, all the pricing probabilities have non-negative values, the row vector \( q = (1/3 1/3 1/3) \) represents a linear pricing measure.

• The line \( 3h_1 + 3h_2 = 0 \) always lies outside the region above the two bold lines. Therefore, we cannot find \( (h_1 h_2) \) such that \( 3h_1 + 3h_2 = 0 \) together with \( h_1 \) and \( h_2 \) satisfying all these inequalities.
Theorem

There exists a linear pricing measure if and only if there are no dominant trading strategies.

The above linear pricing measure theorem can be seen to be a direct consequence of the Farkas Lemma.

Farkas Lemma

There does not exist \( \mathbf{h} \in \mathbb{R}^M \) such that

\[
\hat{S}^*(1; \Omega)\mathbf{h} > 0 \quad \text{and} \quad \hat{S}^*(0)\mathbf{h} = 0
\]

if and only if there exists \( \mathbf{q} \in \mathbb{R}^K \) such that

\[
\hat{S}^*(0) = \mathbf{q}\hat{S}^*(1; \Omega) \quad \text{and} \quad \mathbf{q} \geq 0.
\]
Conditions for sufficiency of law of one price

Given a securities model, we can deduce that law of one price holds by observing either either

(i) null space of $S^*(1)$ has zero dimension, or
(ii) existence of a linear pricing measure.

Both (i) and (ii) represent the various conditions of sufficiency for law of one price.

Remarks

1. Condition (i) is equivalent to non-existence of redundant securities.

2. Condition (i) and condition (ii) are not equivalent, nor one condition implies the other.
Various conditions that give sufficiency for law of one price

- $q \geq 0$
  - existence of linear pricing measure

- $\dim(\eta(S^*(1))) = 0$
  - non-existence of redundant securities
4.2 No-arbitrage theory and risk neutral probability measure — Fundamental Theorem of Asset Pricing

- An *arbitrage opportunity* is a trading strategy that has the following properties: (i) $V_0^* = 0$, (ii) $V_1^*(\omega) \geq 0$ with strict inequality at least for one state.

- The existence of a dominant strategy requires a portfolio with initial zero wealth to end up with a *strictly* positive wealth in all states.

- The existence of a dominant trading strategy implies the existence of an arbitrage opportunity, but the converse is not necessarily true.
Risk neutral probability measure

A probability measure $Q$ on $\Omega$ is a risk neutral probability measure if it satisfies

(i) $Q(\omega) > 0$ for all $\omega \in \Omega$, and

(ii) $E_Q[\Delta S^*_m] = 0, m = 0, 1, \ldots, M$, where $E_Q$ denotes the expectation under $Q$. The expectation of the discounted gain of any security in the securities model under $Q$ is zero.

Note that $E_Q[\Delta S^*_m] = 0$ is equivalent to $S^*_m(0) = \sum_{k=1}^{K} Q(\omega_k)S^*_m(1; \omega_k)$.

Fundamental Theorem of Asset Pricing

No arbitrage opportunities exist if and only if there exists a risk neutral probability measure $Q$. 

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Remark When we consider the linear pricing measure $q$ and martingale pricing measure $Q$, the associated securities model has to contain the riskfree asset.
• In a one-period model, given the information on the initial prices and terminal payoff values of the security prices at \( t = 0 \), the risk neutral pricing measure satisfies

\[
S^*_m(0) = E_Q[S^*_m(1; \Omega)] = \sum_{k=1}^{K} S^*_m(1; \omega_k)Q(\omega_k), \quad m = 1, 2, \ldots, M. \tag{1}
\]

The discounted security price process \( S^*_m(t) \) is said to be a martingale* under \( Q \).

Martingale is an adapted stochastic process associated with the wealth process of a gambler in a fair game. In a fair game, the expected value of the gambler’s wealth after any number of plays is always equal to her initial wealth.

*Martingale property with respect to \( Q \) and \( \mathbb{F} \):

\[
S^*_m(t) = E_Q[S^*_m(s + t)|\mathcal{F}_t] \text{ for all } t \geq 0, s \geq 0.
\]
Equivalent martingale measure

- The risk neutral probability measure $Q$ is commonly called the equivalent martingale measure. “Equivalent” refers to the equivalence between the physical measure $P$ and martingale measure $Q$ [observing $P(\omega) > 0 \iff Q(\omega) > 0$ for all $\omega \in \Omega$]. The linear pricing measure falls short of this equivalence property since $q(\omega)$ can be zero.

