

Mathematics and Social Choice Theory

Topic 1 – Voting systems and power indexes

- 1.1 Weighted voting systems and yes-no voting systems
- 1.2 Power indexes: Shapley-Shubik index and Banzhaf index
- 1.3 Case studies of power indexes calculations
- 1.4 Probabilistic characterizations of power indexes

1.1 Weighted voting systems and yes-no systems

Voting procedure: “support” or “object” a given motion/bill (no “abstain”).

Weighted voting system – individuals/political bodies can cast more ballots than others. For example, voting by stockholders in a corporation, more votes being held by countries with stronger economic powers in the International Monetary Authority.

Question: How to define voting power in a weighted voting system?

- Power is not a trivial function of one’s strength as measured by his number of votes.
- Voters with pivotal though small number of votes may have higher influential power per each vote held.

Define two indices that indicate the real distribution of influence.

- Useful in evaluating some of the existing democratic institutions in terms of fairness, concealed biases, etc.

Remark

No mathematical theory will suffice to reveal all of the behind-the-scene nuances - influences of party loyalty, persuasion, lobbying, fractionalism, bribes, gratuities, campaign financing, etc.

A new field called GOVERNMENTRICS 計量管治學

Weighted majority voting game is characterized by a voting vector

$$[q; w_1, w_2, \dots, w_n]$$

where there are n voters, w_i is the voting weight of player i ; $N = \{1, 2, \dots, n\}$ be the set of all n voters; q is the quota (minimum number of votes required to pass a bill).

Let S be a typical coalition, which is a subset of N . A coalition wins a bill (called winning) whenever

$$\sum_{i \in S} w_i \geq q.$$

It is natural to observe “complement of a winning coalition should be losing”. As a result, we require the quota to observe $q > \frac{1}{2} \sum_{i \in N} w_i$ in order to avoid two disjoint coalitions that are both winning.

Examples

1. $[51; 28, 24, 24, 24]$; the first voter is much stronger than the last 3 since he needs only one other to pass an issue, while the other three must all combine in order to win.
2. $[51; 26, 26, 26, 22]$, the last player seems powerless since any winning coalition containing him can just as well win without him (a dummy).
3. In the game $[q; 1, 1, \dots, 1]$, each player has equal power. This is called a pure bargaining game.
4. $[51; 40, 30, 20, 10]$ and $[51; 30, 25, 25, 20]$ seem identical in terms of voting influence, since the same set of coalitions are winning in both voting vectors.

5. Games such as $[3; 2, 2, 1]$, $[8; 7, 5, 3]$ and $[51; 49, 48, 3]$ are identical to $[2; 1, 1, 1]$ in terms of power, since they give rise to the same collection of winning coalitions.
6. If we add to the game $[3; 2, 1, 1]$ the rule that player 2 can cast an additional vote in the case of 2 to 2 tie, then it is effectively $[3; 2, 2, 1]$.
If player 1 can cast the tie breaker, then it becomes $[3; 3, 1, 1]$ and he is the dictator. He forms a winning coalition by himself.
7. In the game $[50(n - 1) + 1; 100, 100, \dots, 100, 1]$, the last player has the same power as the others when n is odd; the game is similar to one in which all players have the same weights. For example, when $n = 5$, we have $[201; 100, 100, 100, 100, 1]$. Any 3 of the 5 players can form a winning coalition.

Dummy players

Any winning coalition that contains such an impotent voter could win just as well without him.

Examples

- Player 4 in $[51; 26, 26, 26, 22]$.
- Player n in $[50(n - 1) + 1; 100; 100, \dots, 100, 1]$ is a dummy when n is even. For example, take $n = 4$, we have $[151; 100, 100, 100, 1]$. Obviously, the last player is a dummy.
- In $[10; 5, 5, 5, 2, 1, 1]$, the 4th player with 2 votes is a dummy. The 5th and 6th players with only one vote are sure to be dummies. The collection of dummies remains to be a dummy collection. This is because one cannot turn a losing coalition into a winning coalition by adding a dummy one at a time.

Example

Consider $[16; 12, 6, 6, 4, 3]$, player 5 with 3 votes is a dummy since no subset of the numbers 12, 6, 6, 4 sums to 13, 14 or 15. Therefore, player 5 could never be pivotal in the sense that by adding his vote a coalition would just reach or surpass the quota of 16.

Example

If we add the 8th player with one vote into $[15; 5, 5, 5, 5, 2, 1, 1]$ so that the new game becomes $[15; 5, 5, 5, 5, 2, 1, 1, 1]$, the 4th player in the new voting game is not a dummy since sum of votes of some coalition may assume the value of 13.

Notion of Power

- The index should indicate one's relative influence, in some numerical way, to bring about the *passage or defeat of some bill*.
- The index should depend upon the number of players involved, on one's fraction of the total weight, and upon how the remainder of the weight is distributed (a critical swing-man that causes a desired outcome).
- A winning coalition is said to be *minimal winning* if no proper subset of it is winning. A coalition that is not winning is called *losing*. Technically, the one who is the 'last' to join a minimal winning coalition is particularly influential.
- A voter i is a dummy if every winning coalition that contains him is also winning without him, that is, he is in no minimal winning coalition. A dummy has ZERO power.

Example

Party	Leader	No. of seats
Liberals	Pierre E. Trudeau	109
Tories	Robert L. Stanfield	107
New Democrats	David Lewis	31
Others		17

- Though Liberals has the largest number of seats, none of the parties has the majority. Any two of the three leading parties can form a coalition and obtain the majority, so the first three parties had equal power. Other small parties are all dummies.

Veto Power and Dictators

A player or coalition is said to have *veto power* if no coalition is able to win a ballot without his or their consent. A subset S of voters is a blocking coalition or has *veto power* if and only if its complement $N - S$ is not winning. We also require that S itself is losing in order to be a blocking coalition, otherwise S is a dictatorial coalition (see “dictator” below).

Given that $q > 50\%$ of total votes, a player i is a dictator if he forms a winning coalition $\{i\}$ by himself.

- If the dictator says “yes”, then the bill is passed. If the dictator says “no”, then the bill is not passed (veto power).
- If a dictator exists, then all other players are dummies.

Example

Player 1 has veto power in $[51; 50, 49, 1]$ and $[3; 2, 1, 1]$. In the last case, if he is the chairman with additional power to break ties, then the game becomes $[3; 3, 1, 1]$ and now he becomes a dictator.

Example

The ability of an individual to break tie votes in the pure bargaining game

$$\begin{cases} \left[\frac{n}{2} + 1; 1, 1, \dots, 1 \right] & \text{when } n \text{ is even} \\ \left[\frac{n+1}{2}; 1, 1, \dots, 1 \right] & \text{when } n \text{ is odd} \end{cases}$$

adds power when n is even and adds nothing when n is odd. Actually, when n is odd, tie votes will not occur.

Properties on dummies

A collection of dummies can never turn a losing coalition into a winning coalition. 如足夠多傀儡聯合起來, 這團隊可否改變傀儡地位

In other words, it is not possible that $S \cup \{D_1, \dots, D_m\}$ is winning but S is losing since the dummies can be successively deleted while the coalition remains to be winning.

Corollary

If both “ d ” and “ ℓ ” are dummies, then the coalition $\{d, \ell\}$ is dummy.

Theorem

In a weighted voting game, let “ d ” and “ ℓ ” be two voters with votes x_d and x_ℓ , respectively. Suppose “ d ” is a dummy and $x_\ell \leq x_d$, then “ ℓ ” is also a dummy.

Proof

Assume the contrary. Suppose “ ℓ ” is not a dummy, then there exists a coalition S that does not contain the dummy “ d ” such that S is losing but $S \cup \{\ell\}$ is winning. As a remark, we just require S not to contain “ d ” while S may contain dummies other than “ d ”. Now, $n(S) < q$ while $n(S \cup \{\ell\}) \geq q$. Since $x_\ell \leq x_d$, so $n(S \cup \{d\}) \geq q$, contradicting that “ d ” is a dummy.

Corollary

If the coalition $\{d, \ell\}$ is dummy, then both “ d ” and “ ℓ ” are dummies. This is obvious since $n(\{d, \ell\}) \geq \max(x_d, x_\ell)$.

Yes-no voting system

A yes–no voting system is simply a set of rules that specify exactly which collections of “yes” votes yield passage of a bill.

Under what condition that a yes–no voting system is a weighted system (with weights assigned for the voters and quota)?

Example

Bill to be passed: Grades of this course are based on “absolute grading”.

Set of players, $N = \{\text{professor, tutor, Chan, Lee, Cheung, Wong, Ho}\}$.

Yes–no rule requires professor, at least one from “tutor and Chan” and number of students must be at least 3. Each of the following player/coalition, “professor” and “tutor and Chan”, has veto power.

United States federal system

- 537 voters in this system: 435 House of Representatives, 100 Senate members, Vice-President and President.
- Vice President plays the role of tie breaker in the Senate
- President has veto power that can be overridden by a two-thirds vote of both the House and the Senate.

To pass a bill, it must be supported by either one of the following three sets of votes

1. 218 or more representatives and 51 or more senators and President
2. 218 or more representatives and 50 senators and both Vice President and President
3. 290 or more representatives and 67 or more senators.

System to amend the Canadian constitution (since 1982)

- In addition to the House of Commons and the Senate, approval by two-thirds majority of provincial legislatures, that is, at least 7 of the 10 Canadian provinces subject to the proviso that the approving provinces have among them at least half of Canada's population.
- Based on 1961 census
 - Prince Edward Island (1%)
 - Newfoundland (3%)
 - New Brunswick (3%)
 - Nova Scotia (4%)
 - Manitoba (5%)
 - Saskatchewan (5%)
 - Alberta (7%)
 - British Columbia (9%)
 - Quebec (29%)
 - Ontario (34%)

Swap robustness and trade robustness

Definition (swap robust)

A yes–no voting system is said to be swap robust if a “swap” of one player for one between two winning coalitions leaves at least one of the two coalitions winning.

- Start with two arbitrary winning coalitions X and Y
 - arbitrary player x in X (not in Y) and arbitrary player y in Y (not in X)
 - let X' and Y' be the result of exchanging x and y
 - either X' or Y' is winning for swap robustness

Proposition

Every weighted voting system is swap robust.

