Mathematics and Social Choice Theory

Topic 2 – Analysis of powers in voting systems

2.1 Potential blocs, quarreling paradoxes and bandwagon effects

2.2 Power distribution in weighted voting systems

2.3 Incomparability and desirability

2.1 Potential blocs, quarelling paradoxes and bandwagon effects

Power of Potential blocs

- Groups of voters with similar interests and values who might consider joining together and casting their votes in common.

Question If a potential bloc decides to organize and vote as an actual bloc, does it really gain power?

Under the Shapley-Shubik model, we are comparing the chance that the organized bloc will pivot against the chance that one of the unorganized members will pivot.

- If we are in a majority game where each player has just one vote, the answer is "yes", provided that no other potential blocs organize.

- Union does not always mean strength in other simple games.

Example – disunity adds strength

Consider the weighted majority game $[5; 3, 3, 1, 1, 1]$

Here $\phi = \left(\frac{9}{30}, \frac{9}{30}, \frac{4}{30}, \frac{4}{30}, \frac{4}{30}\right)$ and $\boldsymbol{\beta} = \left(\frac{2}{7}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right)$.

If the 3 small players unite to form a bloc of three, that bloc will have power $1/3$, which is less than the total of what the members originally had, measured by either index.

Example

In a 40-member council, 11 members from Town B, 14 from Town J and 15 from other towns and rural areas.

Shapley-Shubik power index calculations

1. Both towns do not organize
   If the 11 members in Town B vote independently, then each supervisor will have $1/40$ of the total power and Town B together will have $\frac{11}{40} = 27.5\%$.

2. Town B organizes but Town J does not organize

- If the 11 members in Town B organize, then there are effectively 30 voters. Since 21 votes are needed to pass a measure, Town B will pivot if it joins a coalition after the 10$^{\text{th}}$ through 20$^{\text{th}}$ i.e. 11/30 of the time. Town B will have $11/30 = 36\frac{2}{3}\%$ of the power.
- Town J who had $14/40 = 35\%$ of the power before Town B organized, would have $(14/29)(19/30) \approx 30.5\%$ of the power after Town B organizes. Note that 29 council members from Town J and other towns share the remaining power of $1 - \dfrac{11}{30} = \dfrac{19}{30}$. Since the power of $\dfrac{19}{30}$ is shared equally among 29 members, so the total power of the 14 independent councils from Town $J = \dfrac{14}{29} \times \left(1 - \dfrac{11}{30}\right)$.

3. Both town organize

   As a result, we have a game of 17 voters: 15 casting a single vote. Town J pivots 100/272 of the time, for 37% of the power, while Town B pivots 49/272 of the time, for 18% of the power.

There are 15 other members, other than Town B and Town J.

Let Town B joins right after position i and Town J joins right after position j, $0 \leq i \leq 15$ and $0 \leq j \leq 15$.

(a) When Town J enters first, what is the number of possible orderings?

   Since $i \geq j$, we have $j = 0, 1, 2, \cdots, 15, i = j, j+1, \cdots, 15$, so number of orderings $= 16 + 15 + \cdots + 1 = \frac{16 \times 17}{2} = 136$.

- Suppose $j \geq 7$, then J always pivots since $[j+1, j+2, \cdots, j+14]$ contains the pivotal 21st position.

- On the other hand, with $j < 7$, and Town B enters after position i with $i \geq j$, then B pivots when $[i+14, i+15, \cdots, i+24]$ contains the pivotal 21st position. This occurs when $i < 7$.

Summary

(i) J pivots when $j = 7, 8, \cdots, 15, i = j, j+1, \cdots, 15$;

number of orderings $= 9 + 8 + \cdots + 1 = \frac{9 \times 10}{2} = 45$.

(ii) B pivots when $j < 7$, $i \leq j$ and $i < 7$. We have

$$j = 0, 1, 2, \cdots, 6; i = j, j+1, \cdots, 6;$$

the number of orderings $= 7 + 6 + \cdots + 2 + 1 = 28$.

(b) We consider the another case where Town B enters earlier, where $i \leq j$.

The total number of possible orderings remains to be 136. Following similar arguments, we can show that

(i) the number of orderings that J pivots is 55;

(ii) the number of orderings that B pivots is 21.

Finally, combining the two cases, we have

$$\phi_B = \frac{28 + 21}{272} \simeq 18\% \text{ and } \phi_J = \frac{45 + 55}{272} \approx 37\%$$

First entry is the power index for B, second entry is the power index for J.

	Town J does not organize		Town J organizes	
Town B does not organize	27.5	35	20	52
Town B organizes	36.7	30.5	18	37

- Town J will prefer to organize regardless of what Town B does since Town J increases its power upon organizing. "Organize" is a dominant strategy of Town J.

- Once Town J organizes, Town B is actually better off not organizing.

- Thus the natural outcome is Town B not organizes while Town J organizes. Town B supervisors should be "cunning" to choose uncooperative behavior. Town B should be very happy not to rock the boat.

Example

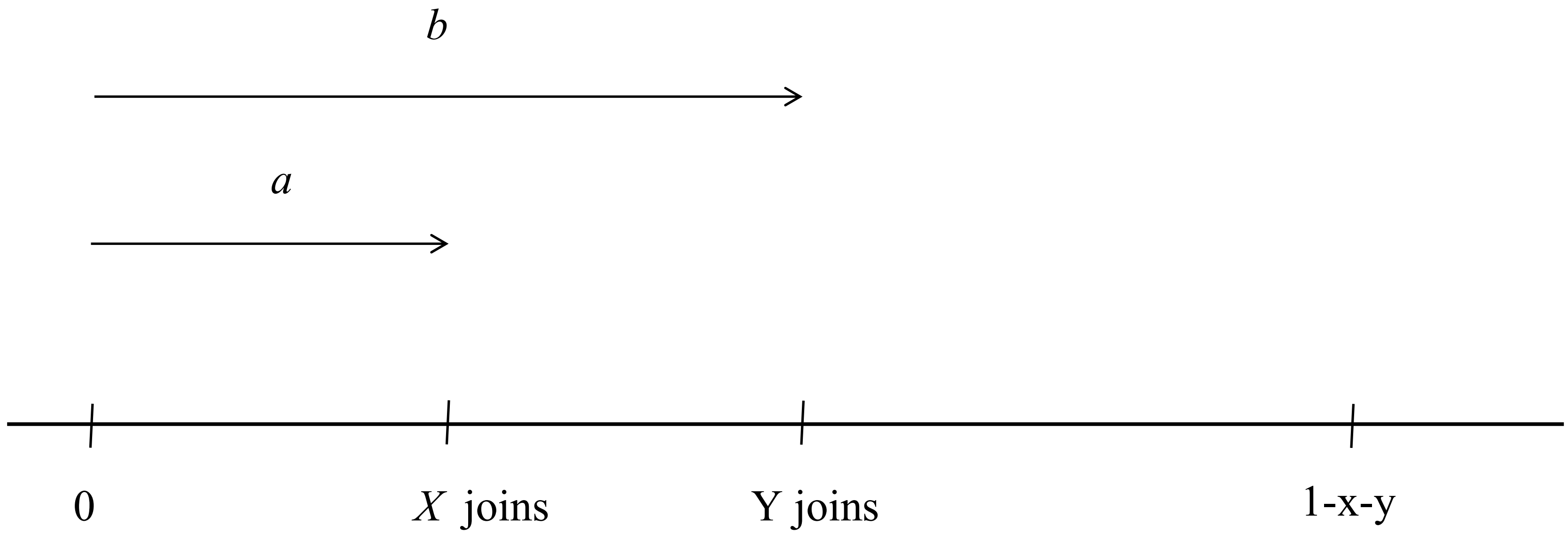
Assuming $0 < x < 1/2$ and $0 < y < 1/2$. If Player X controls a fraction of x of the vote and Player Y controls a fraction y, then

$$\phi_X(x,y) = \begin{cases} \frac{\left(\frac{1}{2}-y\right)^2}{(1-x-y)^2}, & \text{if } x+y \geq \frac{1}{2} \\ \frac{x(1-x-2y)}{(1-x-y)^2}, & \text{if } x+y \leq \frac{1}{2} \end{cases}.$$

The earlier example on P.65 (Topic 1) corresponds to $x = 3/9$ and $y = 2/9$, where $x + y > 1/2$. The area of the two triangles that correspond to X being pivotal is $\left(\frac{1}{2} - y\right)^2$. The area of the square is $(1 - x - y)^2$, so

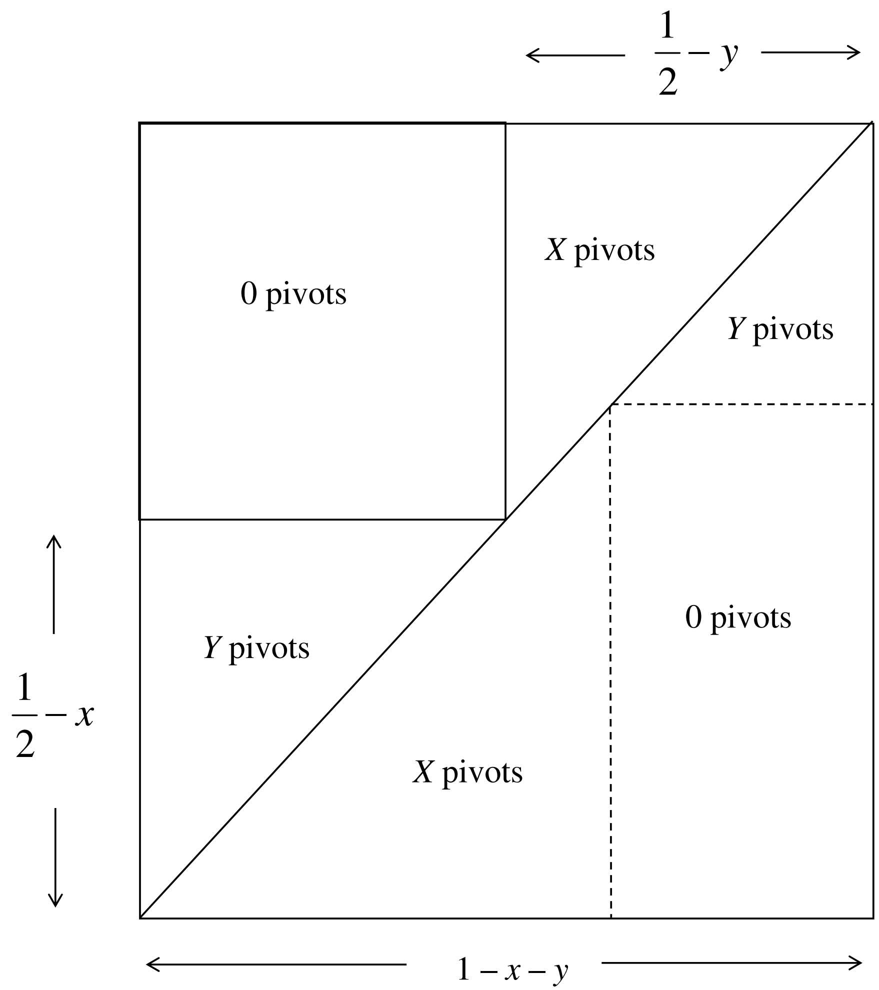
$$\phi_X(x,y) = \frac{\left(\frac{1}{2} - y\right)^2}{(1 - x - y)^2}, \quad x + y \geq \frac{1}{2}.$$

If the members of X and Y vote independently, the members of X will have power equal to their fraction of the vote, namely, x.