* $P$ and $Q$ may not agree on the assignment of probability values to individual events, but they always agree as to which events are possible or impossible.
Martingale property of discounted portfolio value (assuming the existence of $Q$, or equivalently, the absence of arbitrage in the securities model)

- Let $V_1^*(\Omega)$ denote the discounted payoff of a portfolio. Since $V_1^*(\Omega) = \hat{S}^*(1; \Omega)h$ for some trading strategy $h = (h_0 \cdots h_M)^T$, by Eq. (1),

$$V_0^* = (S_0^*(0) \cdots S_M^*(0))h$$

$$= (E_Q[S_0^*(1; \Omega)] \cdots E_Q[S_M^*(1; \Omega)])h$$

$$= \sum_{m=0}^{M} \left[ \sum_{k=1}^{K} S_m^*(1; \omega_k)Q(\omega_k) \right] h_m$$

$$= \sum_{k=1}^{K} Q(\omega_k) \left[ \sum_{m=0}^{M} S_m^*(1; \omega_k)h_m \right] = E_Q[V_1^*(\Omega)].$$

- The equivalent martingale measure $Q$ is not necessarily unique. Since “absence of arbitrage opportunities” implies “law of one price”, the expectation value $E_Q[V_1^*(\Omega)]$ is single-valued under all equivalent martingale measures.
Finding the set of risk neutral measures

Consider the earlier securities model with the riskfree security and only one risky security, where \( \hat{S}(1; \Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} \) and \( \hat{S}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \). The risk neutral probability measure

\( Q = (Q(\omega_1) \quad Q(\omega_2) \quad Q(\omega_3)), \)

if exists, will be determined by the following system of equations

\[
(Q(\omega_1) \quad Q(\omega_2) \quad Q(\omega_3)) \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.
\]

Since there are more unknowns than the number of equations, the solution is not unique. The solution is found to be \( Q = (\lambda \quad 1 - 2\lambda \quad \lambda), \) where \( \lambda \) is a free parameter. Since all risk neutral probabilities are all strictly positive, we must have \( 0 < \lambda < 1/2. \)
Under market completeness, if the set of risk neutral measures is non-empty, then it must be a singleton.

Under market completeness, column rank of $\hat{S}(1; \Omega)$ equals the number of states. Since column rank $= row$ rank, then all rows of $\hat{S}(1; \Omega)$ are independent. If solution $Q$ exists and $Q > 0$ for the linear system

$$Q\hat{S}^*(1; \Omega) = S^*(0),$$

implying that the set of risk neutral measures is non-empty, then it must be unique.

Conversely, suppose the set of risk neutral measures is a singleton, one can show that the securities model is complete (see later discussion).
Numerical example

Suppose we add the second risky security with discounted payoff \( S_2^*(1) = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \) and current discounted value \( S_2^*(0) = 3 \). With this new addition, the securities model becomes complete.

With the new equation \( 3Q(\omega_1) + 2Q(\omega_2) + 4Q(\omega_3) = 3 \) added to the linear system, this new securities model is seen to have the unique risk neutral measure \( Q = (1/3 \ 1/3 \ 1/3) \).

Indeed, when the securities model is complete, all Arrow securities are replicable. Furthermore, given that the securities model is arbitrage-free, their prices (called state prices) are simply equal to the risk neutral probabilities. In this example, we have

\[
s_1 = Q(\omega_1) = \frac{1}{3}, \quad s_2 = Q(\omega_2) = \frac{1}{3}, \quad s_3 = Q(\omega_3) = \frac{1}{3}.
\]
Subspace of discounted gains

Let $W$ be a subspace in $\mathbb{R}^K$ which consists of vectors of discounted gains corresponding to some trading strategy $h$. Note that $W$ is spanned by the set of vectors representing the discounted gains of the risky securities.