Proof

1. If x and y have the same weight, then both X' and Y' are winning.
2. If x is heavier than y , then Y' weighs strictly more than Y . The weight of Y' certainly exceeds the quota, and thus Y' is winning.
3. If y is heavier than x , then a similar argument as in (2) holds.

To show a voting system to be not swap robust, we produce two winning coalitions X and Y and a trade between them that renders both losing. Intuitively, X and Y should both be “almost losing” and make both actually losing by a one-for-one trade. We seek for the appropriate coalitions X and Y among the *minimal winning coalitions*.

Proposition

The US federal system is not swap robust.

Proof

$X = \{\text{President, 51 shortest senators and 218 shortest House Representatives}\}$

$Y = \{\text{President, 51 tallest senators and 218 tallest House Representatives}\}$

Let x be shortest senator and y be the tallest House Representative.

Both X and Y are winning coalitions; $x \in X$ but $x \notin Y$; $y \in Y$ but $y \notin X$. After swapping, X' is a losing coalition since it has only 50 senators and Y' is a losing coalition because it has only 217 Representatives.

Corollary The US federal system is not a weighted voting system.

Proposition

The procedure to amend the Canadian Constitution is swap robust (later shown to be not weighted).

Proof

Suppose X and Y are winning coalitions, $x \in X, x \notin Y$ while $y \in Y$ but $y \notin X$. We must show that at least one of X' and Y' is still a winning coalition. That is, at least one of X' and Y' still satisfies both conditions:

- (i) It contains at least 7 provinces.
- (ii) The provinces represent at least half of the Canadian population.

The first condition is obvious. If x has more population than y , then Y' is a winning coalition. Similar argument when x has a smaller population than y .

Definition (trade robust)

A yes–no voting system is said to be trade robust if an arbitrary exchange of players (a series of trades involving groups of players) among several winning coalitions leaves at least one of the coalitions winning.

1. The exchanges of players are not necessarily one-for-one as they are in swap robustness.
2. The trades may involve more than two coalitions.

For example, consider 3 winning coalitions X , Y and Z .

- (i) 3 players are exchanged out from X , 2 to Y and 1 to Z ;
- (ii) 4 players are exchanged out from Y , 1 to X and 3 to Z ;
- (iii) 3 players are exchanged out from Z , 2 to X and 1 to Z ;

Proposition

Every weighted voting system is trade robust.

Proof

- A series of trades among several winning coalitions leaves the total weight of all these coalitions added together unchanged. Hence, the average weight of these coalitions is unchanged.
- Suppose we start with several winning coalitions in a weighted voting system, then their average weight at least meets quota. After the trades, the average weight of the coalitions is unchanged and at least meets quota. Hence, at least one of the coalitions must meet quota (at least one of the resulting coalitions is winning).

Proposition The procedure to amend the Canadian Constitution is not trade robust.

Proof

X	Y
Prince Edward Island (1%)	New Brunswick (3%)
Newfoundland (3%)	Nova Scotia (4%)
Manitoba (5%)	Manitoba (5%)
Saskatchewan (5%)	Saskatchewan (5%)
Alberta (7%)	Alberta (7%)
British Columbia (9%)	British Columbia (9%)
Quebec (29%)	Ontario (34%)

- Let X' and Y' be obtained by trading Prince Edward Island and Newfoundland for Ontario.
- X' is a losing coalition because it has too few provinces (having given up two provinces in exchange for one).
- Y' is a losing coalition because the eight provinces in Y' represent less than half of Canada's population.

Corollary

The procedure to amend the Canadian Constitution is not a weighted voting system.

Theorem (proof omitted)

A yes–no voting system is weighted if and only if it is trade robust.

Minority veto

For example, we have a majority group of 5 voters and a minority group of 3 voters. The passage requires not only approval of at least 5 out of the 8 voters, but also approval of at least 2 of the 3 minority voters. This system is swap robust but not trade robust.

Hint Consider how to make one coalition to have more total number of voters but do not have enough minority voters while the other coalition has less total number of voters but more minority voters after the trades.

Illustration

Majority group of 5 voters minority group of 3 voters

To form a winning coalition, we must have

- at least 5 out of 8 total voters
- at least 2 out of 3 minority voters

$$X = \{M_1, M_2, M_3, m_1, m_2\}$$

$$Y = \{M_3, M_4, M_5, m_2, m_3\}$$

$$X' = \{M_1, M_2, M_3, M_4, M_5, m_2\}$$

$$Y' = \{M_3, m_1, m_2, m_3\}$$

Both X & Y are winning coalitions

Both X' & Y' are losing coalitions

Hence, the system is NOT trade robust since X' does not satisfy the minority requirement and Y' does not satisfy the “total votes” requirement.

Swap robustness is easily seen since the requirement of 5 votes out of 8 voters is satisfied under one-for-one trade and the minority requirement is always satisfied by at least one of the coalitions after one-for-one trade.

Intersection of weighted voting systems and dimension theory

The procedure to amend the Canadian Constitution can be constructed by “putting together” two weighted systems.

- W_1 = collection of coalitions with 7 or more provinces
- W_2 = collection of coalitions representing at least half of Canada’s population

A coalition is winning if and only if it is winning in both System I and System II.

$$W = W_1 \cap W_2.$$

Question How to construct a non-weighted system in terms of intersection of weighted ones?

Dimension of a yes-no voting system

A yes–no voting system is said to be dimension k if and only if it can be represented as the intersection of exactly k weighted voting systems, but not as the intersection of $k - 1$ weighted voting systems.

For example, the procedure to amend the Canadian Constitution is of dimension 2 since the passage requires two separate weighted voting systems with regard to “population” and “provinces”.

Proposition

The US federal system has dimension 2.

System I: quota = 67

System II: quota = 290

weight 0 to House Representative

weight 1 to House Representative

weight 1 to Senator

weight 0 to Senator

weight 0.5 to Vice President

weight 0 to Vice President

weight 16.5 to President

weight 72 to President

System I is meant for the Senate (veto power of the President and tie breaker role of the Vice President; sum of weights of P & VP = 17, weight of $P \geq 16$ and weight of VP > 0). System II is meant for the House, with veto power of the President.

Let X be a minimal winning coalition. Then X is one of the following 3 kinds of coalitions:

1. X consists of 218 House members, 51 Senators and the President
2. X consists of 218 House members, 50 Senators, the Vice President and the President
3. X consists of 290 House members and 67 Senators.

All these 3 kinds of coalitions achieve quota in both Systems I and II.

Conversely, how to find the minimal winning coalitions that satisfy both weighted voting systems?

Hint (With the President)

In System I, weight of the President is 16.5, so the other members of X must contribute at least weight 50.5 to the total System I. X must contain either 51 (or more) senators or at least 50 senators and the Vice President.

Looking at System II, which is at least 290 including the 72 contributed by the President. So X must also contain at least $290 - 72 = 218$ House Representatives.

Proposition

Suppose S is a yes–no voting system for the set V of voters, and let m be the number of losing coalitions in S . It is then possible to find m weighted voting systems with the same set V of voters such that a coalition is winning in S if and only if it is winning in every one of these m weighted systems.

Proof

For each losing coalition L in S , we construct an associated weighted voting system. Let $|L|$ be the number of voters in L . Every voter in L is given weight -1 . Every voter not in L is given weight 1 . Quota is set at $-|L| + 1$. The possibility that a coalition is losing if and only if it contains exactly all the voters in this particular losing coalition L . Then every other coalition is a winning coalition in this weighted voting system (including any proper subset of L).

- If a coalition is winning in S , then it is winning in each of these weighted systems.
- Conversely, if a coalition is winning in each of these weighted systems, then it is a winning coalition in S (a losing coalition in S must lose in one of these weighted voting systems).

Unfortunately, this procedure is enormously inefficient since there may be too many losing coalitions.

Puzzle: Since we allow negative values for votes and quota, every proper subset of the losing coalition L also wins in the weighted voting system associated with L .

Alternative proof

- Define a maximal losing coalition to be a losing coalition that becomes winning with the addition of any one voter outside the losing coalition (implicitly all dummies are included in any of these maximal losing coalitions). Any subset of a maximal losing coalition is also losing.
- We identify all these distinct maximal losing coalitions as L_1, \dots, L_m .
- For each maximal losing coalition L_i , we construct a weighted voting system. Let $|L_i|$ be the number of voters in L_i . Every voter in L_i is assigned weight 1, while every voter not in L_i is assigned weight $|L_i| + 1$. The quota is set at $|L_i| + 1$. A subset of L_i is losing under the associated weighted system since the total weight is less than the quota $|L_i| + 1$.
- Under the intersection of these weighted voting systems:
 - coalition C in S is losing (one of these L_i 's or a subset of some L_i)
 \Leftrightarrow coalition C is losing in at least one of these weighted voting system

Example – Yes-no voting system to decide on grading policy

The professor, tutor and 33 students vote to decide whether “absolute grading” should be applied in assigning the grades of this course.

Voting rule 1

1. The professor, tutor and a particular student A each has veto power.
2. To pass the proposal, the voting rule requires yes votes from the professor, tutor, student A and at least 20 other students.

Note that all winning coalitions must contain the professor, tutor and student A . Any arbitrary trades among the winning coalitions involve only the remaining 32 students.

To show trade robustness, since there are at least 20 other students in the winning coalitions on average, this average remains unchanged after any series of arbitrary trades. There must exist a coalition with at least 20 other students after the trades. This represents a winning coalition.

Hence, the yes-no voting system is trade robust, so it can be represented as a weighted voting system. To find the weighted voting vector, we let w represent the weight of each of the other students is one. The quota q must observe

$$2w + 32 < q \quad \text{and} \quad 3w + 20 \geq q$$

so that $3w + 20 > 2w + 32 \Leftrightarrow w > 12$. Suppose we pick $w = 13$, then

$$59 = 3 \times 13 + 20 \geq q > 2 \times 13 + 32 = 58 \quad \text{so that} \quad q = 59.$$

A possible representation of the weighted vector is $[59; 13, 13, 13, 1, 1, \dots, 1, 1]$.