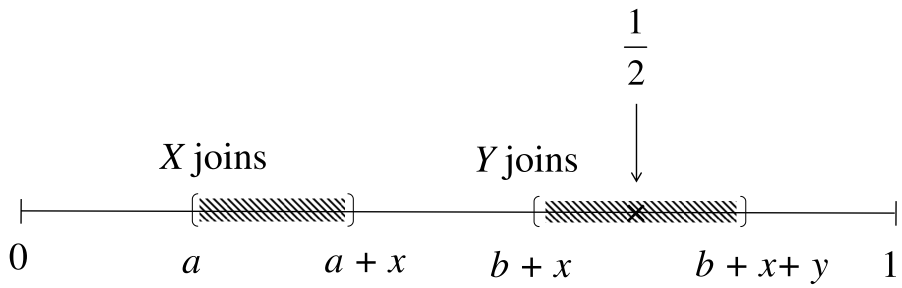


Consider the case where X enters earlier than Y, so that $a < b$; the total length of segment for oceanic voters is $1 - x - y$.

(i) $x + y \geq \dfrac{1}{2}$



Distinguish the various cases to determine which one of the two intervals $[a, a + x]$ or $[b + x, b + x + y]$ includes the pivotal point $\frac{1}{2}$.



For $x + y \geq \frac{1}{2}$; by assuming $a < b, X$ pivots when $a + x \geq \frac{1}{2}$ i.e. $a \geq \frac{1}{2} - x$. Otherwise, for $a + x < \frac{1}{2}$, we have

$$\begin{cases} Y \text{ pivots when } b + x \leq \frac{1}{2} & \text{i.e. } b \leq \frac{1}{2} - x \\ O \text{ pivots when } b + x > \frac{1}{2} & \text{i.e. } b > \frac{1}{2} - x. \end{cases}$$

The power index of X and Y are, respectively,

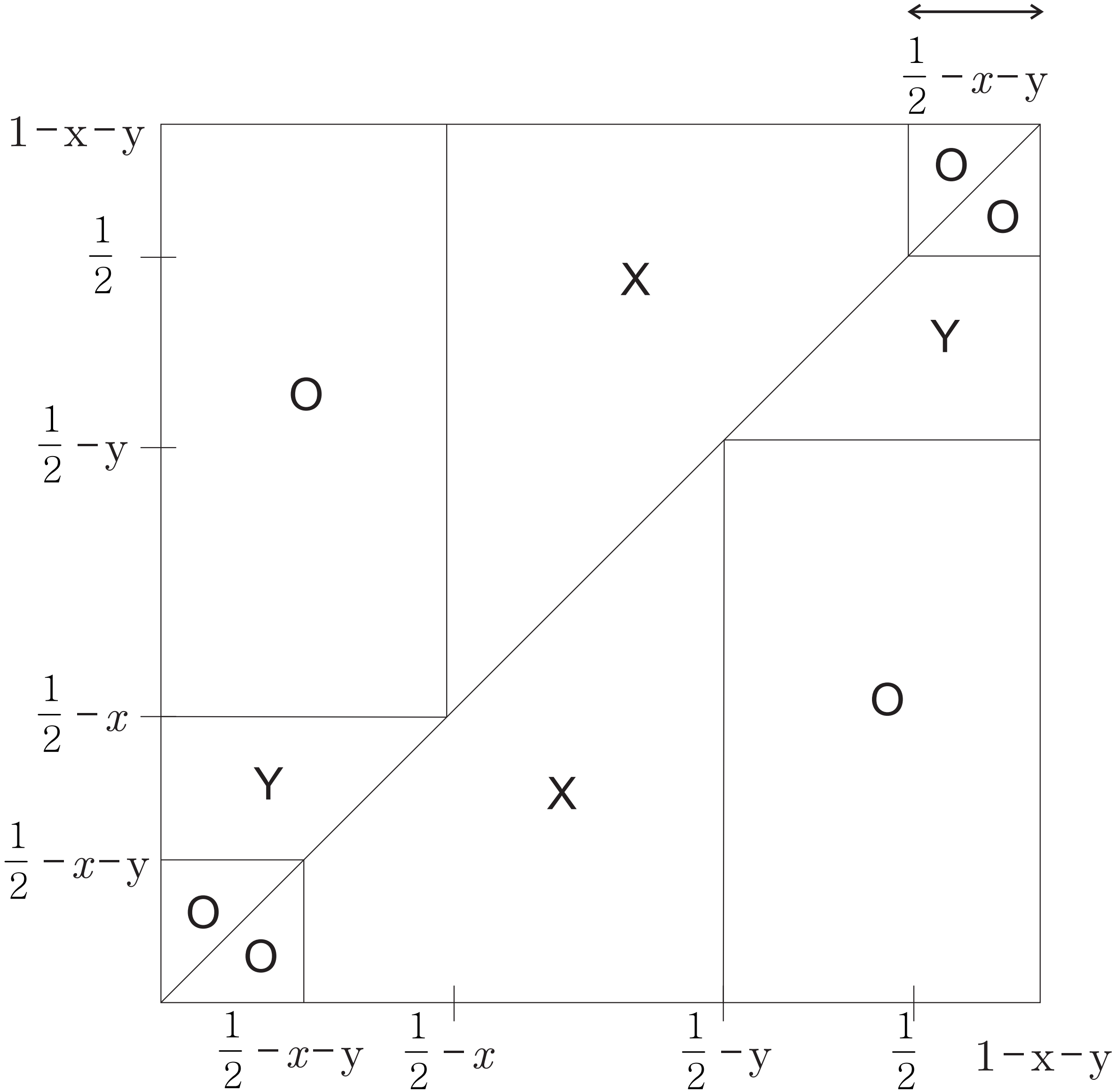
$$\phi_X(x,y) = \frac{\left(\frac{1}{2} - y\right)^2}{(1-x-y)^2} \quad \text{and} \quad \phi_Y(x,y) = \frac{\left(\frac{1}{2} - x\right)^2}{(1-x-y)^2}.$$

Since the power of any player holding x votes is the same across all players, so we expect

$$\phi_X(x,y) = \phi_Y(y,x).$$

Lastly, $\phi_X(x,y) + \phi_Y(x,y) + \phi_O(x,y) = 1$.

(ii) $x + y \leq \frac{1}{2}$

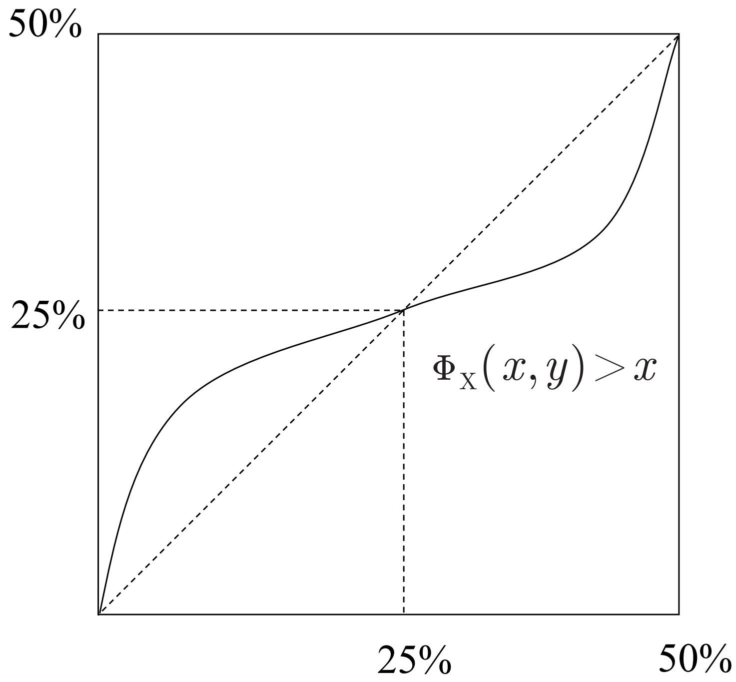


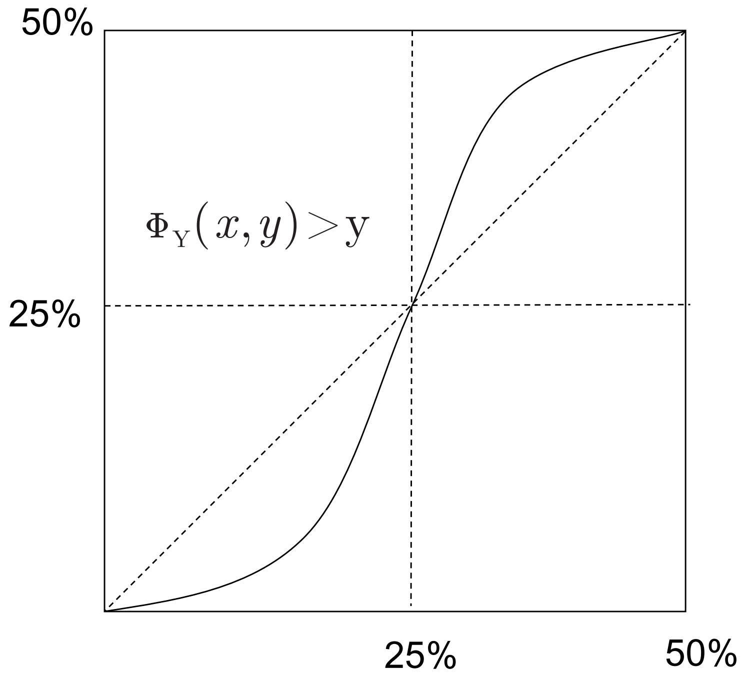
Assuming $a < b$, a slight modification is required when $x + y \leq \frac{1}{2}$ since X pivots if and only if $\frac{1}{2} - x \leq a \leq \frac{1}{2}$. For $a > \frac{1}{2}$, O pivots. Similarly, when $a < \frac{1}{2} - x - y$ and $b < \frac{1}{2} - x - y$, O pivots. We obtain

$$\begin{aligned}
\phi_X(x,y) &= \frac{\left(\frac{1}{2} - y\right)^2 - \left(\frac{1}{2} - x - y\right)^2}{(1-x-y)^2} = \frac{x(1-x-2y)}{(1-x-y)^2}; \\
\phi_Y(x,y) &= \frac{y(1-y-2x)}{(1-x-y)^2}; \\
\phi_U &= 1 - \phi_X - \phi_Y.
\end{aligned}$$