In the above securities model, the discounted gains of the first and second risky securities are

$$
\begin{pmatrix}
4 \\
3 \\
2
\end{pmatrix} - \begin{pmatrix}
3 \\
3 \\
3
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
3 \\
2 \\
4
\end{pmatrix} - \begin{pmatrix}
3 \\
3 \\
3
\end{pmatrix} = \begin{pmatrix}
0 \\
-1 \\
1
\end{pmatrix},
$$

respectively.

The discounted gain subspace is given by

$$
W = \left\{ h_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + h_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \text{ where } h_1 \text{ and } h_2 \text{ are scalars} \right\}.
$$
Orthogonality of discounted gain vectors and $Q$

Let $G^*$ denote the discounted gain of a portfolio. For any risk neutral probability measure $Q$, we have

$$E_QG^* = \sum_{k=1}^{K} Q(\omega_k) \left[ \sum_{m=1}^{M} h_m \Delta S_m^*(\omega_k) \right]$$

$$= \sum_{m=1}^{M} h_m E_Q[\Delta S_m^*] = 0.$$

Under the absence of arbitrage opportunities, the expected discounted gain from any risky portfolio is simply zero. Apparently, there is no risk premium derived from the risky investment. Therefore, the financial economics term “risk neutrality” is adopted under this framework of asset pricing.

For any discounted gain vector $G^* = (G(\omega_1) \cdots G(\omega_K))^T \in W$, we have

$$QG^* = 0, \text{ where } Q = (Q(\omega_1) \cdots Q(\omega_K)).$$
Characterization of the set of risk neutral measures

Since the sum of risk neutral probabilities must be one and all probability values must be positive, the risk neutral probability vector $Q$ must lie in the following subset (but not a subspace) of $\mathbb{R}^K$:

$$P^+ = \{ y \in \mathbb{R}^K : y_1 + y_2 + \cdots + y_K = 1 \text{ and } y_k > 0, k = 1, \ldots, K \}.$$

Also, the risk neutral probability vector $Q$ must lie in the orthogonal complement $W^\perp$. Let $R$ denote the set of all risk neutral measures, then $R = P^+ \cap W^\perp$.

In the above numerical example, $W^\perp$ is the line through the origin in $\mathbb{R}^3$ which is perpendicular to $(1 \ 0 \ -1)$ and $(0 \ -1 \ 1)$. The line should assume the form $\lambda(1 \ 1 \ 1)$ for some scalar $\lambda$. Setting $\lambda = 1/3$ to satisfy the property of summing to one, we obtain the risk neutral probability vector $Q = (1/3 \ 1/3 \ 1/3)$. 
4.3 Valuation of contingent claims and complete markets

- A contingent claim can be considered as a random variable $Y$ that represents the corresponding terminal payoff whose value depends on the occurrence of a particular state $\omega_k$, where $\omega_k \in \Omega$.

- Suppose the holder of the contingent claim is promised to receive the preset contingent payoff, how much should the writer of such contingent claim charge at $t = 0$ so that the price is fair to both parties.

- Consider the securities model with the riskfree security whose values at $t = 0$ and $t = 1$ are $S_0(0) = 1$ and $S_0(1) = 1.1$, respectively, and a risky security with $S_1(0) = 3$ and $S_1(1) = \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$. 
The set of $t = 1$ payoffs that can be generated by certain trading strategy is given by $h_0 \begin{pmatrix} 1.1 \\ 1.1 \\ 1.1 \\ 1.1 \end{pmatrix} + h_1 \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$ for some scalars $h_0$ and $h_1$.

For example, the contingent claim $\begin{pmatrix} 5.5 \\ 4.4 \\ 3.3 \end{pmatrix}$ can be generated by the trading strategy: $h_0 = 1$ and $h_1 = 1$, while the other contingent claim $\begin{pmatrix} 5.5 \\ 4.0 \\ 3.3 \end{pmatrix}$ cannot be generated by any trading strategy associated with the given securities model.
Attainable contingent claims

A contingent claim $Y$ is said to be *attainable* if there exists some trading strategy $h$, called the *replicating portfolio*, such that $V_1 = Y$ for all possible states occurring at $t = 1$.