Voting rule 2

We modify voting rule 1 on setting “veto power”. Now the veto power is held by the professor or “tutor and student A” together. That is, tutor and student A each does not hold veto power individually. To pass the proposal, the new rule requires yes votes from at least one of tutor and student A, and 20 other students.

It is easily seen that the new voting rule is NOT swap robust. To show the claim, we consider the following pair of winning coalitions:

$$X_1 = \{P, T, S_1, S_2, \dots, S_{20}\} \quad \text{and} \quad X_2 = \{P, A, S_{13}, S_{14}, \dots, S_{32}\}.$$

Suppose we transfer S_1 from X_1 to X_2 and A from X_2 to X_1 , the two remaining coalitions are both losing. Therefore, this yes-no system is not a weighted voting system.

However, it can be represented as the intersection of 3 weighted voting systems.

System W_1

$$q_1 = 1$$

$$w_P = 1, w_T = w_A = 0$$

$$w_{S_1} = w_{S_2} = \dots = w_{S_{32}} = 0$$

System W_2

$$q_2 = 1$$

$$w_P = 0, w_T = w_A = 1$$

$$w_{S_1} = w_{S_2} = \dots = w_{S_{32}} = 0$$

System W_3

$$q_3 = 20$$

$$w_P = w_T = w_A = 0$$

$$w_{S_1} = w_{S_2} = \dots = w_{S_{32}} = 1$$

$$W = W_1 \cap W_2 \cap W_3$$

1.2 Power indexes

Shapley-Shubik power index

1. One looks at all possible orderings of the n players, and consider this as all of the potential ways of building up toward a winning coalition. For each one of these permutations, some unique player joins and thereby turns a losing coalition into a winning one, and this voter is called the *pivot*.
2. In the sequence of player $x_1, x_2, \dots, x_{i-1}, x_i, \dots, x_n$, $\{x_1, x_2, \dots, x_i\}$ is a winning coalition but $\{x_1, x_2, \dots, x_{i-1}\}$ is losing, then i is in the *pivotal position*.
3. What is the probability that a particular voter is the pivot? The expected frequency with which a voter is the pivot, over all possible orderings of the voters, is taken to be a good indication of his voting power.

Example – 4-player weighted voting game

The 24 permutations of the four players 1, 2, 3 and 4 in the weighted majority game $[51; 40, 30, 20, 10]$ are listed below. The “*” indicates which player is pivotal in the corresponding ordering.

1 2*3 4	2 1*3 4	3 1*2 4	4 1 2*3
1 2*4 3	2 1*4 3	3 1*4 2	4 1 3*2
1 3*2 4	2 3 1*4	3 2 1*4	4 2 1*3
1 3*4 2	2 3 4* 1	3 2*4 1	4 2*3 1
1 4 2* 3	2 4 1* 3	3 4 1* 2	4 3 1* 2
1 4 3* 2	2 4 3* 1	3 4 2* 1	4 3 2* 1

For Player 1 winning coalitions consisting of 2 players.

winning coalitions consisting of 3 players.

Shapley-Shubik power index for the i^{th} player is

$$\phi_i = \frac{\text{number of sequences in which player } i \text{ is a pivot}}{n!}$$

and we write $\phi = (\phi_1, \dots, \phi_n)$.

Here, we assume that each of the $n!$ alignments is *equiprobable*.

The power index can be expressed as

$$\phi_i = \sum \frac{(s-1)!(n-s)!}{n!} \quad \left(\text{with } \sum_{i \in N} \phi_i = 1 \right)$$

where $s = |S|$ = number of voters in set S . The summation is taken over all winning coalitions S for which $S - \{i\}$ is losing.

Counting permutations for which a player is pivotal in achieving minimal winning coalitions

- Player 1 is pivotal in 3 coalitions (namely $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$) consisting of 3 players and in 2 coalitions (namely $\{1, 2\}$, $\{1, 3\}$) consisting of two players.

$$\phi_1 = 3 \frac{(3-1)!(4-3)!}{4!} + 2 \frac{(2-1)!(4-2)!}{4!} = \frac{10}{24}.$$

- For player 2, she is pivotal in $\{1, 2, 3\}$ and $\{1, 2, 4\}$ with 3 players and $\{1, 2\}$ with 2 players. Therefore

$$\phi_2 = 2 \frac{(3-1)!(4-3)!}{4!} + \frac{(2-1)!(4-2)!}{4!} = \frac{6}{24}.$$

We have

$$\phi = \frac{(10, 6, 6, 2)}{24}.$$

Example – A bloc versus singles

Suppose we have n players and that a single block of size b forms. Consider the resulting weighted voting system: $[q; b, \underbrace{1, 1, \dots, 1}_{n-b \text{ of these}}]$.

- $n - b + 1$ is just the number of distinct orderings. The b bloc will be pivotal precisely when the initial sequence of ones is of length at least $q - b$ but not more than $q - 1$.
- The b bloc is pivotal when the initial sequence of ones is any of the following lengths:

$$q - 1, q - 2, \dots, q - b.$$

Note that there are $n - b$ ones available, so the above statement is valid provided that $n - b \geq q - 1$ and $q > b$ (which is equivalent to $b < q \leq n - b + 1$).

- Under the assumption of $b < q \leq n - b + 1$, there are b possible initial sequences of ones that make the bloc pivotal, so

$$\begin{aligned} & \text{Shapley-Shubik index of the block of size } b \\ &= \frac{\text{number of orderings in which } b \text{ is pivotal}}{\text{total number of distinct orderings}} \\ &= \frac{b}{n - b + 1}. \end{aligned}$$

The Shapley-Shubik index is higher than the percentage of votes of b/n .

The formation of a block increases the voting power.

Banzhaf index

- Consider all significant combinations of “yes” or “no” votes, rather than permutations of the players as in the Shapley-Shubik index.
- A player is said to be marginal, or a swing or critical, in a given combination of “yes” and “no” if he can change the outcome.
- Let b_i be the number of voting combinations in which voter i is marginal; then $\beta_i = \frac{b_i}{\sum b_i}$.

Assuming that all voting combinations are equally probable.

The game is $[51; 40, 30, 20, 10]$. For the second case, if Player 1 changes from Y to N , then the outcome changes from “Pass” to “Fail”.

Computation of the Banzhaf Index									
Players				Pass/Fail		Marginal			
1	2	3	4	P	F	1	2	3	4
Y	Y	Y	Y	P					
Y	Y	Y	N	P		X			
Y	Y	N	Y	P		X	X		
Y	N	Y	Y	P		X		X	
N	Y	Y	Y	P			X	X	X
Y	Y	N	N	P		X	X		
Y	N	Y	N	P		X		X	

<i>N</i>	<i>Y</i>	<i>Y</i>	<i>N</i>	<i>F</i>	<i>X</i>		<i>X</i>
<i>Y</i>	<i>N</i>	<i>N</i>	<i>Y</i>	<i>F</i>		<i>X</i>	<i>X</i>
<i>N</i>	<i>Y</i>	<i>N</i>	<i>Y</i>	<i>F</i>	<i>X</i>		<i>X</i>
<i>N</i>	<i>N</i>	<i>Y</i>	<i>Y</i>	<i>F</i>	<i>X</i>	<i>X</i>	
<i>Y</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>F</i>		<i>X</i>	<i>X</i>
<i>N</i>	<i>Y</i>	<i>N</i>	<i>N</i>	<i>F</i>	<i>X</i>		
<i>N</i>	<i>N</i>	<i>Y</i>	<i>N</i>	<i>F</i>	<i>X</i>		
<i>N</i>	<i>N</i>	<i>N</i>	<i>Y</i>	<i>F</i>			
<i>N</i>	<i>N</i>	<i>N</i>	<i>N</i>	<i>F</i>			

$$24 \times \beta = (10, 6, 6, 2)$$

Looking at *YYNN* (pass) and *NYNN* (fail), Player 1 can serve as the defector who gives the swing from Pass to Fail in the first case and Fail to Pass in the second case. We expect that the number of swings of winning into losing effected by a particular player is the same as the number of swings of losing into winning by the same player.

Example

Sometimes symmetry can save us writing out all $n!$ orderings. For example, consider the weighted majority game

$$[5; 3, 2, 1, 1, 1, 1].$$

Since the “1” players are all alike, we need to write out only $6 \cdot 5 = 30$ distinct orderings (instead of $6! = 720$):

3 <u>2</u> 1111	2 <u>3</u> 1111	21 <u>3</u> 111	211 <u>3</u> 11	211 <u>1</u> 31	211 <u>1</u> 13
31 <u>2</u> 111	13 <u>2</u> 111	12 <u>3</u> 111	121 <u>3</u> 11	121 <u>1</u> 31	121 <u>1</u> 13
31 <u>1</u> 211	13 <u>1</u> 211	11 <u>3</u> 211	112 <u>3</u> 11	112 <u>1</u> 31	112 <u>1</u> 13
31 <u>1</u> 121	13 <u>1</u> 121	11 <u>3</u> 121	111 <u>3</u> 21	111 <u>2</u> 31	111 <u>2</u> 13
31 <u>1</u> 112	13 <u>1</u> 112	11 <u>3</u> 112	111 <u>3</u> 12	111 <u>1</u> 32	111 <u>1</u> 23

Notice that the 1's pivot 12/30 of the time, but since there are four of them, each 1 pivots only 3/30 of the time. We get

$$\begin{aligned}\text{Shapley-Shubik index} &= \phi = \left(\frac{12}{30}, \frac{6}{30}, \frac{3}{30}, \frac{3}{30}, \frac{3}{30}, \frac{3}{30} \right) \\ &= (0.4, 0.2, 0.1, 0.1, 0.1, 0.1).\end{aligned}$$

Power as measured by the Shapley-Shubik index in a weighted voting game is *not* proportional to the number of votes cast. For instance, the player with $3/9 = 33\frac{1}{3}\%$ of the votes has 40% of the power.

Use the same game for the computation of the *Banzhaf index*

Types of winning coalitions with	Number of ways this can occur	Number of swings for		
		3	2	1
5 votes: <u>32</u>	1	1	1	
<u>311</u>	$6 = {}_4C_2$	6		12
<u>2111</u>	$4 = {}_4C_3$		4	12
6 votes: <u>321</u>	$4 = {}_4C_1$	4	4	
<u>3111</u>	$4 = {}_4C_3$	4		
<u>21111</u>	$1 = {}_4C_4$		1	
7 votes: <u>3211</u>	$6 = {}_4C_2$	6		
<u>31111</u>	$1 = {}_4C_4$	1		
		22	10	24

We do not need to include those winning coalitions of 8 or 9 votes, since not even the player with 3 votes can be critical to them.