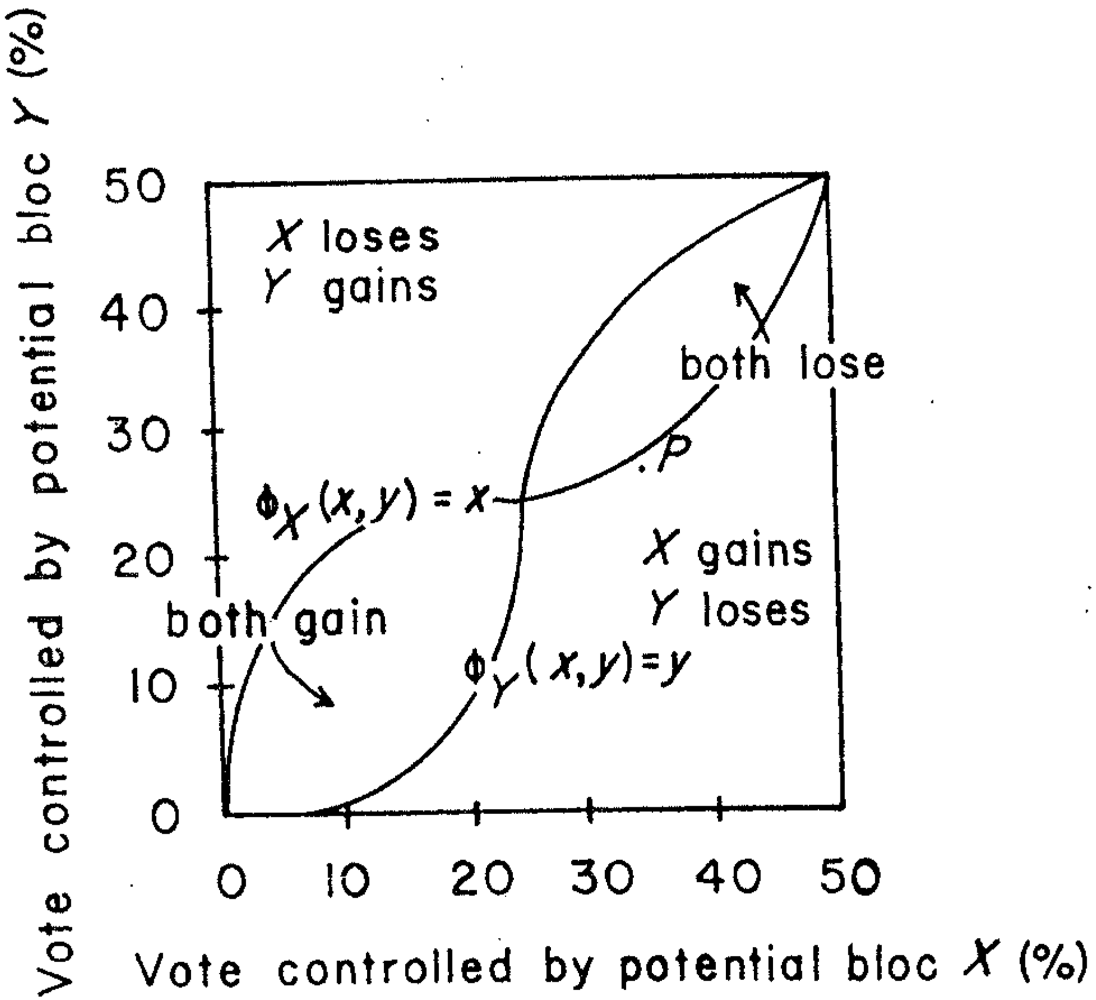
By symmetry, $\phi_X(x,y) = \phi_Y(y,x)$. This is because $\phi_X(x,y)$ is the power of X when he holds x votes and the other holds y votes; while $\phi_Y(y,x)$ gives the power of Y when he holds x votes and the other holds y votes.

We compare the power of x when both blocs organize or both do not organize by examining the relative magnitude of $\phi_X(x,y)$ and x.





Note that $0 \le x \le \frac{1}{2}$ and $0 \le y \le \frac{1}{2}$. It is seen that $\phi_X(x,y) = x$ when $x = y = \frac{1}{4}$, $x = y = 0$ or $x = y = \frac{1}{2}$. When both x and y are less than $\frac{1}{4}$, the organization of bloc X is beneficial when bloc Y is not substantially larger than bloc X.



The members of X will be better off if both X and Y organize precisely when $\phi_X(x,y) > x$ [the region below the curve $\phi_X(x,y) = x$]. If X represents Town J and Y represents Town B, then point P corresponds to the above example.

Quarreling paradoxes

- What happens if the two players quarrel, and refuse to enter into a coalition together?
- We normally think that we maximize our power by keeping as many options open as possible, and that restricting our freedom to act lessens our influence.
- Quarreling, of course, restricts our freedom to act.

Example

Consider the weighted voting game

$$\begin{array}{c} [5; 3, 2, 2] \\ \quad A \;\; B \;\; C \end{array}$$

For this game, $\phi = (2/3, 1/6, 1/6)$ and $\beta = (3/5, 1/5, 1/5)$, as seen by writing out the orderings with the pivots underlined:

$$\begin{array}{lll} A\underline{B}C & B\underline{A}C & C\underline{A}B \\ A\underline{C}B & *\,BC\underline{A} & *\,CB\underline{A} \end{array}$$

and the winning coalitions with the critical defectors underlined:

$$\underline{AB} \quad \underline{AC} \quad *\,\underline{A}BC.$$

Suppose members B and C quarrel. What is the effect on the Shapley-Shubik index?

In considering the orders in which the players might join a coalition in support of a proposal, we must now rule out those orderings in which B and C join together to help put the coalition over the top, i.e. those orderings in which both B and C join at or before the pivot.

There are two orderings in which this happens, marked by an $*$. In the four other orderings, the coalition becomes winning with the help of only one of B or C. By the original Shapley-Shubik assumption, these four orderings are equally likely. We obtain the *Shapley-Shubik index with quarreling* as

$$\phi_{BC}^{Q} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Banzhaf model for quarreling

We merely eliminate from consideration those winning coalitions containing both B and C (just ABC above) and compute proportions of critical defections in the remaining winning coalitions. We obtain

$$\beta^{Q}_{BC} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

with the same qualitative effect of an increase in B and C's share of the power at the expense of A.

What are the possibilities?

1. If two members quarrel, they may both gain power (as measured by ϕ or β). See the above example.

2. If two members quarrel, they may both lose power.

   This, of course, seems much more natural than (1).

3. If two members quarrel, one may gain power while the other loses power. (A quarrel might hurt you while helping your opponent, or vice versa $\cdots$.)

   Example: A and D quarrel in $[5; 3, 2, 2, 1]$

$$\phi = \beta = \left(\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}\right)$$

$$\phi^Q_{AD} = \beta^Q_{AD} = \left(\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right).$$

4. A quarrel may not affect the power of the quarrelers at all, but change the power of innocent bystanders. (神仙打架，禍及凡人)

   Example: B and C quarrel in the game of the last example.

$$\phi_{BC}^{Q} = \beta_{BC}^{Q} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0\right)$$

   Poor D, whose only chance to become part of a minimal winning coalition was with BC, has become a dummy.

5. The presence of dummies does not change the relative proportion of pivotal orderings among the players. However, quarreling with a dummy always hurts you. This is because the quarreler with the dummy would lose more on the number of pivotal orderings that involve the dummy compared to other players. It is still possible for a bystander to lose in power (the loss is less than that of the quarreler) if this bystander derives most of her power via "joining with the quarreler".

   Example: A quarrels with D in $[4; 2, 2, 2, 1]$

$$\phi = \beta = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$$

$$\phi_{AD}^{Q} = \beta_{AD}^{Q} = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, 0\right)$$

   It may be worthwhile staying on friendly terms even with those who have no power.

One-way quarrel (hostility)

A one-way quarreler can help or hurt his victim, and he may either help or hurt himself.

For example, in $[5; 3, 2, 2, 1]$, if B or C is hostile to D, D loses the chance of forming a minimal winning coalition, making D to be a dummy. In this case, the victim D is hurt.

Suppose that, in the weighted voting game $\underset{A\ \ B\ \ C\ D}{[7; 4, 3, 2, 1]}$ player B hates player C and refuses to join any coalition in support of a proposal that C has already joined. Player C has no such hostile feeling about B. What is the effect upon the power of B and C?

Orderings and pivots are

$A\underline{B}CD$	$B\underline{A}CD$	$^*CA\underline{B}D$	$DA\underline{B}C$
$A\underline{B}DC$	$B\underline{A}DC$	$CA\underline{D}B$	$DA\underline{C}B$
$^*AC\underline{B}D$	$^+BC\underline{A}D$	$^*CB\underline{A}D$	$DB\underline{A}C$
$AC\underline{D}B$	$^+BCD\underline{A}$	$^*CBD\underline{A}$	$^+DBC\underline{A}$
$AD\underline{B}C$	$BD\underline{A}C$	$CD\underline{A}B$	$DC\underline{A}B$
$AD\underline{C}B$	$^+BDC\underline{A}$	$^*CDB\underline{A}$	$^*DCB\underline{A}$

1. Consider $^*AC\underline{B}D$, C joins earlier. Since B is hostile to C, B will not join later to form a winning coalition.

2. Consider $^+BC\underline{A}D$, B joins earlier. Since C is hostile to B, C will not join later to form a winning coalition.

Both orderings will be ruled out if B and C quarrel.

With no quarreling, we have

$$\phi = \left(\frac{14}{24}, \frac{6}{24}, \frac{2}{24}, \frac{2}{24}\right) \approx (0.58, 0.25, 0.08, 0.08).$$

B's hostility to C rules out orderings marked by $*$, giving

$$\phi^Q_{B\to C} = \left(\frac{10}{18}, \frac{4}{18}, \frac{2}{18}, \frac{2}{18}\right) \approx (0.56, 0.22, 0.11, 0.11).$$

B has hurt himself and helped his victim.

If we reversed the situation and had C hating B, the orderings marked by$^+$ would be ruled out, giving

$$\phi^Q_{C\to B} \left(\frac{10}{20}, \frac{6}{20}, \frac{2}{20}, \frac{2}{20}\right) = (0.50, 0.30, 0.10, 0.10)$$

C would help B, and also help herself!

Example

Weighted voting game: $[5; 4, 2, 1, 1, 1]$; the first two voters quarrel. Note that the first two players can form a winning coalition together.

$$
\begin{array}{ccccc}
4 & \underline{2} & 1 & 1 & 1 \\
4 & \underline{1} & 2 & 1 & 1 \\
4 & \underline{1} & 1 & 2 & 1 \\
4 & \underline{1} & 1 & 1 & 2
\end{array}
\qquad
\begin{array}{ccccc}
2 & \underline{4} & 1 & 1 & 1 \\
1 & \underline{4} & 2 & 1 & 1 \\
1 & \underline{4} & 1 & 2 & 1 \\
1 & \underline{4} & 1 & 1 & 2
\end{array}
\qquad
\begin{array}{ccccc}
2 & 1 & \underline{4} & 1 & 1 \\
1 & 2 & \underline{4} & 1 & 1 \\
1 & 1 & \underline{4} & 2 & 1 \\
1 & 1 & \underline{4} & 1 & 2
\end{array}
$$

$$
\begin{array}{ccccc}
2 & 1 & 1 & \underline{4} & 1 \\
1 & 2 & 1 & \underline{4} & 1 \\
1 & 1 & 2 & \underline{4} & 1 \\
1 & 1 & 1 & \underline{4} & 2
\end{array}
\qquad
\begin{array}{ccccc}
2 & 1 & 1 & \underline{1} & 4 \\
1 & 2 & 1 & \underline{1} & 4 \\
1 & 1 & 2 & \underline{1} & 4 \\
1 & 1 & 1 & \underline{2} & 4
\end{array}
$$

20 distinct orderings without consideration of quarrel

$$\phi = \left(\frac{6}{10} \; \frac{1}{10} \; \frac{1}{10} \; \frac{1}{10} \; \frac{1}{10} \right).$$

Rule out 7 orderings when the first two voters quarrel

$$\begin{array}{ccccc} 4 & \underline{2} & 1 & 1 & 1 \\ 2 & \underline{4} & 1 & 1 & 1 \\ 2 & 1 & 1 & \underline{4} & 1 \\ 2 & 1 & \underline{4} & 1 & 1 \end{array} \qquad \begin{array}{ccccc} 1 & 2 & \underline{4} & 1 & 1 \\ 1 & 2 & 1 & \underline{4} & 1 \\ 1 & 1 & 2 & \underline{4} & 1 \end{array}$$

These "illegal" orderings must lie in the set of pivotal orderings held by the two quarreling players. In other words, none of these "illegal" orderings are pivotal orderings of the other players.