The price at $t = 0$ of the replicating portfolio is given by

$$V_0 = h_0 S_0(0) + h_1 S_1(0) = 1 \times 1 + 1 \times 3 = 4.$$

Suppose there are no arbitrage opportunities (equivalent to the existence of a risk neutral probability measure), then the law of one price holds and so $V_0$ is unique.
Consider a given attainable contingent claim $Y$ which is generated by certain trading strategy. The associated discounted gain $G^*$ of the trading strategy is given by $G^* = \sum_{m=1}^{M} h_m \Delta S^*_m$. Now, suppose a risk neutral probability measure $Q$ associated with the securities model exists, we have

$$V_0 = E_Q V_0^* = E_Q[V_1^* - G^*].$$

Since $E_Q[G^*] = 0$ and $V_1^* = Y/S_0(1)$, we obtain

$$V_0 = E_Q[Y/S_0(1)].$$

*Risk neutral valuation principle*:

The price at $t = 0$ of an attainable contingent claim $Y$ is given by the expectation under any risk neutral measure $Q$ of the discounted terminal value of the contingent claim.
Recall that the existence of the risk neutral probability measure implies the law of one price. Does $E_Q[Y/S_0(1)]$ assume the same value for every risk neutral probability measure $Q$?

Provided that $Y$ is attainable, this must be true by virtue of the law of one price since we cannot have two different values for $V_0$ corresponding to the same attainable contingent claim $Y$.

**Theorem** (Attainability of a contingent claim and uniqueness of $E_Q[Y^*]$)

Suppose the securities model admits no arbitrage opportunities. The contingent claim $Y$ is attainable if and only if $E_Q[Y^*]$ takes the same value for every $Q \in M$, where $M$ is the set of risk neutral measures.
Proof

Attainability $\implies$ uniqueness

Recall existence of $Q \iff$ absence of arbitrage $\implies$ law of one price. Given that $Y$ is attainable, so $E_Q[Y^*]$ is constant with respect to all $Q \in M$, otherwise this leads to violation of the law of one price.

Non-attainability $\implies$ non-uniqueness

It suffices to show that if the contingent claim $Y$ is not attainable then $E_Q[Y^*]$ does not take the same value for all $Q \in M$. 
Let \( y^* \in \mathbb{R}^K \) be the discounted payoff vector corresponding to \( Y^* \). Since \( Y \) is not attainable, then there is no solution to

\[
\hat{S}^*(1)h = y^*
\]

(non-existence of trading strategy \( h \)). It then follows that there must exist a non-zero row vector \( \pi \in \mathbb{R}^K \) such that

\[
\pi \hat{S}^*(1) = 0 \quad \text{and} \quad \pi y^* \neq 0.
\]

Remark

Recall that the orthogonal complement of the column space is the left null space. The dimension of the left null space equals \( K \)-column rank, and it is non-zero when the column space does not span the whole \( \mathbb{R}^K \). Suppose there exists \( y^* \) that does not lie in the column space of \( \hat{S}^*(1) \), then there exists a non-zero vector \( \pi \) in the left null space of \( \hat{S}^*(1) \) such that \( y^* \) and \( \pi \) are not orthogonal.
Write $\pi = (\pi_1 \cdots \pi_K)$. Let $\hat{Q} \in M$ be arbitrary, and let $\lambda > 0$ be small enough such that

$$Q(\omega_k) = \hat{Q}(\omega_k) + \lambda \pi_k > 0, \quad k = 1, 2, \cdots, K.$$ 

We then show that $Q(\omega_k)$ is also a risk neutral measure by applying the relation: $\pi \hat{S}^*(1) = 0$. The first column of $\hat{S}^*(1)$ is 1.

1. Note that $\pi \mathbf{1} = \sum_{k=1}^K \pi_k = 0$, so $\sum_{k=1}^K Q(\omega_k) = 1$.

2. For the discounted price process $S^*_n$ of the $n$th risky securities in the securities model, we have

$$E_Q[S^*_n(1)] = \sum_{k=1}^K Q(\omega_k)S^*_n(1; \omega_k)$$

$$= \sum_{k=1}^K \hat{Q}(\omega_k)S^*_n(1; \omega_k) + \lambda \sum_{k=1}^K \pi_kS^*_n(1; \omega_k)$$

$$= \sum_{k=1}^K \hat{Q}(\omega_k)S^*_n(1; \omega_k) = S_n(0).$$
The above property is valid for any risky asset, so $Q$ satisfies the martingale property. Together with $Q(\omega_k) > 0$ and $\sum_{k=1}^{K} Q(\omega_k) = 1$, it is also a risk neutral measure.