Remark

It suffices to consider the swings only in winning coalitions in the calculation of the Banzhaf index. A defector that turns a winning coalition into a losing coalition also gives the symmetric swing that turns a losing coalition into a winning coalition.

The numbers in the second column are derived from the theory of combinations. For instance, the number of ways that you could choose 311 from 321111 is ${}_4C_2 = 6$.

$$\begin{aligned}\beta &= \left(\frac{22}{56}, \frac{10}{56}, \frac{6}{56}, \frac{6}{56}, \frac{6}{56}, \frac{6}{56} \right) \\ &\approx (0.392, 0.178, 0.107, 0.107, 0.107, 0.107).\end{aligned}$$

Comparing this with ϕ , we see that the two indices turn out to be quite close in this case, with β giving slightly less power to the two large players and slightly more to the small players.

1.3 Case studies of power indexes calculations

United Nations Security Council

1. Big “five” – permanent member 常任理事國 each has veto power; ten “small” countries whose (non-permanent) membership rotates.
2. It takes 9 votes, the “big five” plus at least 4 others to carry an issue.

For simplicity, we assume no “abstain” votes. The game is $[39; 7, 7, 7, 7, 7, 1, 1, \dots, 1]$. Why? Let x be the weight of any of the permanent member and q be the quota. Then

$$4x + 10 < q \quad \text{and} \quad q \leq 5x + 4$$

so that $4x + 10 < 5x + 4$ giving $x > 6$. Taking $x = 7$, we then have $38 < q \leq 39$. We take $q = 39$.

3. A “small” country i can be pivotal in a winning coalition if and only if S contains exactly 9 countries including the big “five”. There are 9C_3 such different S that contain i since the remaining 3 “small” countries are chosen from 9 “small” countries (other than country i itself). For each such S , the corresponding coefficient in the Shapley-Shubik formula for this 15-person game is $\frac{(9-1)!(15-9)!}{15!}$. Hence, $\phi_S = {}^9C_3 \times \frac{8!6!}{15!} \approx 0.001863$. Any “big-five” has index $\phi_b = \frac{1-10\phi_S}{5} = 0.1963$.

4. Old Security Council before 1963, which was

$$[27; 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1].$$

What is the corresponding yes-no voting system?

Answer for ϕ : $\phi_b = \frac{1}{5} \cdot \frac{76}{77}$; $\phi_S = \frac{1}{6} \cdot \frac{1}{77}$.

Canadian Constitutional Amendment

Investigate the voting powers exhibited in a 10-person game between the provinces, and to compare the results with the provincial populations.

The winning coalitions or those with veto power can be described as follows. In order for passage, approval is required of

- (a) any province that has (or ever had) 25% of the population,
- (b) at least two of the four Atlantic provinces, and
- (c) at least two of the four western provinces that currently contain together at least 50% of the total western population.

Veto power

Recall that a blocking coalition (holding veto power) is a subset of players whose complement is not winning. Using the current population figures, the veto power is held by

- (i) Ontario (O) and Quebec (Q),
- (ii) any three of the four Atlantic (A) provinces [New Brunswick (NB), Nova Scotia (NS), Prince Edward Island (PEI), and Newfoundland (N)],
- (iii) British Columbia (BC) plus any one of the three prairie (P) provinces [Alberta (AL), Saskatchewan (S), and Manitoba (M)], and
- (iv) the three prairie provinces taken together.

In calculating the Shapley-Shubik index of Quebec or Ontario, it is necessary to list all possible winning coalitions since any of these winning coalitions must contain Quebec and Ontario.

Winning Provincial Coalitions

Type	S	s	No. of such S
1	$1P, 2A, BC, Q, O$	6	18
2	$2P, 2A, BC, Q, O$	7	18
3	$3P, 2A, Q, O$	7	6
4	$1P, 3A, BC, Q, O$	7	12
5	$3P, 2A, BC, Q, O$	8	6
6	$2P, 3A, BC, Q, O$	8	12
7	$3P, 3A, Q, O$	8	4
8	$1P, 4A, BC, Q, O$	8	3
9	$3P, 3A, BC, Q, O$	9	4
10	$2P, 4A, BC, Q, O$	9	3
11	$3P, 4A, Q, O$	9	1
12	$3P, 4A, BC, Q, O$	10	1
Total:			88

Ontario's Shapley-Shubik index

$$\varphi_O = \frac{[18(5!4!) + 36(6!3!) + 25(7!2!) + 8(8!1!) + 1(9!0!)]}{10!} = \frac{53}{168}$$

- There are 18 winning coalitions that contain 6 provinces. In order that Ontario serves as the pivotal player, 5 provinces are in front of her and 4 provinces are behind her. This explains why there are altogether $18(5!4!)$ permutations in these 6-province winning coalitions.
- Ontario and Quebec are equivalent in terms of influential power (though their populations are different).

British Columbia

Listing of all winning coalitions that upon deleting British Columbia the corresponding coalition becomes losing. These are the winning coalitions that British Columbia can serve as the pivotal player.

Type	S	s	No. of such S
1	1P, 2A, BC, Q, O	6	${}_3C_1 \times {}_4C_2 = 18$
2	1P, 3A, BC, Q, O	7	${}_3C_1 \times {}_4C_3 = 12$
3	1P, 4A, BC, Q, O	8	${}_3C_1 \times {}_4C_4 = 3$
4	2P, 2A, BC, Q, O	7	${}_3C_2 \times {}_4C_2 = 18$
5	2P, 3A, BC, Q, O	8	${}_3C_2 \times {}_4C_3 = 12$
6	2P, 4A, BC, Q, O	9	${}_3C_2 \times {}_4C_4 = 3$

- Note that we exclude those coalitions with 3 prairie provinces since the deletion of British Columbia does not cause the coalition to become losing.

$$\phi_{BC} = \frac{18(5!4!) + 30(6!3!) + 15(7!2!) + 3(8!1!)}{10!}.$$

Atlantic provinces

We consider winning coalitions that contain a particular Atlantic province and one of the three other Atlantic provinces.

Type	S	s	No. of such S
1	$A_{sp}, 1A, 1P, BC, Q, O$	6	${}_3C_1 \times {}_3C_1 = 9$
2	$A_{sp}, 1A, 2P, BC, Q, O$	7	${}_3C_1 \times {}_3C_2 = 9$
3	$A_{sp}, 1A, 3P, BC, Q, O$	8	${}_3C_1 = 3$
4	$A_{sp}, 1A, 3P, Q, O$	7	${}_3C_1 = 3$

$$\phi_{A_{sp}} = \frac{9(5!4!) + 12(6!3!) + 3(7!2!)}{10!}.$$

Prairie provinces

We consider winning coalitions that contain

- (i) a particular prairie province and British Columbia
- (ii) a particular prairie province and two other prairie provinces

Type	S	s	No. of such S
1	$P_{sp}, 2A, BC, Q, O$	6	6
2	$P_{sp}, 3A, BC, Q, O$	7	4
3	$P_{sp}, 4A, BC, Q, O$	8	1
4	$P_{sp}, 2P, 2A, Q, O$	7	6
5	$P_{sp}, 2P, 3A, Q, O$	8	4
6	$P_{sp}, 2P, 4A, Q, O$	9	1

$$\phi_{P_{sp}} = \frac{6(5!4!) + 10(6!3!) + 5(7!2!) + 8!1!}{10!}.$$

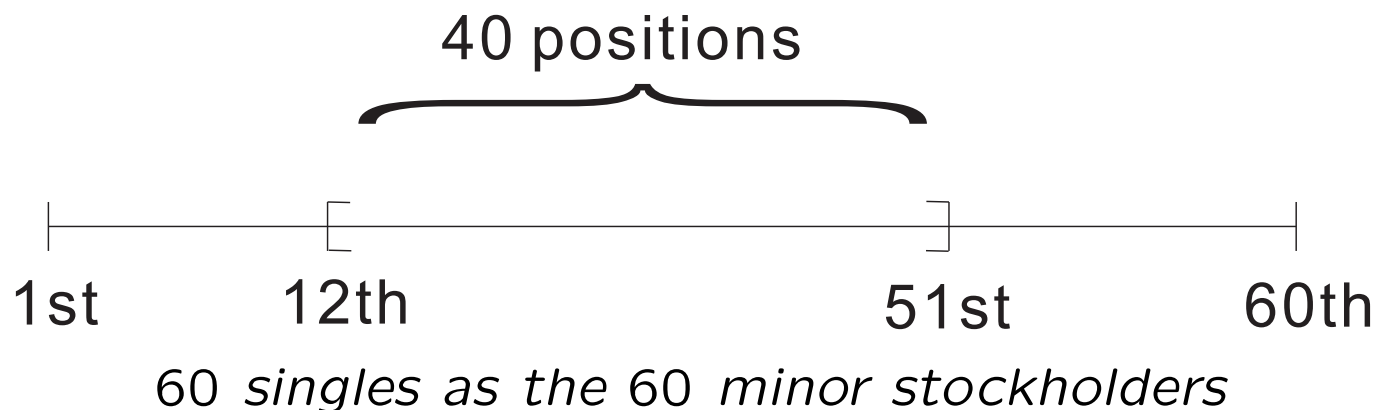
Shapley-Shubik Index Provinces

Province	ϕ (in %)	% Population	ϕ /Population
<i>BC</i>	12.50	9.38	1.334
<i>AL</i>	4.17	7.33	0.570
<i>S</i>	4.17	4.79	0.872
<i>M</i>	4.17	4.82	0.865
(4 Western)	(25.01)	(26.32)	(0.952)
<i>O</i>	31.55	34.85	0.905
<i>Q</i>	31.55	28.94	1.092
<i>NB</i>	2.98	3.09	0.965
<i>NS</i>	2.98	3.79	0.786
<i>PEI</i>	2.98	0.54	5.53
<i>N</i>	2.98	2.47	1.208
(4 Atlantic)	(11.92)	(9.89)	(1.206)

- British Columbia has a higher index value per capita compared to other Western provinces.