After ruling out 7 "illegal" orderings, the remaining 13 coalitions are:

$$\begin{array}{ccccc}
4 & \underline{1} & 2 & 1 & 1 \\
4 & \underline{1} & 1 & 2 & 1 \\
4 & \underline{1} & 1 & 1 & 2 \\
1 & \underline{4} & 2 & 1 & 1 \\
1 & \underline{4} & 1 & 2 & 1 \\
1 & \underline{4} & 1 & 1 & 2
\end{array}
\qquad\qquad
\begin{array}{ccccc}
1 & 1 & \underline{4} & 2 & 1 \\
1 & 1 & \underline{4} & 1 & 2 \\
1 & 1 & 1 & \underline{4} & 2 \\
2 & 1 & 1 & \underline{1} & 4 \\
1 & 2 & 1 & \underline{1} & 4 \\
1 & 1 & 2 & \underline{1} & 4 \\
1 & 1 & 1 & \underline{2} & 4
\end{array}$$

Out of these 13 orderings, 6 orderings of which "4"-voter pivots, only one ordering of which "2"-voter pivots, 6 orderings of which either one of the three "1"-voter pivots. The new power indexes with quarreling is

$$\phi = \left(\frac{6}{13} \quad \frac{1}{13} \quad \frac{2}{13} \quad \frac{2}{13} \quad \frac{2}{13}\right).$$

Lemma

Assume that the two players A and B in a N-person voting game can always form a winning coalition just with themselves. Suppose A and B now quarrel, show that the other players in the game enjoy an increase in power.

Proof

We let n_i and n_i' denote the number of orderings that player i is pivotal without and with A and B quarreling, respectively. The Shapley-Shubik power index of player i without and with quarreling are

$$\phi_i = \frac{n_i}{\sum_{i=1}^N n_i} \quad \text{and} \quad \phi_i' = \frac{n_i'}{\sum_{i=1}^N n_i'},$$

respectively. It is easily seen that $n_A' < n_A$ and $n_B' < n_B$ due to the constraint imposed by quarreling.

However, $n_i = n'_i$, $i \neq A, B$, since quarrel between A and B does not change the number of pivotal orderings of player i. This is because A and B together can form a winning coalition, so the ruling out of orderings with both A and B together would not reduce the pivotal orderings held by player i. We then have

$$\begin{aligned}
\phi'_i = \frac{n'_i}{\sum_{i=1}^N n'_i} &> \frac{n'_i}{n_A + n_B + \sum_{\substack{i=1 \\ i \neq A,B}}^N n'_i} \\
&= \frac{n_i}{n_A + n_B + \sum_{\substack{i=1 \\ i \neq A,B}}^N n_i} = \phi_i,\ i \neq A, B.
\end{aligned}$$

Therefore, all other players other than the quarrelers in the game enjoy an increase in power.

Warning on the results

1. The conclusion is true, a subtlety of political situations that precise analysis has thrown light upon.
2. The conclusion is a peculiarity of the power indices, showing that they have strange properties that should make us wary of where and how we use them.
3. The conclusion is a peculiarity not of the indices but of the model of quarreling we made using the indices. The model does not adequately reflect properties of real world quarrels.

Bandwagon effect (跟紅頂白)

- Two opposing blocs compete for the support of uncommitted voters in an attempt to achieve winning sizes. Uncommitted voters *suddenly* find it *advantageous* to begin committing themselves to the larger bloc, quickly enlarging it to winning size.

  "be (jump) on the bandwagon" – join in what seems likely to be a successful enterprise.

Two opposing blocs, X and Y, and a collection of uncommitted voters U.

- If X and Y are opposing, we rule out orderings in which they join together to win. We consider only orderings in which exactly one of X or Y is present when the coalition first becomes winning. Also, we rule out the possibility of uncommitted voters uniting to win without either X or Y.

Example

Consider $[5; 3, 2, 1, 1, 1, 1]$

- Of the 30 orderings, 9 of them are ruled out by the restriction that exactly one of the "3" or "2" should appear at or before the pivot:

$$
\begin{array}{lllll}
3\ \underline{2}\ 1\ 1\ 1\ 1 & 2\ \underline{3}\ 1\ 1\ 1\ 1 & 2\ 1\ \underline{3}\ 1\ 1\ 1 & 2\ 1\ 1\ \underline{3}\ 1\ 1 & \\
3\ 1\ \underline{2}\ 1\ 1\ 1 & 1\ 3\ \underline{2}\ 1\ 1\ 1 & 1\ 2\ \underline{3}\ 1\ 1\ 1 & 1\ 2\ 1\ \underline{3}\ 1\ 1 & 1\ 1\ 2\ \underline{3}\ 1\ 1
\end{array}
$$

- The use of the remaining 21 orderings leads to a modified Shapley-Shubik index
$$\left(\frac{6}{21}, \frac{3}{21}, \frac{3}{21}, \frac{3}{21}, \frac{3}{21}, \frac{3}{21}\right) \approx (0.286, 0.143, 0.143, 0.143, 0.143, 0.143).$$

- An uncommitted voter should commit to X if the increment of power he will add to X is larger than the power he would have if he remains uncommitted (毀滅自我，成就他人霸業).

- If an uncommitted voter commits to X, then the new game is

$$\underset{}{[\,5;} \quad \underset{X}{3,} \quad \underset{Y}{2,} \quad 1, \quad \underset{U}{1,} \quad 1, \quad 1\,] \longrightarrow [\,5; \quad 4, \quad 2, \quad 1, \quad 1, \quad 1\,]$$

with only 3 uncommitted voters.

To calculate the modified Shapley-Shubik index by writing down the 20 possible orderings for this game, crossing out the 7 "illegal" orderings since X and Y are not supposed to appear together in a winning coalition. We obtain

$$\left(\frac{6}{13}, \frac{1}{13}, \frac{2}{13}, \frac{2}{13}, \frac{2}{13}\right) \approx (0.462, 0.077, 0.154, 0.154, 0.154).$$

- The uncommitted voter raises the power of X from 0.286 to 0.462, an increment of 0.176, which is more than 0.143 he would have by remaining uncommitted.

Example - 1956 Democratic Vice-Presidential Nomination

Jack Kennedy	618	45%
Estes Kefauver	551.5	40%
Albert Gore	110.5	8%
Others	92	7%
Total	1372	(687 is needed to be nominated)

As a model, we consider the weighted voting game

$$[50;\ \underset{\text{J}}{45},\ \underset{\text{E}}{40},\ \underset{\text{G}}{8},\ \underbrace{1,1,1,1,1,1,1}_{\text{uncommitted}}]$$

JE should be ruled out. We also rule out JG and allow EG.

Background

- Gore was friendly with Kennedy, but Gore and Kefauver were both senators from Tennessee.
- Gore's supporters would not follow Gore to Kennedy. Gore was under constant pressure to withdraw in favor of Kefauver. In fact, Kefauver went on to win the nomination.

We would like to examine the degree that Kennedy is disadvantaged by not being able to form an alliance with Gore.

Under the above alliance conditions, 408 of the $10!/7! = 720$ possible orderings were ruled out. For example,

$$40\ 1\ 1\ \underline{45}\ 1\ 1\ 8\ 1\ 1\ 1;\ \ 1\ 8\ 40\ \underline{45}\ 1\ 1\ 1\ 1\ 1\ 1$$

are ruled out since J and E should not appear together to form a winning coalition.

Of the remaining orderings, J has 20 pivots, E and G both have 98, and uncommitted delegates have 96, giving the following modified SS index

(0.064 0.314 0.314 0.043 0.043 0.043 0.043 0.043 0.043 0.043).

Interestingly, E and G have equal power under the above alliance conditions. This is because E cannot form a winning coalition even with the inclusion of all uncommitted voters. Poor J, since he quarrels with both E and G, his power is much undermined.

Consider the following possible changes:

1. G joins E, giving $[50; 45, 48, 1, 1, 1, 1, 1, 1, 1]$. The new power indexes are now

   (0.111 0.389 0.071 0.071 0.071 0.071 0.071 0.071 0.071).

   G can contribute only 0.075 to E, much less than he has by not joining E.

2. An uncommitted delegate joins J, giving

$$[50; 46, 40, 8, 1, 1, 1, 1, 1, 1]$$

with power indices

(0.088 0.285 0.285 0.057 0.057 0.057 0.057 0.057 0.057).

She has contributed only 0.024 to J, less than she had while uncommitted.

3. An uncommitted delegate joins E, giving

$$[50; 45, 41, 8, 1, 1, 1, 1, 1, 1]$$

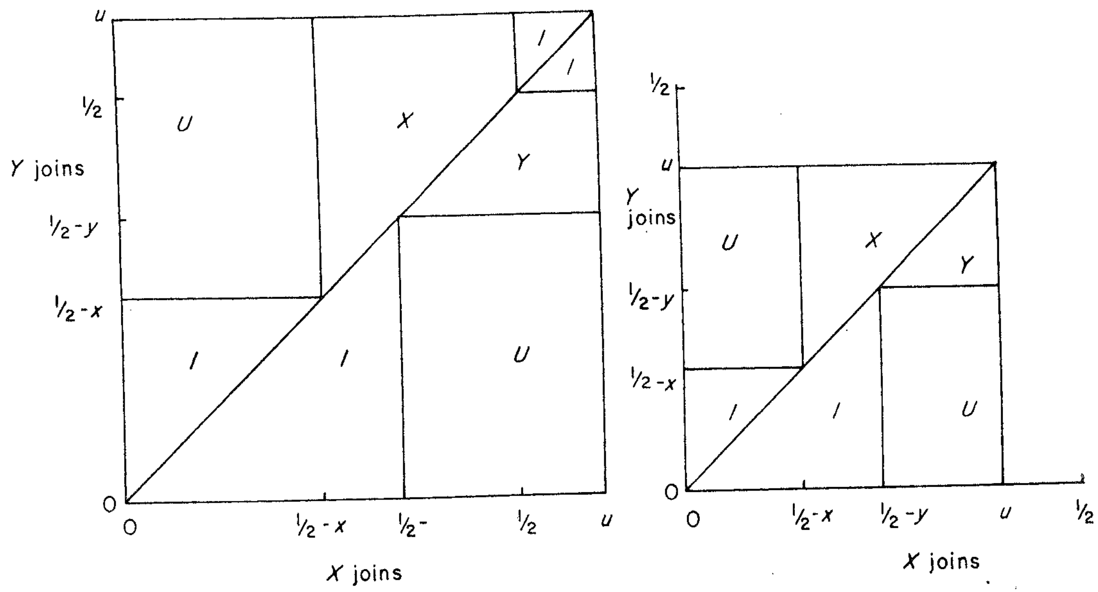
with power indexes

(0.039 0.377 0.377 0.034 0.034 0.034 0.034 0.034 0.034).

She has committed 0.063 to E, more than 0.043 she had while uncommitted. A bandwagon effect for E occurs.

Example

Let x be the fraction of votes controlled by bloc X, y the fraction controlled by bloc Y, and $u = 1 - x - y$ the fraction held by an ocean of uncommitted voters. Assume majority rule and $x < \frac{1}{2}, y < \frac{1}{2}$.



(a) $x + y \leq 1/2$ ($u \geq 1/2$). (b) $x + y \geq 1/2$ ($u \leq 1/2$).

The label Y in the figure means player Y pivots; U means an uncommitted voter pivots; I means an illegal ordering.