Lastly, we consider

$$E_Q[Y^*] = \sum_{k=1}^{K} Q(\omega_k)Y^*(\omega_k)$$

$$= \sum_{k=1}^{K} \hat{Q}(\omega_k)Y^*(\omega_k) + \lambda \sum_{k=1}^{K} \pi_k Y^*(\omega_k).$$

The last term is non-zero since $\pi y^* \neq 0$ and $\lambda > 0$. Therefore, we have

$$E_Q[Y^*] \neq E_{\hat{Q}}[Y^*].$$

Thus, when $Y$ is not attainable, $E_Q[Y^*]$ does not take the same value for all risk neutral measures.
Corollary Given that the set of risk neutral measures $R$ is non-empty. The securities model is complete if and only if $R$ consists of exactly one risk neutral measure.

An earlier proof of “$\Rightarrow$ part” has been shown on p.50. Alternatively, we may prove by contradiction: non-uniqueness of $Q \Rightarrow$ non-completeness.

Suppose there exist two distinct $Q$ and $\hat{Q}$, that is, $Q(\omega_k) \neq \hat{Q}(\omega_k)$ for some state $\omega_k$. Let $Y^* = \begin{cases} 1 & \text{if } \omega = \omega_k \\ 0 & \text{otherwise} \end{cases}$, which is the $k^{th}$ Arrow security. Obviously,

$$E_Q[Y^*] = Q(\omega_k) \neq \hat{Q}(\omega_k) = E_{\hat{Q}}[Y^*],$$

so $E_Q[Y^*]$ is not unique. By the theorem, $Y^*$ is not attainable so the securities model is not complete.

$\Leftarrow$ part: If the risk neutral measure is unique, then for any contingent claim $Y$, $E_Q[Y^*]$ takes the same value for any $Q$ (actually single $Q$). By the theorem, any contingent claim is attainable so the market is complete.
Remarks

- When the securities model is complete and admits no arbitrage opportunities, all Arrow securities lie in the asset span and risk neutral measures exist. The state price of state $\omega_k$ exists for any state and it is equal to the unique risk neutral probability $Q(\omega_k)$. This represents the best scenario of applying the risk neutral valuation procedure for pricing any contingent claim (which is always attainable due to completeness).

- On the other hand, suppose there are two risk neutral probability values for the same state $\omega_k$, the state price of that state cannot be defined properly without contradicting the law of one price. By the theorem, the Arrow security of that state would not be attainable, so the securities model cannot be complete.
Example

Suppose

\[
Y^* = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \hat{S}^*(1; \Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix},
\]

\(Y^*\) is seen to be attainable. We have seen that the risk neutral probability is given by

\[
Q = (\lambda \quad 1 - 2\lambda \quad \lambda), \quad \text{where} \quad 0 < \lambda < 1/2.
\]

The price at \(t = 0\) of the contingent claim is given by

\[
V_0 = 5\lambda + 4(1 - 2\lambda) + 3\lambda = 4,
\]

which is independent of \(\lambda\). This verifies the earlier claim that \(E_Q[Y/S_0(1)]\) assumes the same value for any risk neutral measure \(Q\).

Suppose \(Y^*\) is changed to \((5 \quad 4 \quad 4)^T\), then \(V_0 = E_Q[Y^*] = 4 + \lambda\), which is not unique. This is expected since the new \(Y^*\) is non-attainable.
Complete markets - summary of results

Recall that a securities model is complete if every contingent claim $Y$ lies in the asset span, that is, $Y$ can be generated by some trading strategy.

Consider the augmented terminal payoff matrix

$$
\hat{S}(1; \Omega) = \begin{pmatrix}
S_0(1; \omega_1) & S_1(1; \omega_1) & \cdots & S_M(1; \omega_1) \\
\vdots & \vdots & & \vdots \\
S_0(1; \omega_K) & S_1(1; \omega_K) & \cdots & S_M(1; \omega_K)
\end{pmatrix},
$$

$Y$ always lies in the asset span if and only if the column space of $\hat{S}(1; \Omega)$ is equal to $\mathbb{R}^K$.