Power of the major stockholders

- Consider a corporation with one major stockholder X who controls 40% of the stock, and suppose the remainder is split evenly among 60 other stockholders, each having 1%.
- There are 61 players. Since the 60 minor stockholders are symmetric, there are only 61 distinct orderings, depending only on the position of X .
- Of these 61 orderings, X will pivot if he appears in positions 12 – 51 inclusive (if we assume that approval must be by an amount strictly over 50%), i.e. 40/61 of the time.



- Now suppose X still controls 40% of the stock, but the remainder is split evenly among 600 other stockholders, each controlling 0.1%.
- Of the 601 distinct orderings, X will pivot if he appears in positions 102 – 501, i.e., $400/601$ of the time. Clearly, as the number of minor stockholders gets very large, X 's share of the power (as measured by the Shapley-Shubik index) approaches $2/3$.

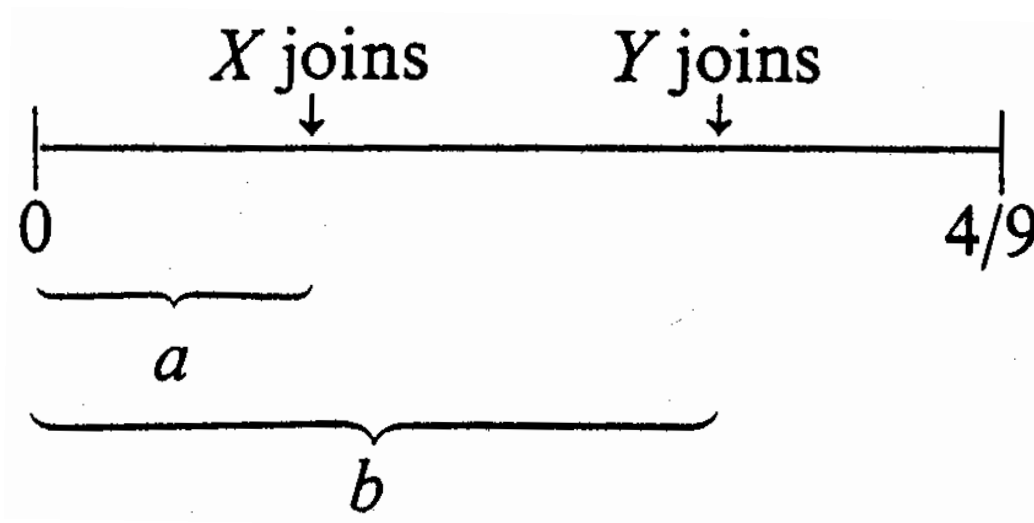
Oceanic weighted voting game

- Let there be one major player X controlling 40% of the vote, with the remaining 60% held by an infinite “ocean” of minor voters.
- Think of the minor voters lined up as points in a line segment of length 0.6, as they come to join a coalition in support of some proposal.
- Voter X can join at any point along this line segment. He will pivot if he joins after 0.1 and before (or at) 0.5. His Shapley-Shubik index is

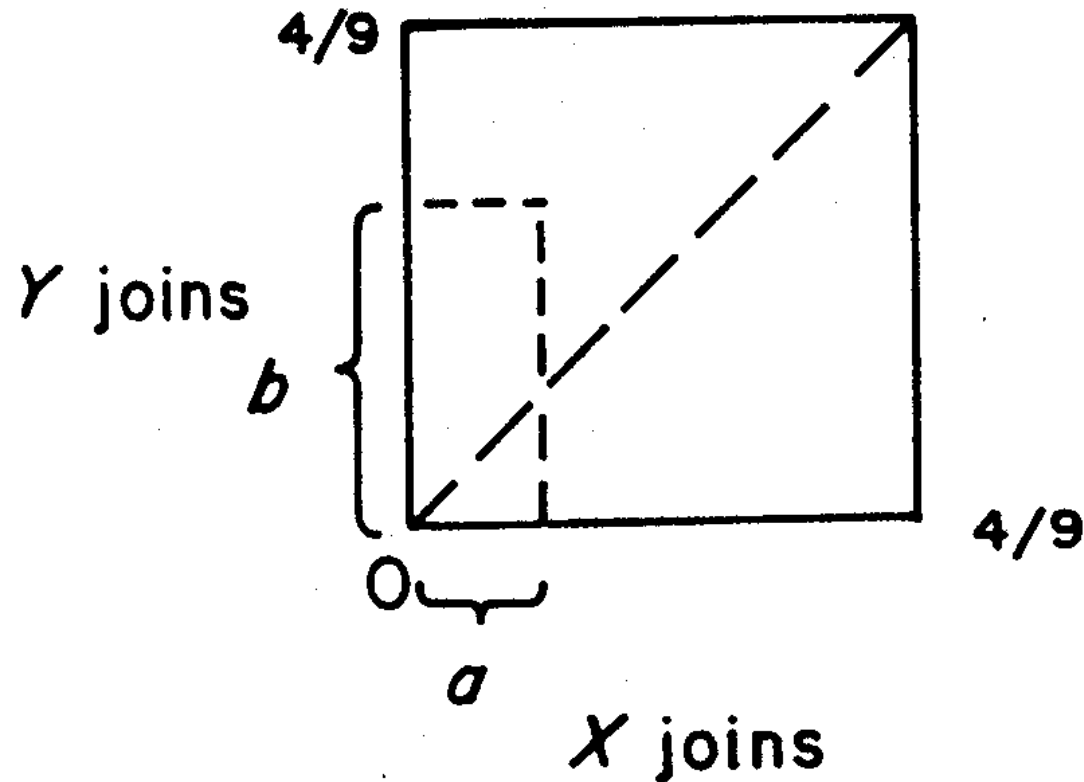
$$\phi_X = \frac{\text{Length of segment in which } X \text{ pivots}}{\text{Total length of segment}} = \frac{0.5 - 0.1}{0.6} = \frac{2}{3}.$$

Example

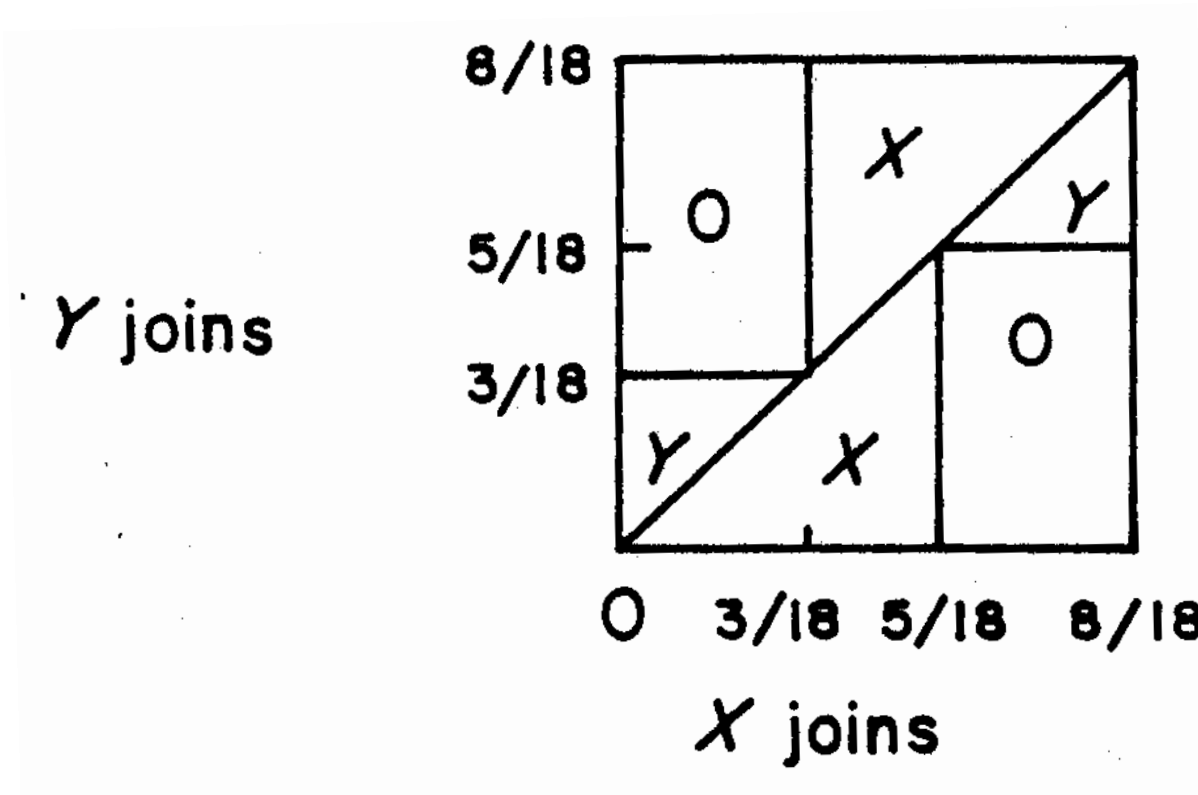
There are two major voters and an ocean of minor voters. Suppose voter X holds $3/9$ of the total vote, and voter Y holds $2/9$, with the other $4/9$ held by the ocean of minor voters. The minor voters line up along a line segment of length $4/9$. X and Y can join at any point along this line segment:



- We can represent geometrically the positions at which X and Y join by giving a single point in a square of side $4/9$, whose horizontal coordinate is X 's position and whose vertical coordinate is Y 's:
- The point is *above* the diagonal of the square if X joins before Y , and below the diagonal if Y joins before X .