- Ordering represented by points in the lower left-hand corners and the upper right-hand corner are ruled out since we require exactly one of X or Y to appear at or before the pivot. For example, considering the case $x+y \geq 1/2$, the left bottom triangle above the diagonal corresponds to Y being pivotal if X and Y do not oppose each other. Since this triangle corresponds to the scenario where X joins before Y and Y pivots, this should be ruled out as "illegal". Similarly, for the case $x+y \leq 1/2$, the box at the top right corner corresponds to the scenario where U pivots without the participation of either X or Y. The power index of X is found to be

$$\phi_X(x,y) = \begin{cases} \frac{x(1-x-2y)}{1-x-y-x^2-y^2}, & \text{if } x+y \leq \frac{1}{2} \\ \frac{\left(\frac{1}{2}-y\right)^2}{(1-x-y)^2+2\left(\frac{1}{2}-x\right)\left(\frac{1}{2}-y\right)}, & \text{if } x+y \geq \frac{1}{2} \end{cases}.$$

The formulas for ϕ_Y are symmetric to this.

The total power of the uncommitted voters:

$$\phi_U(x,y) = \begin{cases} \frac{(1-2x)(1-2y)}{1-x-y-x^2-y^2}, & \text{if } x+y \le \frac{1}{2}, \\ \\ \frac{(1-2x)(1-2y)}{(1-x-y)^2+2\left(\frac{1}{2}-x\right)\left(\frac{1}{2}-y\right)}, & \text{if } x+y \ge \frac{1}{2}. \end{cases}$$

- Consider a small bloc of uncommitted voters, comprising a fraction Δx of the total vote, considering whether to join X.

- If they do, the power increment they will contribute is $\phi_X(x+\Delta x, y) - \phi_X(x,y)$. If they remain uncommitted, as they are independent and sharing equal power, their total power will be $(\Delta x/u)\phi_U(x,y)$.

- They should join X if

$$\phi_X(x+\Delta x, y) - \phi_X(x,y) > \frac{\Delta x}{u}\phi_U(x,y).$$

Taking the limit $\Delta x \to 0$, we obtain

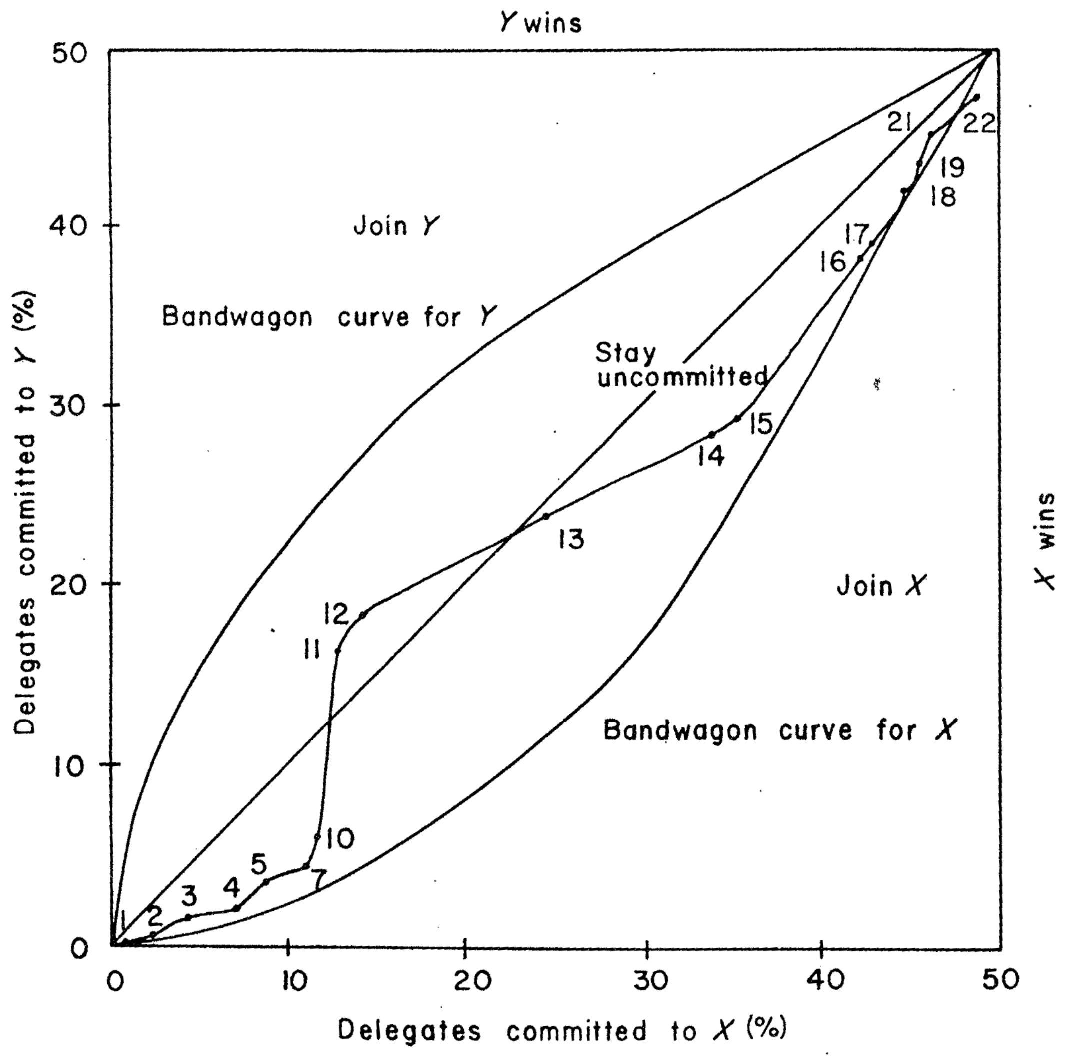
$$\frac{\partial \phi_X}{\partial x} > \frac{\phi_U}{u}.$$

- When $x + y \geq \frac{1}{2}$, the curve where $\frac{\partial \phi_X}{\partial x} = \frac{\phi_U}{u}$ is a straight line

$$\frac{1}{2} - y = a\left(\frac{1}{2} - x\right), \quad a \approx 1.78.$$

- When $x + y \leq \frac{1}{2}$, the bandwagon curves for X and Y are plotted in the figure. It is advantageous to join X when (x, y) lies below the bandwagon curve for X. When x and y are both small, x has to be several times larger than y in order that the uncommitted voters join X (see the bandwagon curve at the lower left corner).

1976-Race between Ford (player X) and Reagan (player Y) for the Republican nomination. The small numbers along the curve represent the number of weeks lapsed since the race started. By week 22, the uncommitted voters joined Ford to take advantage of the bandwagon effect.



2.2 Power distribution in weighted voting systems

Meeting the target

- Suppose the players are the countries of a federal union, one may wish the power of a player to be proportional to its population.

- Given the "population" vector $\boldsymbol{p}$ representing the ideal influence of the players, find the distribution of weights $(w_1, \cdots, w_n)$ and the quota of such that the sum of the differences between the target and the power is minimized.

- Let $d(x, y)$ be the distance between the power index and the target:

$$d(x, y) = \sum_{i=1}^{n} (x_i - y_i)^2, \quad \boldsymbol{x} \in \mathbb{R}^n \quad \text{and} \quad \boldsymbol{y} \in \mathbb{R}^n.$$

  We write d^{SS} = distance between the Shapley-Shubik index and the target, and similar definition for d^B.

3-player case

Given $w_1 \geq w_2 \geq w_3$, the four different weighted quota games for $n = 3$, with examples, are presented below.

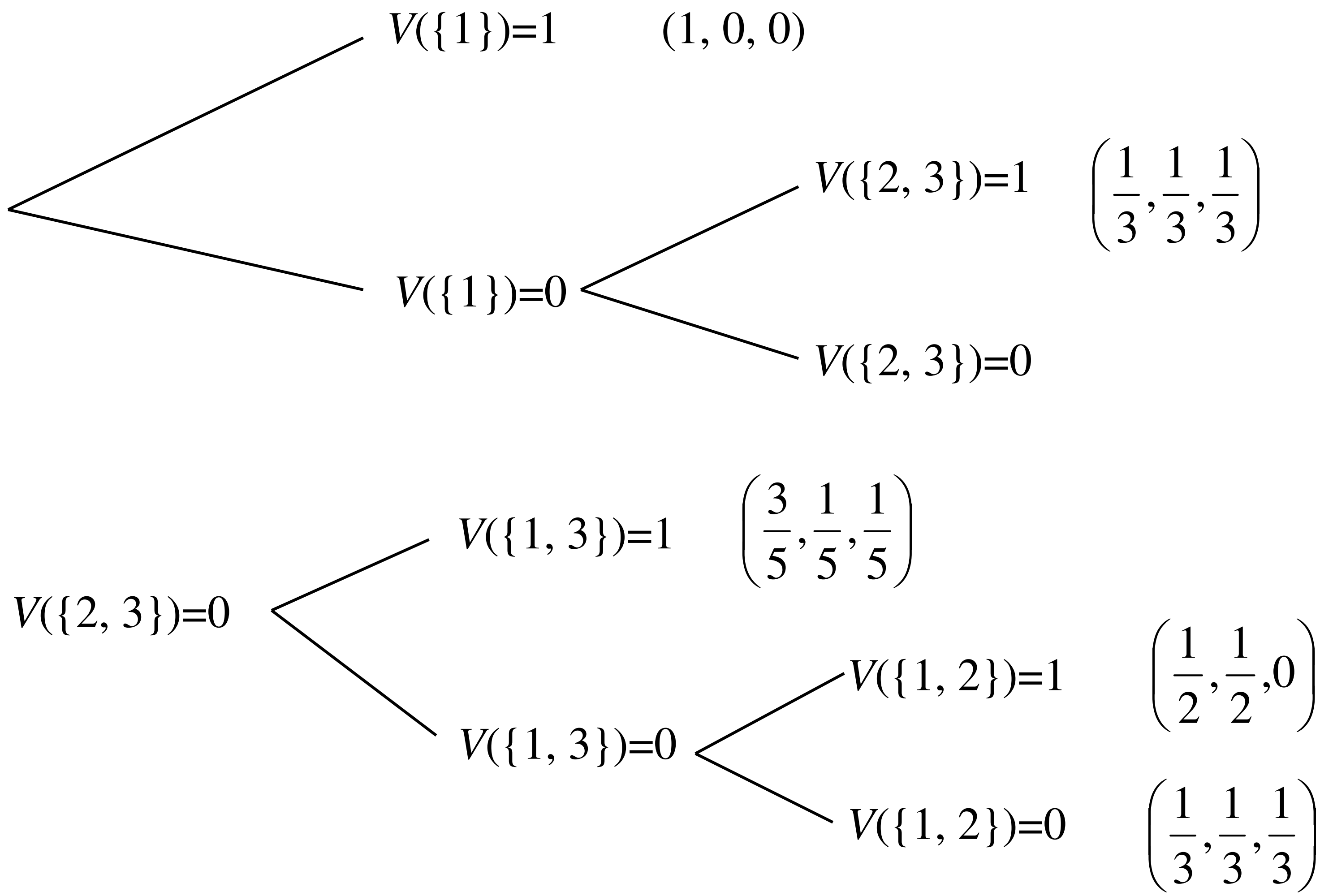
Conditions on w	Banzhaf	Shapley-Shubik	Example
$w_1 \geq q$	$\beta^1 = (1, 0, 0)$	$\phi^1 = (1, 0, 0)$	$G_1 = (2; 2, 0, 0)$
$w_1 + w_3 < q$ and $w_1 + w_2 \geq q$	$\beta^2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$\phi^2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$G_2 = (2; 1, 1, 0)$
$w_1 < q$ and $w_2 + w_3 \geq q$	$\beta^3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\phi^3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$G_3 = (4; 2, 2, 2)$
$w_1 < q$ and $w_2 + w_3 < q$ and $w_1 + w_3 \geq q$	$\beta^4 = \left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right)$	$\phi^4 = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$	$G_4 = (5; 4, 2, 2)$

For example, suppose $w_1 < q$ and $w_2 + w_3 \geq q$, then the winning coalitions are $\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$. All players then have equal power.