- Since the dimension of the column space of $\hat{S}(1; \Omega)$ cannot be greater than $M + 1$, a necessary condition for market completeness is that $M + 1 \geq K$. 

• When $\tilde{S}(1; \Omega)$ has independent columns and the asset span is the whole $\mathbb{R}^K$, then $M + 1 = K$. Now, the trading strategy that generates $Y$ must be unique since there are no redundant securities. In this case, any contingent claim is replicable and its price is unique.

• When the asset span is the whole $\mathbb{R}^K$ but some securities are redundant, the trading strategy that generates $Y$ would not be unique.

• Provided that a risk neutral measure exists, the price at $t = 0$ of the contingent claim is unique under arbitrage pricing, independent of the chosen trading strategy. This is a consequence of the law of one price, which holds since a risk neutral measure exists.

• Non-existence of redundant securities is a sufficient but not necessary condition for the law of one price.
Non-attainable contingent claim

A non-attainable contingent claim cannot be priced using arbitrage pricing theory. However, we may specify an interval \((V_-(Y), V_+(Y))\) where a reasonable price at \(t = 0\) of the contingent claim should lie. The lower and upper bounds are given by

\[
V_+(Y) = \inf\{E_Q[\tilde{Y}/S_0(1)] : \tilde{Y} \geq Y \text{ and } \tilde{Y} \text{ is attainable}\}
\]
\[
V_-(Y) = \sup\{E_Q[\tilde{Y}/S_0(1)] : \tilde{Y} \leq Y \text{ and } \tilde{Y} \text{ is attainable}\}.
\]

Here, \(V_+(Y)\) is the minimum value among all prices of attainable contingent claims that dominate the non-attainable claim \(Y\), while \(V_-(Y)\) is the maximum value among all prices of attainable contingent claims that are dominated by \(Y\).
Suppose $V(Y) > V_+(Y)$, then an arbitrageur can lock in riskless profit by selling the contingent claim to receive $V(Y)$ and use $V_+(Y)$ to construct the replicating portfolio that generates the attainable $\tilde{Y}$. The upfront positive gain is $V(Y) - V_+(Y)$.

How to solve for $V_+(Y)$ and $V_-(Y)$? Based on linear programming duality theory, we have the following results.

If $R \neq \phi$, then for any contingent claim $Y$, we have

$$V_+(Y) = \sup \{ E_Q[Y^*] : Q \in R \},$$

$$V_-(Y) = \inf \{ E_Q[Y^*] : Q \in R \}.$$

If $Y$ is attainable, then $V_+(Y) = V_-(Y)$. 
Example

Consider the securities model: $\tilde{S}(0) = (1 \ 3)$ and $\tilde{S}^*(1; \Omega) = 
\begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}$, and the non-attainable contingent claim $Y^* = 
\begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}$. The risk neutral measure is

$$Q = (\lambda \quad 1 - 2\lambda \quad \lambda), \quad \text{where} \ 0 < \lambda < 1/2.$$  

Note that $E_Q[Y^*] = 4 + \lambda$ so that the upper and lower bounds of the no-arbitrage price are given by

$$V_+ = \sup\{E_Q[Y^*] : Q \in R\} = 9/2 \quad \text{and} \quad V_- = \inf\{E_Q[Y^*] : Q \in R\} = 4.$$

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The attainable contingent claim corresponding to the no-arbitrage price $V_+$ is

$$Y_+^* = \begin{pmatrix} 5 \\ 4.5 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0.5 \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix},$$

while the attainable contingent claim corresponding to the no-arbitrage price $V_-$ is

$$Y_-^* = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}.$$

Any reasonable initial price of the non-attainable contingent claim $Y^* = (5 \ 4 \ 4)^T$ should lie between the interval $(4, 4.5)$. 
Law of one price revisited

Take $\hat{S}^*(1; \Omega) = \begin{pmatrix} 1 & 2 & 6 & 9 \\ 1 & 3 & 3 & 7 \\ 1 & 6 & 12 & 19 \end{pmatrix}$, the sum of the first 3 columns gives the fourth column. The first column corresponds to the discounted terminal payoff of the riskfree security under the 3 possible states of the world. The third risky security is a redundant security.