- Which points in the square correspond to orderings for which X or Y pivots?
- Divide the square into regions where X pivots, Y pivots, or voters in the ocean (O) pivot:

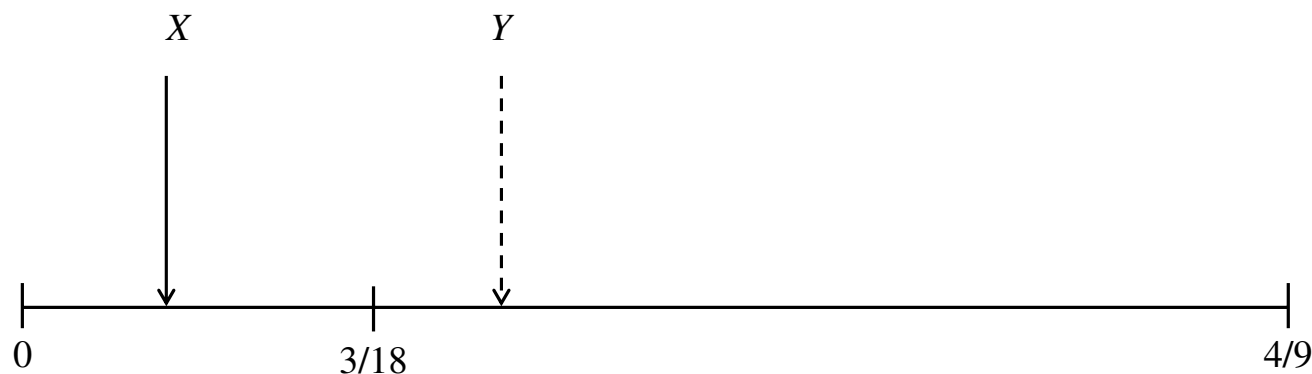


X joins before Y (points that lie above the diagonal $\Leftrightarrow a < b$)

1. If X joins before $3/18$, it can never be pivotal since

$$a + 3/9 < 1/2 \quad \text{for } a < 3/18.$$

Y can be pivotal provided that $b < 3/18$.



When Y joins after X and the point of joining is after $3/18$, then the oceanic voters have pivoted already.

2. The oceanic voters all combined together cannot pass the bill since they hold $4/9$ which is less than 50% of the votes. If X joins after $1/2 - 3/9 = 3/18$, then X pivots.

Calculate the Shapley-Shubik index for X or Y by calculating the *area* of the region in which X or Y pivots and dividing by the total area of the square. We have

$$\phi_X = \frac{(5/18)^2}{(4/9)^2} = \frac{25}{64} \approx 0.391$$

$$\phi_Y = \frac{(3/18)^2}{(4/9)^2} = \frac{9}{64} \approx 0.141;$$

with the other $1 - \frac{25}{64} - \frac{9}{64} = \frac{30}{64}$ being shared by the oceanic players. Interestingly, the major stockholder X has a higher power relative to his percentage holding. The gain comes at the expense of Y .
(做老大總是好一些)

- If there are three major players in an oceanic game, we represent orderings as points in a cube. We then calculate the volumes of the regions where each of the major players pivots.

Shapley-Shubik index of the President

Actual federal system (with the Vice President ignored)

When the number of Representatives ≥ 290 and the number of Senator ≥ 67 , the President cannot be pivotal.

House	Senate	
218	51 to 100	Without the two-thirds majority in both the Senate and the House, the President can veto.
⋮	⋮	
289	51 to 100	
290	51 to 66	Once the House has the two-thirds majority, the President is pivotal only when the Senate lies between one-half and two-thirds majority
⋮	⋮	
435	51 to 66	

$$\begin{aligned}
& \binom{435}{218} \left[\binom{100}{51} (218 + 51)! (535 - 218 - 51)! + \dots \right. \\
& \qquad \qquad \qquad \left. + \binom{100}{100} (218 + 100)! (535 - 218 - 100)! \right] \\
& + \dots \\
& + \binom{435}{289} \left[\binom{100}{51} (289 + 51)! (535 - 289 - 51)! + \dots \right. \\
& \qquad \qquad \qquad \left. + \binom{100}{100} (289 + 100)! (535 - 289 - 100)! \right] \\
& + \binom{435}{290} \left[\binom{100}{51} (290 + 51)! (535 - 290 - 51)! + \dots \right. \\
& \qquad \qquad \qquad \left. + \binom{100}{66} (290 + 66)! (535 - 290 - 66)! \right] \\
& + \dots \\
& + \binom{435}{435} \left[\binom{100}{51} (435 + 51)! (535 - 435 - 51)! + \dots \right. \\
& \qquad \qquad \qquad \left. + \binom{100}{66} (435 + 66)! (535 - 435 - 66)! \right]
\end{aligned}$$

When divided by $536!$, we obtain the Shapley-Shubik index of the President as $\phi = 0.16047$.

Inclusion of the Vice President

We need to add the case where 50 Senators and the Vice President say “yes”. The following terms should be added:

$$\begin{aligned} & \binom{435}{218} \binom{100}{50} (218 + 51)! (535 - 218 - 50)! \\ & + \binom{435}{219} \binom{100}{50} (219 + 51)! (535 - 219 - 50)! \\ & + \cdots + \binom{435}{435} \binom{100}{50} (435 + 51)! (535 - 435 - 50)! \end{aligned}$$

The denominator is modified to be 537!

- For the first term, we choose 218 Representatives from 435 of them and 50 Senators from 100 of them. There are $218 + 50 + 1$ “yes” voters and $535 - 218 - 50$ “no” voters. We have $(218 + 50 + 1)!$ orderings before the President and $(535 - 218 - 50)!$ orderings after the President.

Banzhaf index of the President

Let S denote the number of coalitions within the Senate that contain more than two-thirds of the members of the Senate:

$$S = \binom{100}{67} + \dots + \binom{100}{100}.$$

Let s denote the number of coalitions within the Senate that contain equal and more than one-half of the members of the Senate:

$$s = \binom{100}{50} + \binom{100}{51} + \dots + \binom{100}{100}.$$

Let H denote the number of coalitions within the House that contain more than two-thirds of the members of the House:

$$H = \binom{435}{290} + \dots + \binom{435}{435}.$$

Let h denote the number of coalitions within the House that contain more than one-half of the members of the House:

$$h = \binom{435}{218} + \cdots + \binom{435}{435}.$$

- We count the number of winning coalitions with the President such that the defection of the President turns winning into losing. Write N_P = total number of winning coalitions in which the president is critical.

$$\begin{aligned} N_P &= \left[\binom{435}{218} + \cdots + \binom{435}{289} \right] \left[\binom{100}{50} + \binom{100}{51} + \cdots + \binom{100}{100} \right] \\ &\quad + \left[\binom{435}{290} + \cdots + \binom{435}{435} \right] \left[\binom{100}{50} + \binom{100}{51} + \cdots + \binom{100}{66} \right] \\ &= (h - H) \times s + H \times (s - S) = h \times s - H \times S. \end{aligned}$$

- The total number of winning coalitions in which the Vice President is critical = $N_V = \binom{100}{50} \times h$.

- Given a particular senator, we find the number of winning coalitions such that this senator is critical.

Without the President, the number of critical swings effected by this senator is $\binom{99}{66} \times H$; and with the President, (but without the Vice president), the number of critical swings is $\binom{99}{50} \times h$. The last term corresponds to the presence of both the President and Vice President. The total number of winning coalitions to which the chosen senator is critical = $N_S = \binom{99}{66} \times H + \binom{99}{50} \times h + \binom{99}{49} \times h$.

- In a similar manner, the total number of winning coalitions to which a particular Representative is critical = $N_R = \binom{434}{289} \times S + \binom{434}{217} \times s$.

Banzhaf indexes calculations

$$\begin{aligned} & \text{Total number of critical swings by all players} \\ = N &= 100 \times N_S + 435 \times N_R + N_P + N_V \end{aligned}$$

$$\begin{aligned} & \text{Banzhaf index of any senator} \\ = & \frac{\text{number of winning coalitions to which the chosen senator is critical}}{\text{total number of critical swings}} \\ = & N_S/N \end{aligned}$$

$$\text{Banzhaf index for the President} = N_P/N.$$

1.4 Probabilistic characterization of power indexes

Question of Individual Effect. *What is the probability that my vote will make a difference, that is, that a proposal will pass if I vote for it, but fail if I vote against it?*

Question of Individual-Group Agreement. *What is the probability that the group decision will agree with my decision on a proposal? (如己所願)*

- The answers depend on both the decision rule of the body and the probabilities that various members will vote for or against a proposal. In some particular political example, we might also be able to estimate voting probabilities of the players for some particular proposal or class of proposals.
- If we are interested in general theoretical questions of power, we cannot reasonably assume particular knowledge about individual players or proposals. We should only make assumptions about voting probabilities which do not discriminate among the players.

Homogeneity Assumption. *Every proposal to come before the decision-making body has a certain probability p of appealing to each member of the body. The homogeneity is among members: they all have the same probability p of voting for a given proposal, but p varies from proposal to proposal.*

The homogeneity assumption does not assume that members will all vote the same way, but it does say something about their similar criteria for evaluating proposals. For instance, some bills that came before a legislature seem to have a high probability of appealing to all members, and pass by large margins: those have high p . Others are overwhelmingly defeated (low p) or controversial (p near $1/2$).

Remark For the Shapley-Shubik index, we further assume the common p to be uniformly distributed between 0 and 1.

Shapley-Shubik index focuses on the *order* in which a winning coalition forms, and defines the power of a player to be proportional to the number of orderings in which she is pivotal. If voters have a certain degree of homogeneity, then ϕ is most appropriate.

Theorem 1. *The Shapley-Shubik index ϕ gives the answer to the question of individual effect under the homogeneity assumption about voting probabilities.*

Remark A swing for player i occurs if a coalition S_i exists such that

$$\sum_{j \in S_i} w_j < q \quad \text{and} \quad w_i + \sum_{j \in S_i} w_j \geq q.$$

Proof of Theorem 1

We randomize the probabilities p_1, \dots, p_N and invoke the conditional independence assumption. Given the realization of $\mathbf{p}_i = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_N)$, the conditional probability that player i 's vote will make a difference is given by

$$\pi_i(\mathbf{p}_i) = \sum_{S_i} \prod_{j \in S_i} p_j \prod_{j \notin S_i} (1 - p_j),$$

where p_j is the voting probability of player j . The sum is taken over all such coalitions where player i is pivotal. The expected frequency where player i is pivotal is obtained by integrating over the probability distribution:

$$E[\pi_i(\mathbf{p}_i)] = \int_0^1 \int_0^1 \cdots \int_0^1 \pi_i(\mathbf{p}_i) f_i(\mathbf{p}_i) dp_1 \cdots dp_{i-1} dp_{i+1} \cdots dp_N$$

where $f_i(\mathbf{p}_i)$ is the joint density function of \mathbf{p}_i . The voting probabilities are randomized.