Notation

Let W be the set of all winning coalitions. A voting game is *proper* if $S \in W$ then $N \backslash S \notin W$, where N is the set of all players. If a coalition $S \in W$, then $V(S) = 1$; otherwise, $V(S) = 0$.

- It is impossible to have $V(\{2\}) = 1$ or $V(\{3\}) = 1$. If otherwise, $V(\{1\}) = 1$, then $V(\{2,3\}) = 0$. This leads to a contradiction.



Vector of power under Banzhaf index

Banzhaf index

1. $V(\{1\}) = 1$

   This means player 1 has all the power, so the vector of power is $(1, 0, 0)$.

2. $V(\{1\}) = 0$ and $V(\{2, 3\}) = 1$

   We have $V(\{1, 2, 3\}) = 1, V(\{1, 3\}) = 1$ and $V(\{1, 2\}) = 1$. There is no decisive player since $V(\{1\}) = V(\{2\}) = V(\{3\}) = 0$. However, in $\{1, 2\}, \{1, 3\}, \{2, 3\}$, every player is decisive. Therefore, the vector of power is $(1/3, 1/3, 1/3)$.

3. $V(\{1\}) = 0, V(\{2, 3\} = 0$ and $V(\{1, 3\}) = 1$.

   We have $V(\{1, 2, 3\}) = 1$ and $V(\{1, 2\}) = 1$.

   In $\{1, 2, 3\}$, only player 1 is decisive while in $\{1, 2\}, \{1, 3\}$, every player is decisive. Therefore, the vector of power is $(3/5, 1/5, 1/5)$.

4. $V(\{1\}) = 0, V(\{2,3\}) = 0, V(\{1,3\}) = 0, V(\{1,2\}) = 1$

   We have $V(\{1,2,3\}) = 1$. Only player 3 is not decisive while in $\{1,2\}$, every player is decisive. Therefore, the vector of power is $(1/2, 1/2, 0)$.

5. $V(\{1\}) = 0, V(\{2,3\}) = 0, V(\{1,3\}) = 0, V(\{1,2\}) = 0$

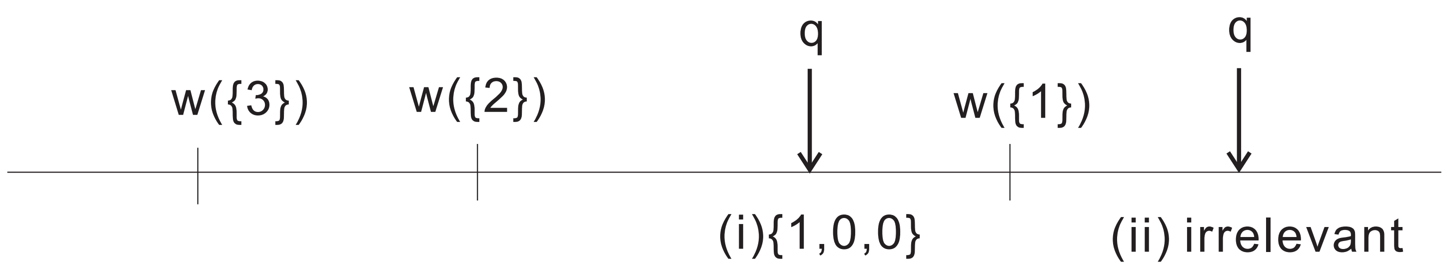
   We have $V(\{1,2,3\}) = 1$. In $\{1,2,3\}$, every player is decisive and the vector of power is $(1/3, 1/3, 1/3)$.

In summary, there are 5 possible cases but only 4 vectors of power. The Banzhaf indexes are:

$$(1/3, 1/3, 1/3), (3/5, 1/5, 1/5), (1/2, 1/2, 0) \text{ and } (1, 0, 0).$$

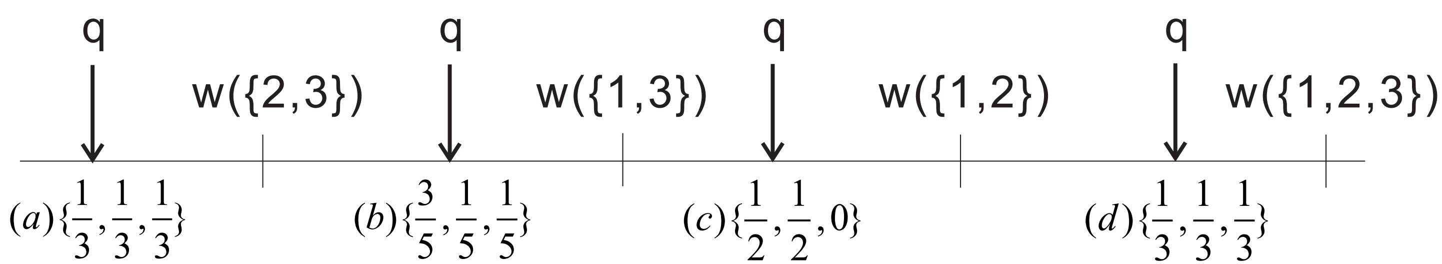
Let $w(S)$ denote the sum of weights in the coalition S.

First, we consider one-player coalitions



The quota q may fall (i) below $w(\{1\})$ and player 1 is the dictator; or (ii) above $w(\{1\})$, and as a result, there is no one-player winning coalition. We rule out the case where q falls below $w(\{2\})$. If otherwise, $\{2,3\}$ is also winning and its complement $\{1\}$ cannot be winning.

Next, we consider two-player and three-player coalitions



Remarks

It is necessary to refine the potential ordering of weights in coalitions according to $w(\{1\}) < w(\{2,3\})$ or otherwise. We avoid the discussion on the special case of $w(\{1\}) = w(\{2,3\})$.

1. Suppose $w(\{1\}) < w(\{2,3\})$, then $\{1\}$ can never be winning. Implicitly, we deduce that $w(\{1\}) < q$, and there will be no one-player winning coalition. We can perform the analysis solely based on the two-player and three-player coalitions.

2. Suppose $w(\{1\}) > w(\{2,3\})$, then $\{2,3\}$ can never be winning. We rule out the scenario where $q < w(\{2,3\})$ and need to include the new cases (i) $w(\{2,3\}) < q < w(\{1\})$ and (ii) $w(\{1\}) < q < w(\{2,3\})$.

   Case (i) gives the power distribution $\{1,0,0\}$ while case (ii) gives $\left\{\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right\}$.

Assuming $w(\{1\}) < w(\{2,3\})$, we examine the various outcomes when q increases gradually.

(a) $w(\{1\}) < q < w(\{2,3\})$
All two-player coalitions and three-player coalitions are winning, so the players have equal power.

(b) $w(\{2,3\}) < q < w(\{1,3\})$
Since $\{2,3\}$ is not winning, player 1 has higher power compared to players 2 and 3, so the power distribution is $\left\{\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right\}$.

(c) $w(\{1,3\}) < q < w(\{1,2\})$
Since $\{1,2\}$ is the only two-player winning coalitions, so player 3 is a dummy. This gives the power distribution $\left\{\frac{1}{2}, \frac{1}{2}, 0\right\}$.

(d) $w(\{1,2\}) < q < w(\{1,2,3\})$
All two-player coalitions are losing and only $\{1,2,3\}$ is the winning coalition. Therefore, the players have equal power.

What would happen when some of the voters have the same weight?

1. Suppose $w_1 = w_2 > w_3$, we rule out the case where player 1 is the dictator of the game. We consider the following cases:

   (a) $w_1 < q < w_1 + w_3$, we have the power distribution $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$. This is because player 3 is critical in $\{1,3\}$ and $\{2,3\}$, while player 1 is critical in $\{1,2\}$ and $\{1,3\}$ (similarly for player 2).

   (b) $w_1 + w_3 < q < 2w_1$, the power distribution is $\left\{\frac{1}{2}, \frac{1}{2}, 0\right\}$.

   (c) $2w_1 < q < 2w_1 + w_3$, the power distribution is $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$.

   Note that the power distribution $\left\{\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right\}$ is ruled out.

2. Suppose $w_1 > w_2 = w_3$, we rule out the power distribution $\left\{\frac{1}{2}, \frac{1}{2}, 0\right\}$. It is still possible for player 1 to be the dictator.

3. Suppose $w_1 = w_2 > w_3$, the 3 players are equal in power.

Shapely-Shubik index

There are 6 possible orders with 3 players

1. $V(\{1\}) = 1$

   Player 1 is the only pivotal, even if it arrives last in the coalition. The vector of power is $(1, 0, 0)$

2. $V(\{1\}) = 0$ and $V(\{2, 3\}) = 1$

   We have $V(\{1, 2, 3\}) = 1, V(\{1, 3\}) = 1$ and $V(\{1, 2\}) = 1$

   For each order, the player in the second position is pivotal, so the vector of power is $(1/3, 1/3, 1/3)$.

3. $V(\{1\}) = 0, V(\{2,3\}) = 0$ and $V(\{1,3\}) = 1$

   We have $V(\{1,2,3\}) = 1$ and $V(\{1,2\}) = 1$.

   When player 1 is not first in the order, it is always pivotal.

   When player 1 is first in the orders, the pivotal is the player which arrives second in the order. Therefore, the vector of power is $(2/3, 1/6, 1/6)$.

4. $V(\{1\}) = 0, V(\{2,3\}) = 0, V(\{1,3\}) = 0, V(\{1,2\}) = 1$

   We have $V(\{1,2,3\}) = 1$. When player 1 is first in the order, player 2 is pivotal. When player 2 is first in the order, player 1 is pivotal. In the orders 312 and 321, the player who arrives last is pivotal. Therefore, the vector of power is $(1/2, 1/2, 0)$.

5. $V(\{1\}) = 0, V(\{2,3\}) = 0, V(\{1,3\}) = 0, V(\{1,2\}) = 0$

We have $V(\{1,2,3\}) = 1$. It is always the player who arrives last in the order the pivotal. The vector of power is $(1/3, 1/3, 1/3)$.

In summary, there are 4 vectors of power using the Shapley-Shubik index:

$$(1/3, 1/3, 1/3), (2/3, 1/6, 1/6), (1/2, 1/2, 0) \text{ and } (1, 0, 0).$$

Remark

There has not existed a general formula to determine all the possible vectors for a given n and a given power index.