Let $\hat{S}^*(0) = (1 \ 2 \ 3 \ k)$. We observe that solution to

$$(1 \ 2 \ 3 \ k) = (\pi_1 \ \pi_2 \ \pi_3) \begin{pmatrix} 1 & 2 & 6 & 9 \\ 1 & 3 & 3 & 7 \\ 1 & 6 & 12 & 19 \end{pmatrix}$$

exists if and only if $k = 6$. That is, $S^*_3(0) = S^*_0(0) + S^*_1(0) + S^*_2(0)$.

When $k \neq 6$, the law of one price does not hold. The last equation: $9\pi_1 + 7\pi_2 + 19\pi_3 = k \neq 6$ is inconsistent with the first 3 equations.
We consider the linear system

\[ \hat{S}^*(0) = \pi \hat{S}^*(1; \Omega), \]

solution exists if and only if \( \hat{S}^*(0) \) lies in the row space of \( \hat{S}^*(1; \Omega) \). Uniqueness follows if the rows of \( \hat{S}^*(1; \Omega) \) are independent.

Since

\[ S_3^*(1; \Omega) = S_0^*(1; \Omega) + S_1^*(1; \Omega) + S_2^*(1; \Omega), \]

the third risky security is replicable by holding one unit of each of the riskfree security and the first two risky securities. The initial price must observe the same relation in order that the law of one price holds.

Here, we have redundant securities. Actually, one may show that the law of one price holds if and only if we have existence of solution to the linear system. In this example, when \( k = 6 \), we obtain

\[ \pi = \begin{pmatrix} 1 & 2 \\ 2 & 3 & -1/6 \end{pmatrix}. \]

This is not a risk neutral measure nor a linear pricing measure.
Consider another example

\[
(1 \ 2 \ 3 \ 6) = (\pi_1 \ \pi_2 \ \pi_3) \begin{pmatrix}
1 & 2 & 3 & 6 \\
1 & 3 & 4 & 8 \\
1 & 6 & 7 & 14
\end{pmatrix},
\]

where the number of non-redundant securities is only 2. Note that

\[
S^*_2(1; \Omega) = S^*_0(1; \Omega) + S^*_1(1; \Omega) \quad \text{and}\]

\[
S^*_3(1; \Omega) = S^*_0(1; \Omega) + S^*_1(1; \Omega) + S^*_2(1; \Omega),
\]

and the initial prices have been set such that

\[
S^*_2(0) = S^*_0(0) + S^*_1(0) \quad \text{and} \quad S^*_3(0) = S^*_0(0) + S^*_1(0) + S^*_2(0),
\]

so we expect to have the existence of solution. However, since 2 = number of non-redundant securities < number of states = 3, we do not have uniqueness of solution. Indeed, we obtain

\[
(\pi_1 \ \pi_2 \ \pi_3) = (1 \ 0 \ 0) + t(3 \ -4 \ 1), \quad t \text{ any value.}
\]

For example, when we take \( t = 1 \), then

\[
(\pi_1 \ \pi_2 \ \pi_3) = (4 \ -4 \ 1).
\]
In terms of linear algebra, we have existence of solution if the equations are consistent. Consider the present example, we have

\[
\begin{align*}
\pi_1 + \pi_2 + \pi_3 &= 1 \\
2\pi_1 + 3\pi_2 + 6\pi_3 &= 2 \\
3\pi_1 + 4\pi_2 + 7\pi_3 &= 3 \\
6\pi_1 + 8\pi_2 + 14\pi_3 &= 6
\end{align*}
\]

Note that the last two redundant equations are consistent. Alternatively, we can interpret that the row vector \( \mathbf{S}^*(0) = (1 \ 2 \ 3 \ 6) \) lies in the row space of \( \mathbf{\tilde{S}}^*(1; \Omega) \), which is spanned by the two row vectors: \( \{(1 \ 2 \ 3 \ 6), (0 \ 1 \ 1 \ 2)\} \).