Remark

Density function $f_X(x)$ of a uniform distribution over $[a, b]$ is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}.$$

Under the homogeneity assumption, a number p is selected from the uniform distribution on $[0, 1]$ and p_j is set equal to p for all j . In this case, $f_i(p) = 1$ since $a = 0$ and $b = 1$ so that

$$E[\pi_i(p)] = \int_0^1 \pi_i(p) dp \quad \text{where} \quad \pi_i(p) = \sum_{S_i} p^{s_i} (1-p)^{N-s_i-1}, \quad s_i = n(S_i).$$

Lastly, making use of the Beta integral:

$$\frac{s_i!(N - s_i - 1)!}{N!} = \int_0^1 p^{s_i}(1 - p)^{N-s_i-1} dp,$$

we obtain

$$E[\pi_i(p)] = \sum_{S_i} \frac{s_i!(N - s_i - 1)!}{N!} = \phi_i = \text{Shapley-Shubik index for player } i.$$

The Beta integral links the probability of being pivotal under the homogeneity assumption of voting probabilities with the expected frequency of being pivotal in various orderings of voters.

Independence Assumption. *Every proposal has a probability p_i of appealing to the i^{th} member. Each of the p_i is chosen independently from the interval $[0, 1]$. Here how one member feels about the proposal has nothing to do with how any other member feels.*

Banzhaf index ignores the question of ordering and looks only at the final coalition which forms in support of some proposal. The power of a player is defined to be proportional to the number of such coalitions. If the voters in some political situation behave completely independently, then β is the most appropriate index.

Theorem 2.

The absolute Banzhaf index β' gives the answer to the question of individual effect under the independence assumption about voting probabilities.

- The absolute Banzhaf index β'_i can be interpreted as assuming that each player votes randomly and independently with a probability of $1/2$. It can be shown mathematically that this is equivalent to assume that voting probabilities are selected randomly and independently from a distribution with mean $1/2$ without regard for the forms of those distributions.
- Each player can be thought of as having probability $1/2$ of voting for any given proposal, so we can think of all coalitions to be equally likely to form. Therefore, the probability of player i 's vote making a difference is exactly the probability that player i will be a swing voter. This is precisely the absolute Banzhaf index.

Proof of Theorem 2

Under the independence assumption, the voting probabilities are selected independently from distributions (not necessarily uniform) on $[0, 1]$ with $E[p_j] = 1/2$. Since p_j are independent, the joint density is

$$f_i(\mathbf{p}) = \prod_{j \neq i} f_j(p_j)$$

where $f_j(p_j)$ is the marginal density for p_j . Consider

$$\begin{aligned} E[\pi_i(\mathbf{p})] &= \sum_{S_i} \int_0^1 \cdots \int_0^1 \prod_{j \in S_i} p_j \prod_{j \notin S_i} (1 - p_j) \prod_{j \neq i} f_j(p_j) dp_1 \cdots dp_N \\ &= \sum_{S_i} \prod_{j \in S_i} \int_0^1 p_j f_j(p_j) dp_j \prod_{j \notin S_i} \int_0^1 (1 - p_j) f_j(p_j) dp_j \\ &= \pi_i \left(\frac{1}{2} \right) = \sum_{S_i} \frac{1}{2^{N-1}} = \frac{\eta_i}{2^{N-1}} = \beta'_i \\ &= \text{absolute Banzhaf index for player } i, \end{aligned}$$

where η_i is the number of swings for player i . Note that the sum of the absolute Banzhaf indexes for all players is not equal to 1.

Example

$$[3; 2, 1, 1]$$
$$A \ B \ C$$

- Each voter will vote for a proposal with probability p . What is the probability that A 's vote will make a difference between approval and rejection?
- If both B and C vote against the proposal, A 's vote will *not* make a difference, since the proposal will fail regardless of what he does.
- If B or C or both vote for the proposal, A 's vote will decide between approval and rejection.

An alternative approach is shown here to compute $\phi_i(\beta_i)$ without resort to counting of pivotal orderings (swings).

The probability that A 's vote will make a difference is given by

$$\pi_A(p) = \underbrace{p(1-p)}_{B \text{ for, } C \text{ against}} + \underbrace{(1-p)p}_{B \text{ against, } C \text{ for}} + \underbrace{p^2}_{\text{both for}} = 2p - p^2.$$

Similarly, B 's vote will make a difference only if A votes for, and C votes against. If they both voted for, the proposal would pass regardless of what B did.

$$\pi_B(p) = \underbrace{p(1-p)}_{A \text{ for, } C \text{ against}} = p - p^2.$$

By symmetry, we also have $\pi_C(p) = p - p^2$.

- Shapley-Shubik index: voting probabilities are chosen by players from a common uniform distribution on the unit interval.
- Banzhaf index: voting probabilities are selected independently from any set of distributions which have a common mean of $1/2$.

1. Homogeneity assumption

We must average the probability of making a difference $\pi_A(p)$ over all p between 0 and 1.

$$\text{for } A: \int_0^1 \pi_A(p) dp = \int_0^1 (2p - p^2) dp = \frac{2}{2} - \frac{1}{3} = \frac{2}{3} = \phi_A$$

$$\text{for } B: \int_0^1 \pi_B(p) dp = \int_0^1 (p - p^2) dp = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \phi_B$$

$$\text{for } C: \int_0^1 \pi_C(p) dp = \int_0^1 (p - p^2) dp = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \phi_C$$

2. Independence assumption

Assume that all players vote with probability 1/2 for or against a proposal. We obtain

$$\pi_A\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{3}{4} = \beta'_A$$

$$\pi_B\left(\frac{1}{2}\right) = \pi_C\left(\frac{1}{2}\right) = \frac{1}{4} = \beta'_B = \beta'_C,$$

thus verifying Theorem 2. Finally, $\beta_A = \frac{3}{5}$, $\beta_B = \frac{1}{5}$, $\beta_C = \frac{1}{5}$.

Theorem 3

The answer to player i 's question of individual-group agreement, under the independence assumption about voting probabilities, is given by $(1 + \beta'_i)/2$.

Theorem 2 says that β'_i gives the probability that player i 's vote will make the difference between approval and rejection. Since his vote makes the difference, in this situation the group decision always agrees with his.

- With probability $1 - \beta'_i$ player i 's vote will *not* make a difference, but in this case the group will still agree with him, by chance, half the time.
- Hence the total probability that the group decision will agree with player i 's decision is

$$(\beta'_i)(1) + (1 - \beta'_i) \left(\frac{1}{2}\right) = \frac{1 + \beta'_i}{2}.$$

Example

Consider the weighted voting game: $[51; 40, 30, 20, 10]$. We list all the marginal cases where defection of a player changes losing to winning.

<u>Players</u>				<u>marginal (losing to winning)</u>			
1	2	3	4	1	2	3	4
N	Y	Y	N	×			×
Y	N	N	Y		×	×	
N	Y	N	Y	×		×	
N	N	Y	Y	×	×		
Y	N	N	N		×	×	
N	Y	N	N	×			
N	N	Y	N	×			

For player 1, we have $\eta_1 = 5$, where the 5 coalitions are

$$S_1^{(1)} = \{2, 3\}, S_1^{(2)} = \{2, 4\}, S_1^{(3)} = \{3, 4\}, S_1^{(4)} = \{2\}, S_1^{(5)} = \{3\}.$$

The conditional probability that player 1 makes a difference:

$$\begin{aligned} \pi_1(p_2, p_3, p_4) = & p_2 p_3 (1 - p_4) + p_2 (1 - p_3) p_4 + (1 - p_2) p_3 p_4 \\ & + p_2 (1 - p_3) (1 - p_4) + (1 - p_2) p_3 (1 - p_4). \end{aligned}$$

Under the independence assumption and expected probabilities all equal $\frac{1}{2}$, the absolute Banzhaf index of Player 1 is given by

$$\begin{aligned}
E[\pi_1(p_2, p_3, p_4)] &= \int_0^1 p_2 f_2(p_2) dp_2 \int_0^1 p_3 f_3(p_3) dp_3 \int_0^1 (1 - p_4) f_4(p_4) dp_4 \\
&+ \int_0^1 p_2 f_2(p_2) dp_2 \int_0^1 (1 - p_3) f_3(p_3) dp_3 \int_0^1 p_4 f_4(p_4) dp_4 \\
&+ \int_0^1 (1 - p_2) f_2(p_2) dp_2 \int_0^1 p_3 f_3(p_3) dp_3 \int_0^1 p_4 f_4(p_4) dp_4 \\
&+ \int_0^1 p_2 f_2(p_2) dp_2 \int_0^1 (1 - p_3) f_3(p_3) dp_3 \int_0^1 (1 - p_4) f_4(p_4) dp_4 \\
&+ \int_0^1 (1 - p_2) f_2(p_2) dp_2 \int_0^1 p_3 f_3(p_3) dp_3 \int_0^1 (1 - p_4) f_4(p_4) dp_4 \\
&= \pi_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{5}{2^3} = \beta'_1.
\end{aligned}$$

Similarly, we obtain

$$\beta'_2 = \frac{3}{8}, \quad \beta'_3 = \frac{3}{8}, \quad \beta'_4 = \frac{1}{8}.$$

The Banzhaf index is

$$\beta = \left(\frac{5}{12}, \frac{3}{12}, \frac{3}{12}, \frac{1}{12}\right).$$

Player 1 - group agreement

Out of $2^4 = 16$ cases, there are $2\eta_1 = 2 \times 5 = 10$ cases where Player 1 is marginal. In the remaining 6 cases (out of 16 cases), Player 1 does not make a difference.

players				pass/fail	
1	2	3	4		
Y	Y	Y	Y	P	} with Y for players 2,3&4 gives "Pass" already, player 1 has equal probability to say Y or N
N	Y	Y	Y	P	
Y	N	N	Y	F	} with N for players 2&3 gives "Fail" already, player 1 has equal probability to say Y or N
N	N	N	Y	F	
Y	N	N	N	F	} with N for players 2,3&4 gives "Fail" already, player 1 has equal probability to say Y or N
N	N	N	N	F	

$$\text{Probability of player 1-group agreement} = \frac{1}{2} \times \left(1 - \frac{5}{8}\right) + 1 \times \frac{5}{8} = \frac{13}{16}.$$

Example

[3; 2, 1, 1]

Look again at $A B C$. What is the probability that, under the independence assumption, the group decision will agree with A 's preference?