Target power distribution

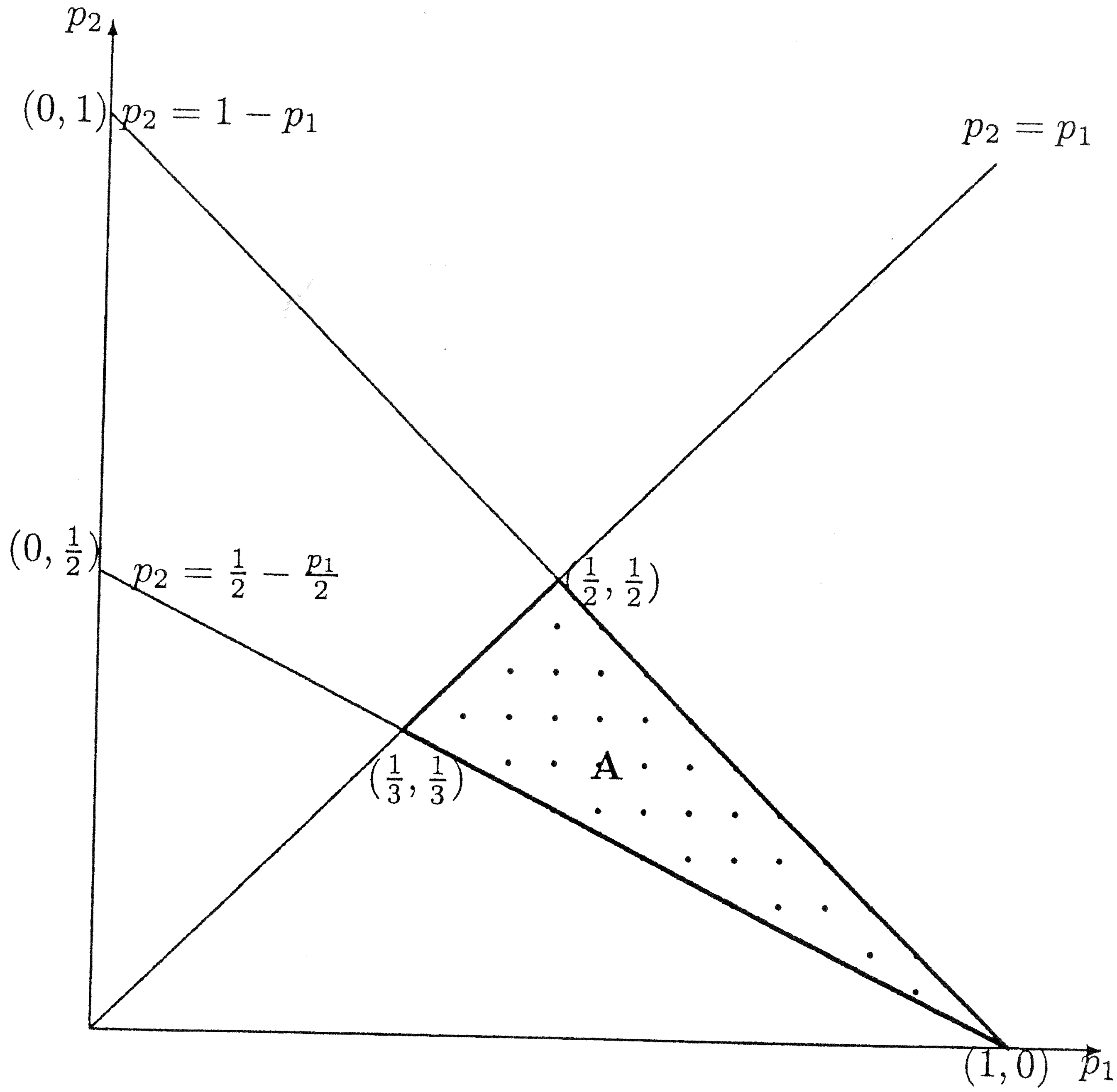
Let p_i denote the power index of player i, $i = 1, 2, 3$. Assume that $p_2 + p_2 + p_3 = 1$, we can represent all the possible targets in a simplex. Further, $p_1 \geq p_2 \geq p_3$ which implies $p_2 \geq \frac{1}{2} - \frac{p_1}{2}$ since $2p_2 + p_1 \geq 1$.

In summary, the simplex that contains feasible power distributions satisfies

$$\begin{aligned} p_1 &\geq p_2 \\ p_2 &\geq \frac{1}{2} - \frac{p_1}{2} \\ p_2 &\leq 1 - p_1. \end{aligned}$$

The area of the admissible region A is equal to

$$\frac{1}{2} \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{3} & 1 \\ 1 & 0 & 1 \end{vmatrix} = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{2} - \frac{1}{3} - \frac{1}{6}\right) = \frac{1}{12} \approx 0.0833.$$



The possible target vectors

If we calculate the distances d^B and d^{SS} between a vector of power and a target p, we obtain easily:

- $$\begin{aligned} d^B = d^{SS} &= (p_1 - \tfrac{1}{3})^2 + (p_2 - \tfrac{1}{3})^2 + (p_3 - \tfrac{1}{3})^2 \\ &= \sum_{i=1}^{3} p_i^2 - \frac{1}{3} \text{ for the vector } (1/3, 1/3, 1/3). \end{aligned}$$

- $d^B = \displaystyle\sum_{i=1}^{3} p_i^2 + \frac{1}{25} - \frac{4p_1}{5}$ for the vector $(3/5, 1/5, 1/5)$.

- $d^{SS} = \displaystyle\sum_{i=1}^{3} p_i^2 + \frac{1}{6} - p_1$ for the vector $(2/3, 1/6, 1/6)$.

- $d^B = d^{SS} = \displaystyle\sum_{i=1}^{3} p_i^2 + \frac{1}{2} - p_1 - p_2$ for the vector $(1/2, 1/2, 0)$.

- $d^B = d^{SS} = \displaystyle\sum_{i=1}^{3} p_i^2 + 1 - 2p_1$ for the vector $(1, 0, 0)$.

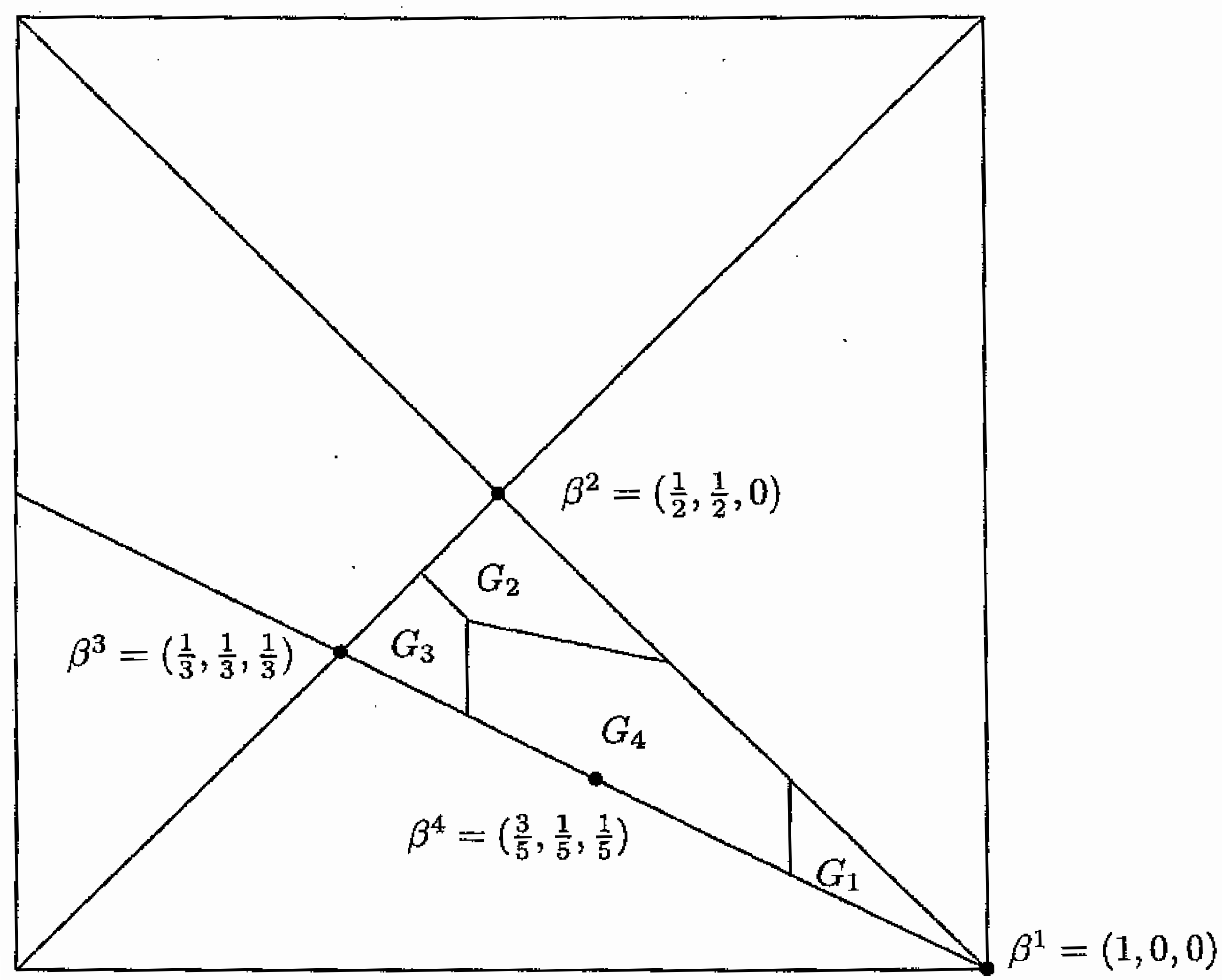
We can compare the distances and obtain for the Banzhaf index:

- The vector of power $\beta^1 = (1,0,0)$ and G_1 minimizes d^B if $p_1 > 5/6$. The result is obtained by comparing $d^B = \sum_{i=1}^{3} p_i^2 + \frac{1}{6} - p_1$ for $(3/5, 1/5, 1/5)$ and $d^B = \sum_{i=1}^{3} p_i^2 + 1 - 2p_1$ for $(1,0,0)$.

- The vector of power $\beta^2 = (1/2, 1/2, 0)$ and G_2 minimizes d^B if $p_2 > 1/3$ and $p_2 > 5/6 - p_1$.

- The vector of power $\beta^3 = (1/3, 1/3, 1/3)$ and G_3 minimizes d^B if $p_2 < 1/2$ and $p_2 < 5/6 - p_1$.

- The vector of power $\beta^4 = (3/5, 1/5, 1/5)$ and G_4 minimizes d^B otherwise. The 3 boundaries are given by: $1/25 - 4p_1/5 < -1/3$, $1/25 - 4p_1/5 < 1/2 - p_1 - p_2$ and $1/25 - 4p_1/5 < 1 - 2p_1$; that is if $7/15 < p_1 < 4/5$ and $p_1 + 5p_2 < 23/10$.