In this securities model, we cannot find a risk neutral measure where \( (Q_1 \ Q_2 \ Q_3) > 0 \). This is easily seen since \( \pi_2 = -4t \) and \( \pi_3 = t \), and they always have opposite sign, except \( t = 0 \). However, a linear pricing measure exists. One such example is given by \( t = 0 \), so \( (q_1 \ q_2 \ q_3) = (1 \ 0 \ 0) \geq 0 \).
Since $Q$ does not exist, the securities model admits arbitrage opportunities. One such example is $h = (-11 \ 1 \ 1 \ 1)^T$, where

$$
S^*(0)h = 0 \quad \text{and} \quad S^*(1; \Omega)h = \begin{pmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 & 8 \\ 1 & 6 & 6 & 14 \end{pmatrix} \begin{pmatrix} -11 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 14 \\ 5 \end{pmatrix}.
$$

However, the securities model does not admit dominant trading strategies since a linear pricing measure exists. This is evidenced by showing that one cannot find a trading strategy $h = (h_0 \ h_1 \ h_2 \ h_3)^T$ such that $h_0 + 2h_1 + 3h_2 + 6h_3 = 0$ while

$$
h_0 + 2h_1 + 3h_2 + 6h_3 > 0, \quad h_0 + 3h_1 + 4h_2 + 8h_3 > 0, \quad h_0 + 6h_1 + 6h_2 + 14h_3 > 0.
$$

The first inequality can never be satisfied when we impose $h_0 + 2h_1 + 3h_2 + 6h_3 = 0$. Indeed, when $S^*(0) = S^*(1; \omega_k)$ for some $\omega_k$, then a linear pricing measure exists, which is given by $q = e^T_k$. 

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Pricing of attainable contingent claims

Let $V_1^*(1; \Omega)$ denote the value of the replicating portfolio that matches with the payoff of the attainable contingent claim at every state of the world. Suppose the associated trading strategy to generate the replicating portfolio is $h$, then

$$V_1^* = \hat{S}^*(1; \Omega)h.$$  

The initial cost of setting up the replicating portfolio is

$$V_0^* = \hat{S}^*(0)h.$$  

Assuming $\pi$ exists, where $\hat{S}^*(0) = \pi \hat{S}^*(1; \Omega)$ so that

$$V_0^* = \pi \hat{S}^*(1; \Omega)h = \pi V_1^*(1; \Omega)$$

$$= \sum_{k=1}^{K} \pi_k V_1^*(1; \omega_k), \text{ independent of } h.$$  

Even when $\pi$ is not a risk neutral measure or linear pricing measure, the above pricing relation remains valid. Suppose $\pi$ is not unique, do we have different values for $V_0^*$?
Using the same $\hat{S}^*(0)$ and $\hat{S}^*(1; \Omega)$ as shown on P.73, we consider the contingent claim \[
\begin{pmatrix}
5 \\
7 \\
13
\end{pmatrix}.
\] The claim is attainable by holding one unit of the first risky security and second risky security. Its price is seen to be

\[
S_1^*(0) + S_2^*(0) = 2 + 3 = 5.
\]

Applying the formula:

\[
V^*(0) = \sum_{k=1}^{3} \pi_k V^*(1; \omega_k)
\]

\[
= 5(1 + 3t) + 7(-4t) + 13t = 5, \text{ independent of } t.
\]

This is not surprising. This is consistent with the law of one price.
Summary \textit{Arbitrage opportunity} 無風險套利機會

An arbitrage strategy is requiring no initial investment, having no probability of negative value at expiration, and yet having some possibility of a positive terminal portfolio value.

- Commonly it is assumed that there are no arbitrage opportunities in well functioning and competitive financial markets.

1. absence of arbitrage opportunities
   $\Rightarrow$ absence of dominant trading strategies
   $\Rightarrow$ law of one price
2. absence of arbitrage opportunities ⇔ existence of risk neutral measure
   absence of dominant trading strategies ⇔ existence of linear pricing measure.

3. The state prices are non-negative when a linear pricing measure exists and they become strictly positive when a risk neutral measure exists.

4. Under the absence of arbitrage opportunities, the risk neutral valuation principle can be applied to find the fair price of an attainable contingent claim.