- With probability $1/2$, A will support a proposal. It will then pass *unless* B and C both oppose it, which will happen with probability $1/4$.
- If A opposes the proposal (probability $1/2$), it will always fail.
- The probability of agreement with A is thus

$$\frac{1}{2} \left(1 - \frac{1}{4}\right) + \frac{1}{2}(1) = \frac{7}{8} = \frac{1 + \frac{3}{4}}{2} = \frac{1 + \beta'_A}{2}.$$

- Similarly, if B supports a proposal (probability $1/2$), it will pass if and only if A supports it (probability $1/2$).
- If B opposes the proposal (probability $1/2$), it will fail unless both A and C support it (probability $1/4$):

$$\frac{1}{2} \left(\frac{1}{2} \right) + \frac{1}{2} \left(1 - \frac{1}{4} \right) = \frac{5}{8} = \frac{1 + \frac{1}{4}}{2} = \frac{1 + \beta'_B}{2}.$$

Example

Consider $[5; 3, 2, 1, 1]$
 $A \ B \ C \ D$. Let $\rho_i(p)$ be the probability that the group decision agrees with player i 's decision, given that all players (including i) vote for a proposal with probability p . Note that A has veto power.

Remark

In the calculation procedure, it is convenient to set $p_A = p_B = p_C = p_D = p$. This is because under the independence assumption and common mean of probabilities of $1/2$, we may set $p = 1/2$ apparently in the calculation of $E[\rho_i(p_A, p_B, p_C, p_D)]$.

(a) It can be shown easily that

$$\rho_A(p) = p [p + (1-p)p^2] + (1-p)(1) = 1 - p + p^2 + p^3 - p^4.$$

A yes B yes B no, C + D yes A no

$$\rho_B(p) = p(p) + (1-p)(1 - p^3) = 1 - p + p^2 - p^3 + p^4$$

B yes A yes B no, not all of

A, C, D yes

$$\rho_C(p) = p [p + (1-p)p] + (1-p)[(1-p) + p(1-p)]$$

C yes A yes B yes B no, D yes C no A no A yes, B no

$$= 1 - p - p^2 + 3p^3 - p^4.$$

(b) Now calculate $\rho_A(1/2)$, $\rho_B(1/2)$, and $\rho_C(1/2)$ and show that these are $(1 + \beta'_A)/2$, $(1 + \beta'_B)/2$, and $(1 + \beta'_C)/2$, thus verifying Theorem 3 for this case.

Example

Consider the majority-minority voting system with 7 voters, where 5 of them are in the majority group and the remaining 2 voters are in the minority group. The passage of a bill requires at least 4 votes from all voters and at least 1 vote from the minority group. Suppose the 5 members in the majority group vote as a homogeneous group and the 2 members in the minority group vote as another homogeneous group.

- (a) Compute the probability that a majority player's vote decides the passage of a bill.
- (b) Compute the probability that a minority player's vote decides the passage of a bill.

Solution

Under the homogeneity assumption, we let p and q denote the homogeneous voting probability of the majority group and minority group, respectively.

- (a) Consider a particular majority member, her vote can decide the passage of a bill if
- (i) 1 minority member and 2 other majority members say “yes” and other members say “no” ;
 - (ii) 2 minority member and 1 other majority members say “yes” and other members say “no” .

$$\begin{aligned}
& P[\text{majority player's vote can decide the passage} | p, q] \\
&= C_1^2 C_2^4 q(1-q)p^2(1-p)^2 + C_1^4 q^2 p(1-p)^3 \\
&= 12q(1-q)p^2(1-p)^2 + 4q^2 p(1-p)^3.
\end{aligned}$$

Assuming independence of the random probabilities p and q , and both of them follow the uniform distribution, we obtain

$$\begin{aligned}
& P[\text{majority player's vote can decide the passage}] \\
&= \int_0^1 \int_0^1 [12q(1-q)p^2(1-p)^2 + 4q^2 p(1-p)^3] dpdq \\
&= 12 \int_0^1 p^2(1-p)^2 dp \int_0^1 q(1-q) dq + 4 \int_0^1 p(1-p)^3 dp \int_0^1 q^2 dq.
\end{aligned}$$

- (b) Consider a particular minority member, her vote can decide the passage of a bill if
- (i) 3 or more majority members say “yes” and other members say “no” ;
 - (ii) 2 majority members and the other minority member say “yes” and other members say “no” .

Using similar assumptions on p and q , we obtain

$$\begin{aligned}
& P[\text{minority player's vote can decide the passage}] \\
&= \int_0^1 \int_0^1 \left[\sum_{k=3}^5 C_k^5 p^k (1-p)^{5-k} (1-q) + C_2^5 p^2 (1-p)^3 q \right] dpdq \\
&= 10 \left[\int_0^1 p^3 (1-p)^2 dp \int_0^1 (1-q) dq + \int_0^1 p^2 (1-p)^3 dp \int_0^1 q dq \right] \\
&\quad + 5 \int_0^1 p^4 (1-p) dp \int_0^1 (1-q) dq + \int_0^1 p^5 dp \int_0^1 (1-q) dq.
\end{aligned}$$

Example

Consider the voting game: $[2; 1, 1, 1]$.

Let p_A, p_B and p_C be the probabilities that A, B and C will vote for a proposal. Assuming independence of the random voting probabilities, we calculate the probabilities of a player's vote making a difference:

$$\pi_A = p_B(1 - p_C) + (1 - p_B)p_C,$$

$$\pi_B = p_A(1 - p_C) + (1 - p_A)p_C,$$

$$\pi_C = p_A(1 - p_B) + (1 - p_A)p_B.$$

- If the p_i s are all independent (β') or all equal (ϕ) as they vary between 0 and 1, then the players have equal power.

Suppose B and C are homogeneous ($p_B = p_C$), but A is independent. Then the answers to the question of individual effect are

$$\text{for } A: \int_0^1 2p_B(1 - p_B) dp_B = \frac{1}{3}$$

$$\begin{aligned} \text{for } B \text{ or } C: & \left(\int_0^1 p_A dp_A \right) \left(\int_0^1 (1 - p_B) dp_B \right) + \left(\int_0^1 (1 - p_A) dp_A \right) \left(\int_0^1 p_B dp_B \right) \\ & = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

With the pair sharing homogeneity in voting probabilities, B and C both have more power than A . In particular, we could normalize $(1/3, 1/2, 1/2)$ to $(1/4, 3/8, 3/8)$ and compare that to $(1/3, 1/3, 1/3)$.

Canadian Constitutional Amendment Scheme revisited

B_1 \otimes B_2 \otimes $M_{4,2}$ \otimes [3; 2, 1, 1, 1]
 Quebec Ontario Atlantic British Columbia and Central.

Intersection of 4 weighted voting systems.

Province	Percentage of power		Percentage of population	
	Shapley-Shubik index	Banzhaf index		
Ontario	31.55	21.78	34.85	} average 31.90
Quebec	31.55	21.78	2.94	
Bristish Columbia	12.50	16.34	9.38	
Central				
Alberta	4.17	5.45	7.33	} average 5.65
Saskatchewan	4.17	5.45	4.79	
Manitoba	4.17	5.45	4.82	
Atlantic				
New Brunswick	2.98	5.94	3.09	} average 2.47
Nova Scotia	2.98	5.94	3.79	
P.E.I.	2.98	5.94	0.54	
Newfoundland	2.98	5.94	2.47	

Observations

- Based on the Shapley-Shubik index calculations, the scheme “produces a distribution of power that matches the distribution of population surprisingly well” .
- However, based on the Banzhaf analysis, the scheme would seriously under-represent Ontario and Quebec and seriously over-represent British Columbia and the Atlantic provinces.
- It is disquieting that the two power indexes actually give different orders for the power of the players. ϕ says the Central Provinces are more powerful than the Atlantic provinces, and β says the opposite.

Which index is more applicable?

- Use ϕ if we believe there is a certain kind of homogeneity among the provinces.
- Use β if we believe there are more likely to act independently of each other.

Actual behavior

- Quebec and British Columbia would likely to behave independently.
- The four Atlantic provinces would more likely to satisfy the homogeneity assumption.

Hybrid approach

If a group of provinces is homogeneous, assign the members of that group the same p , which varies between 0 and 1 (independent of the p assigned to other provinces or groups of provinces).

We obtain $E[\pi_Q] = E[\pi_O] = 24/160$, $E[\pi_C] = 8/160$, $E[\pi_B] = 12/160$, $E[\pi_A] = 5/160$. There are 3C's and 4A's, the π 's sum to 104/160, so we normalize by multiplying the factor 160/104. The final power indexes under this scenarios are tabulated below under "A_s homogeneous and C_s homogeneous".

Table 2

Provinces	All homogeneous (ϕ)	A _s homogeneous C _s and B homogeneous	A _s homogeneous C _s homogeneous	All independent (β)	Average % of population
Quebec or Ontario	31.55	26.09	23.08	21.78	31.90
British Columbia	12.50	13.04	11.54	16.34	9.38
Central province	4.17	4.35	7.69	5.45	5.65
Atlantic province	2.98	5.43	4.81	5.94	2.47

Alternative homogeneity assumption

Quebec seems often to consider itself an island of French culture in the sea of English Canada. Treat all 9 other provinces as homogeneous among themselves, and Quebec as independent.

Quebec: 38.69	British Columbia: 11.61
Ontario: 25.84	Central provinces: 3.87
	Atlantic province: 3.07

Quebec's veto gives it considerable power. Alternatively, by staying homogeneous with other provinces, Ontario loses her power when compared to Quebec.

- Consider the effect of British Columbia's possible homogeneity with the Central provinces. Is it obvious that such homogeneity should give Quebec and Ontario more power?

Apparently, the previous power index calculations indicate that a higher level of homogeneity of other players gives more influential power to the province with veto power.