1. Suppose there are only 2 fully negatively correlated risky assets in the portfolio, whose expected rate of return and variance are $r_i$ and $\sigma_i^2$, respectively, $i = 1, 2$.

(a) Show that it is possible to construct a portfolio that is riskfree. Find this riskfree portfolio and the corresponding expected rate of return. [2]

(b) Next, we assume that these two risky assets now become correlated and let $\rho$ be the coefficient of correlation between their random rates of return, where $-1 < \rho < 1$. Find the corresponding minimum variance portfolio. [3]

2. Suppose there are $N$ risky assets (no riskfree asset) whose covariance matrix of their random rates of return is denoted by $\Omega$. Let $\mu$ denote the vector of expected rates of return of these $N$ risky assets.

(a) Find the global minimum variance portfolio, $g$. [2]

(b) Let $P$ be any portfolio of these $N$ risky assets. Show that

$$\text{cov}(r_g, r_P) = \text{var}(r_g),$$

where $r_g$ and $r_P$ are the random rates of return of $g$ and $P$, respectively. [3]

(c) Suppose $\Omega$ is singular ($\Omega^{-1}$ does not exist). Show that the minimum variance portfolio has zero variance. Find the corresponding riskfree rate of return of this zero-variance portfolio. [3]

3. Consider the universe of $N$ risky assets and one riskfree asset whose riskfree rate of return is $r$. Let $\Omega$ and $\mu$ be the covariance matrix and expected rate of return vector of the $N$ risky assets.

(a) Explain why the Market Portfolio does not exist when $r = b/a$, where $a = 1^T \Omega^{-1} 1$ and $b = \mu^T \Omega^{-1} 1$. State another case where the Market Portfolio does not exist. [3]

(b) Under the scenario of $r = b/a$, show that the efficient frontier in the $\sigma_P - \mu_P$ diagram is given by

$$\mu_P = r + \sqrt{\frac{\Delta}{a}} \sigma_P,$$

where $\Delta = ac - b^2$, $c = \mu^T \Omega^{-1} \mu$. [5]

4. The formulation of the risk tolerance model is given by

$$\text{maximize } \tau \mu_P - \frac{\sigma_P^2}{2}, \text{ with } \tau \geq 0, \text{ subject to } 1^T w = 1.$$

Here, $\tau$ is the risk tolerance factor and $w$ is the portfolio weight vector.
(a) Show that the optimal portfolio weight vector is given by

\[ \mathbf{w}^* = \mathbf{w}_g + \tau \mathbf{z}^*, \]

where \( \mathbf{w}_g \) is the portfolio weight vector of the global minimum variance portfolio and \( \mathbf{z}^* \) is a vector whose sum of components is zero. Find \( \mathbf{z}^* \) explicitly. \[4\]

(b) Let \( \mu_P \) be the expected rate of return of the optimal portfolio. Show that

\[ \tau = \frac{a}{\Delta} \left( \mu_P - \frac{b}{a} \right), \]

where \( a = \mathbf{1}^T \Omega^{-1} \mathbf{1}, \ b = \mathbf{\mu}^T \Omega^{-1} \mathbf{1}, \ c = \mathbf{\mu}^T \Omega^{-1} \mathbf{\mu} \) and \( \Delta = ac - b^2 \). \[3\]

5. Consider the zero-beta Capital Asset Pricing Model (CAPM):

\[ \tau_Q = \tau_{Z_M} + \beta_{QM} (\tau_M - \tau_{Z_M}), \]

where \( Q \) is any portfolio and \( Z_M \) is the uncorrelated counterpart of the Market Portfolio \( M \).

(a) Illustrate the method of constructing \( Z_M \) on a graphical plot of the \( \mu_P - \sigma_P \) diagram. Explain the rationale behind the construction. \[3\]

(b) We may choose any efficient fund \( P \) other than the Market Portfolio \( M \) in the zero-beta version of the CAPM. Explain the rationale behind the CAPM formula:

\[ \tau_Q = \tau_{Z_P} + \beta_{QP} (\tau_P - \tau_{Z_P}), \]

where \( Z_P \) is the uncorrelated counterpart of \( P \). \[2\]

6. (a) Show that

\[ \sigma_P^2 = \beta_{PM}^2 \sigma_M^2 + \frac{\bar{\sigma}^2}{n}, \]

where \( \bar{\sigma}^2 \) is the average of the residual risks of the \( n \) risky assets. Here, \( \sigma_P^2 \) is the portfolio variance, \( \sigma_M^2 \) is the variance of the Market Portfolio \( M \), and \( \beta_{PM} \) is the beta value. State the underlying consumptions in the derivation of the above formula. \[5\]

Hint: Consider

\[ r_i - r = \beta_{iM} (r_M - r) + \epsilon_i, \quad i = 1, 2, \ldots, n, \]

where \( \text{var}(\epsilon_i) \) is the residual risk. Explain why \( \text{cov}(\epsilon_i, \epsilon_j) \approx 0 \). Take the weight of each asset to be equal.

(b) Explain why non-systematic risk is also called diversifiable risk. \[2\]