1. Consider the following one-factor model, where the rate of return \( r_i \) of the risky asset \( i, i = 1, 2 \), is given by
\[
r_i = a_i + b_i f + e_i.
\]
Here, \( f \) is the random factor, \( a_i \) and \( b_i \) are constant parameters and \( e_i \) is the random residual. We assume \( E[e_1] = E[e_2] = 0 \).

(a) Under the assumption of \( \text{cov}(e_1, f) = \text{cov}(e_2, f) = \text{cov}(e_1, e_2) = 0 \) for asset 1 and asset 2, find \( \text{cov}(r_1, r_2) \) in terms of \( b_1, b_2 \) and \( \sigma_f^2 \), where \( r_1 \) and \( r_2 \) are the random rates of return of asset 1 and asset 2, \( \sigma^2_f \) is the variance of \( f \).

(b) Given \( a_1 = 0.10, b_1 = 2, a_2 = 0.08 \) and \( b_2 = 1 \), and assuming \( E[f] = e_1 = e_2 = 0 \), find the factor risk premium \( \lambda \) and the return of the zero-beta portfolio \( \lambda_0 \). How to construct the zero-beta portfolio from these two risky assets?

Hint: \( r_i = E[r_i] = \lambda_0 + b_i \lambda, \quad i = 1, 2 \).

2. Consider the choice set
\[
B = \{(x, y) : x \in [0, \infty) \quad \text{and} \quad y \in [0, \infty)\}
\]
and define the following preference solution \( \succeq \):

For any \((x_1, y_1)\) and \((x_2, y_2)\) in \( B \),
\[
(x_1, y_1) \succeq (x_2, y_2) \text{ if and only if } [x_1 > x_2] \text{ or } [x_1 = x_2 \text{ and } y_1 \geq y_2].
\]

Consider the Order Preserving Axiom, Intermediate Value Axiom and Boundedness Axiom, does the above preference relation satisfy each of the above axioms? Give detailed explanation to your answer.

3. Consider the investment wheel problem with \( m \) sectors whose return vector is given by \( \mathbf{R}(\omega_i) = (0 \cdots a_i \cdots 0)^T \), with \( a_i > 0 \) and \( \omega_i \) corresponds to the event where the pointer falls on the \( i^{\text{th}} \) sector, \( i = 1, 2, \cdots, m \). Let \( p_i, i = 1, 2, \cdots, m \), denote the probability that the pointer lands on the \( i^{\text{th}} \) sector. Let \( w_i, i = 1, 2, \cdots, m \), be the proportion of wealth placed on the \( i^{\text{th}} \) sector, with \( \sum_{i=1}^{m} w_i = 1 \).

(a) Find an optimal betting strategy that maximizes the long-term growth of the capital. Show details of your calculations.
(b) How does the strategy for the long-term growth differ from that of the single-period growth? How would you modify the strategy obtained in part(a) when maximization of the single-period growth is considered?

*Hint:* Use the logarithm utility for maximizing the long-term growth.

4. (a) Suppose all security returns are normal random variables so that for a utility function $U$, the corresponding expected utility value $E[U(y)]$ is a function of $M$ and $\sigma$. Here, $M$ and $\sigma$ are the mean and standard deviation of the random wealth variable $y$. We write

$$
E[U(y)] = \int_{-\infty}^{\infty} U(y) \frac{1}{\sqrt{2\pi}\sigma^2} e^{-(y-M)^2/2\sigma^2} dy = f(M, \sigma).
$$

Explain why $\frac{\partial f}{\partial M} > 0$ and $\frac{\partial f}{\partial \sigma} < 0$. Here, $U(y)$ is concave and an increasing function of $y$.

(b) Suppose we choose the quadratic concave utility function

$$
u(x) = ax - \frac{b}{2}x^2, \quad 0 \leq x \leq a/b, \quad a > 0 \text{ and } b > 0.
$$

Show that

$$
E[\nu(z)] = aE[z] - \frac{b}{2}(E[z])^2 - \frac{b}{2}\text{var}(z),
$$

where $z$ is the random wealth value.

Explain why

(i) for a given value of $E[z]$,

maximizing $E[\nu(z)] \Leftrightarrow$ minimizing $\text{var}(z)$;

(ii) For a given value of $\text{var}(z)$,

maximizing $E[\nu(z)] \Leftrightarrow$ maximizing $E[z]$.

5. Recall that the absolute risk aversion coefficient $A(w)$ is defined by

$$
A(w) = -\frac{u''(w)}{u'(w)},
$$

where $u(w)$ is assumed to be an increasing and concave utility function.

(a) Suppose $\frac{dA(w)}{dw} < 0$, should the investor holds more or less dollars in risky assets when the wealth level $w$ increases? Give an explanation to your answer.

(b) Show that if $\frac{dA(w)}{dw} < 0$, then $u'''(w) > 0$.

(c) Given that $u'(w) > 0$ and $u''(w) < 0$, the certainty equivalent $c$ of the random wealth $X$ as defined by

$$
\nu(c) = E[\nu(X)].
$$

Show that $c$ is less than its mean $E[X]$.

*Hint:* Use the Jensen inequality.
6. Suppose the choice set consists of only two investments $A$ and $B$:

<table>
<thead>
<tr>
<th>Investment $A$</th>
<th>Investment $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>outcome</td>
<td>probability</td>
</tr>
<tr>
<td>5</td>
<td>1/3</td>
</tr>
<tr>
<td>7</td>
<td>1/3</td>
</tr>
<tr>
<td>9</td>
<td>1/3</td>
</tr>
</tbody>
</table>

(a) Find the corresponding probability distribution of Investments $A$ and $B$. Note that the mean of 5 and 7 is the same as the mean of 4 and 8, while the highest outcome from $A$ is less than that from $B$. Does $A$ dominate $B$ by the second order stochastic dominance rule. Give your explanation. \[3\]

(b) Does $A$ dominate $B$ by the first order stochastic dominance rule? Give your explanation. \[2\]

(c) Find the efficient set and inefficient set of the above choice set (which has only two investment choices) under the class of utility functions that are increasing and concave. \[2\]

(d) For the general case (not confined to the choice set in the above), is the efficient set according to first order stochastic dominance a subset of the efficient set according to second order stochastic dominance? If yes, prove it. If not, explain why. \[2\]

7. For each of the following statements, determine whether it is true or false. Provide a proof if it is true and present a counter example if it is false.

(a) Suppose the set of risk neutral measures is non-empty, then the securities model is complete. \[3\]

(b) Suppose redundant securities do not exist, then the securities model has unique risk neutral measure, provided that the set of risk neutral measures is non-empty. \[3\]

(c) Suppose the law of one price holds, then the Arrow securities prices are all positive. \[3\]

8. Consider a one-period securities model with the following initial price vector: 

\[ S(0) = (1 \ 2 \ 4) \] 
and discounted terminal payoff matrix:  
\[ S^*(1) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{pmatrix}. \]

(a) Does the law of one price hold? Do redundant securities exist? Do Arrow securities prices exist? Justify your answers. \[3\]

(b) Find an arbitrage opportunity associated with the above securities model by giving one example that demonstrates arbitrage. \[3\]

(c) How to modify $S(0)$ such that the securities model does not admit arbitrage? Find the corresponding set of risk neutral measures. \[3\]