The different closest games for the Banzhaf index for $n = 3$

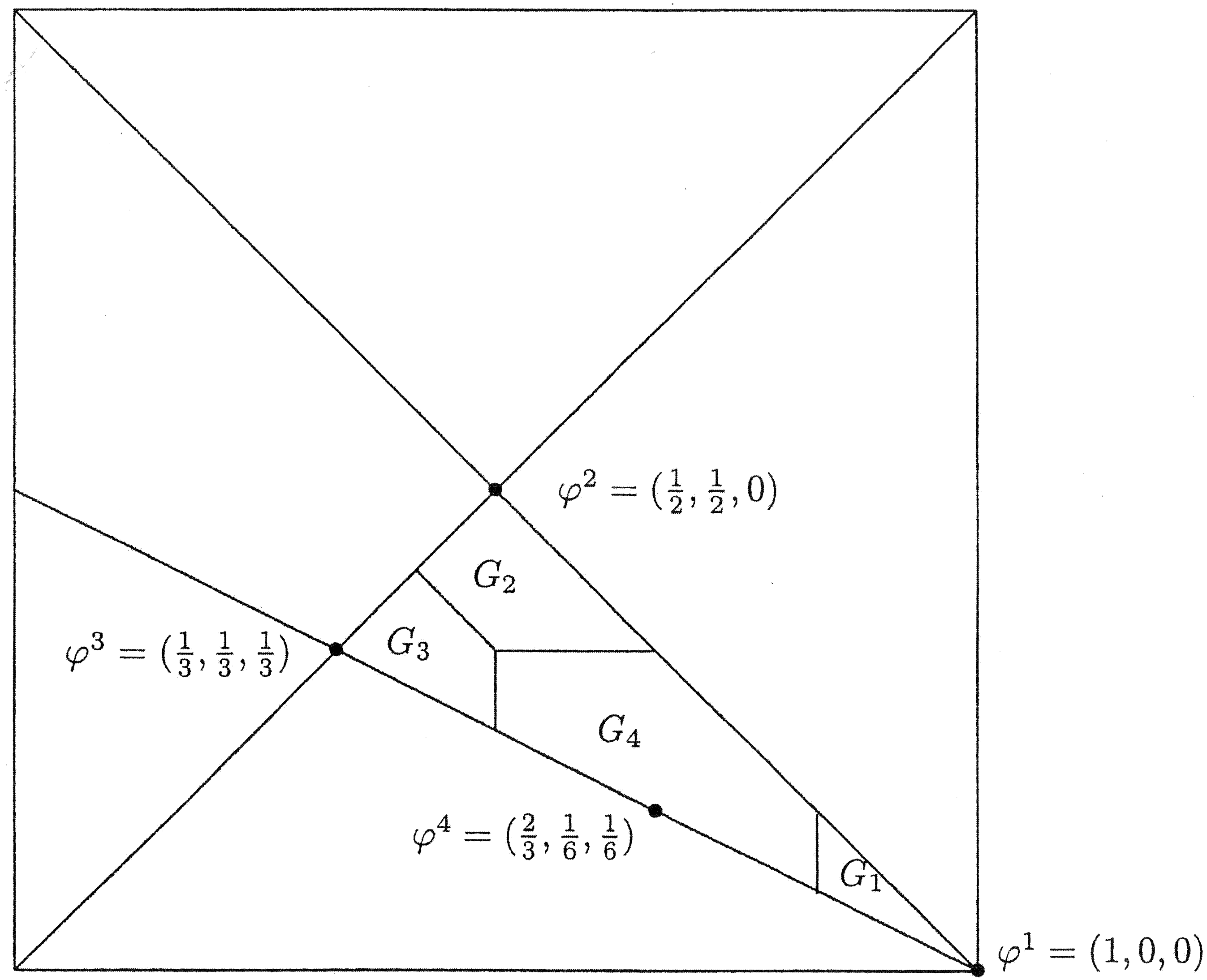
The same reasoning for the Shapley-Shubik index enables us to define the following domains:

- The vector of power $\phi^1 = (1, 0, 0)$ and G_1 minimizes d^{SS} if $p_1 > 4/5$.

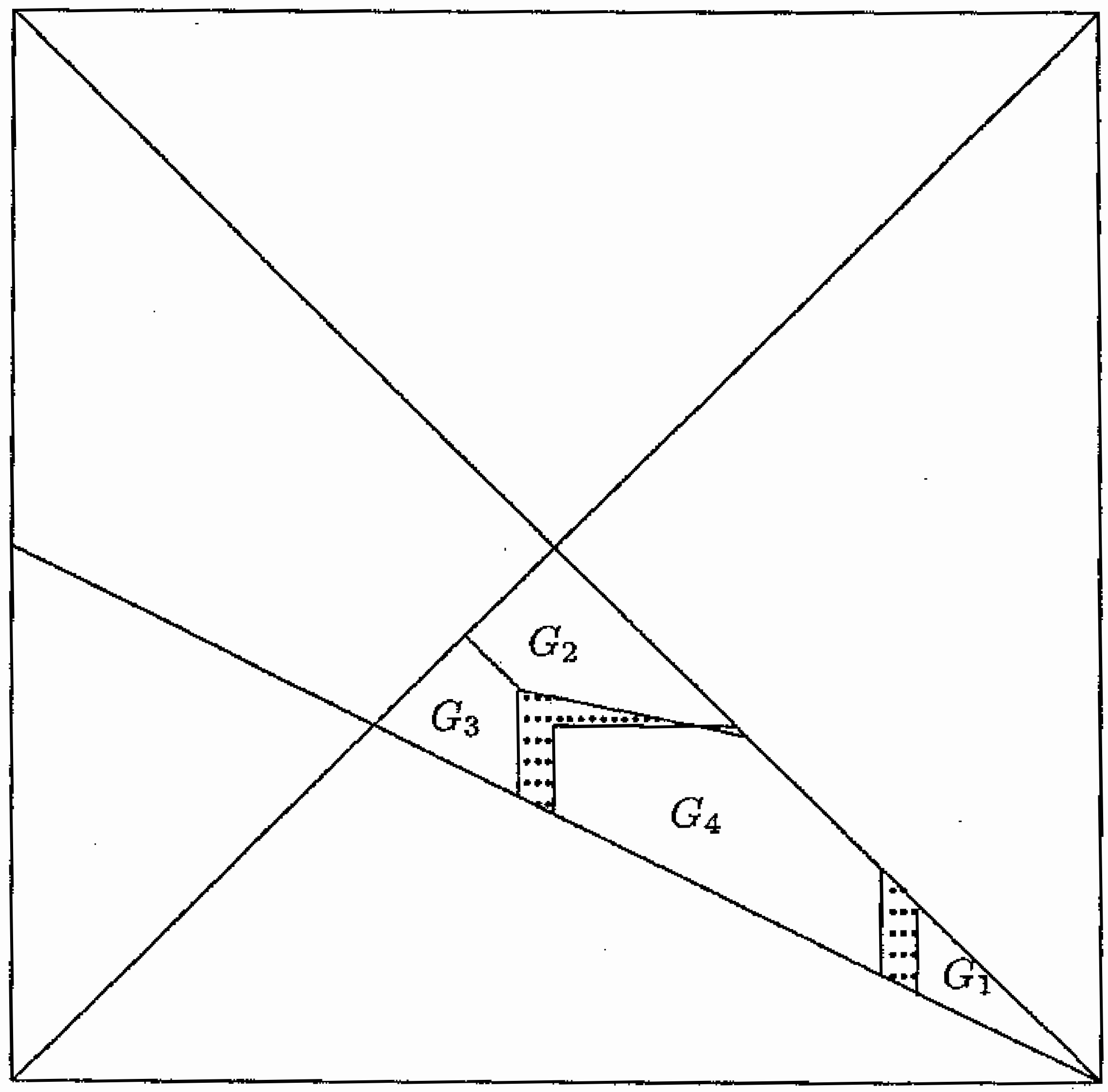
- The vector of power $\phi^2 = (1/2, 1/2, 0)$ and G_2 minimizes d^{SS} if $p_2 > 23/50 - p_1/5$ and $p_2 > 5/6 - p_1$.

- The vector of power $\phi^3 = (1/3, 1/3, 1/3)$ and G_3 minimizes d^{SS} if $p_2 < 7/15$ and $p_2 > 5/6 - p_1$.

- The vector of power $\phi^4 = (2/3, 1/6, 1/6)$ and G_4 minimizes d^{SS} otherwise, that is if $7/5 < p_1 < 4/5$ and $p_1 + 5p_2 < 23/10$.



The different closest games for the Shapley-Shubik index for $n = 3$



Different Shapley-Shubik and Banzhaf inverse games for $n = 3$

Example

For a voting game with 3 players, suppose the target power distribution of the 3 players is

$$\boldsymbol{p}_{\text{tar}} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Based on Shapley-Shubik indexes, find the weighted voting system whose power distribution is closest to the given target distribution.

Solution

Recall that there are only 4 possible power distributions in a 3-player voting game under the Shapley-Shubik indexes: $(1, 0, 0)$, $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$.

$$\text{The distance between } \boldsymbol{p}_{\text{tar}} \text{ and } (1,0,0) = \left(\frac{1}{2}-1\right)^2+\left(\frac{1}{4}-0\right)^2+\left(\frac{1}{4}-0\right)^2 = \frac{3}{8}.$$

$$\text{The distance between } \boldsymbol{p}_{\text{tar}} \text{ and } \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) = \left(\frac{1}{2}-\frac{1}{3}\right)^2+\left(\frac{1}{4}-\frac{1}{3}\right)^2+\left(\frac{1}{4}-\frac{1}{3}\right)^2 = \frac{1}{24}.$$

$$\text{The distance between } \boldsymbol{p}_{\text{tar}} \text{ and } \left(\frac{1}{2},\frac{1}{2},0\right) = \left(\frac{1}{2}-\frac{1}{2}\right)^2+\left(\frac{1}{4}-\frac{1}{2}\right)^2+\left(\frac{1}{4}-0\right)^2 = \frac{1}{16}.$$

$$\text{The distance between } \boldsymbol{p}_{\text{tar}} \text{ and } \left(\frac{2}{3},\frac{1}{6},\frac{1}{6}\right) = \left(\frac{1}{2}-\frac{2}{3}\right)^2+\left(\frac{1}{4}-\frac{1}{6}\right)^2+\left(\frac{1}{4}-\frac{1}{6}\right)^2 = \frac{1}{24}.$$

The closest distance is $\frac{1}{24}$, achieved by either choosing $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ or $\left(\frac{2}{3},\frac{1}{6},\frac{1}{6}\right)$. The corresponding power distribution can be achieved by the respective weighted voting vector $[2;1,1,1]$ or $[5;3,2,2]$.

Four-player weighted voting systems

Editorial Committee

Editor-in-chief has 3 votes; Managing Editor has 2 votes; News Editor and Feature Editor, each has 1 vote. Total votes = 7, quota = 4.

- Given the weighted voting vector $[4; 3, 2, 1, 1]$, we aim to achieve the power distribution that matches with hierarchy of influence, where

$$P_{Chief} > P_{Man} > P_{News} = P_{Feat}.$$

- Take the 4 editors as A,B,C,D for convenience: $S_A^{(1)} = \{B, C\}$, $S_A^{(2)} = \{B, D\}$, $S_A^{(3)} = \{C, D\}$, $S_A^{(4)} = \{B\}$, $S_A^{(5)} = \{C\}$, $S_A^{(6)} = \{D\}$; $S_B^{(1)} = \{A\}$, $S_B^{(2)} = \{C, D\}$; $S_C^{(1)} = \{A\}$, $S_C^{(2)} = \{B, D\}$; etc.

  The corresponding Banzhaf indexes are

$$B_{Chief} = \frac{1}{2}, \quad B_{Man} = \frac{1}{6}, \quad B_{News} = \frac{1}{6}, \quad B_{Feat} = \frac{1}{6}.$$

- Can we design the weights so that the desired hierarchy of influence is achieved?

  Consider a weighted voting system of size 4 as represented by

$$[q; w_1, w_2, w_3, w_4]$$

  with $w_1 \geq w_2 \geq w_3 \geq w_4$, what are the possible power distributions?

  We assume

$$\frac{w_1 + w_2 + w_3 + w_4}{2} < q \leq w_1 + w_2 + w_3 + w_4.$$

  Furthermore, we assume

(a) $w_i < q$ for each i; that is, none of the players form a winning coalition on herself.

(b) $\sum_{j \neq i} w_j \geq q$; that is, none of the players has veto power.

Republic of PAIN (Pennsylvania and its Neighbors)

Total of 90 votes; quota $= 46$

Apportionment according to population:

New York	38
Pennsylvania	25
Ohio	23
West Virginia	4

Any two of New York, Pennsylvania and Ohio can pass a bill. West Virginia is a dummy. The Banzhaf indexes of the 4 states are found to be

$$B_{NY} = B_{Pen} = B_{Ohio} = \frac{1}{3} \quad \text{and} \quad B_{WV} = 0.$$

Any weighted voting system of the form

$$[2m; m, m, m, 1], \quad m \geq 2$$

would yield the same power distribution.

Theorem

In any 4-player weighted voting system with no veto power, there are only 5 possible power distributions:

(a) $B(P_i) = \frac{1}{4}$ for every i.

(b) $B(P_1) = B(P_2) = \frac{1}{3}, B(P_3) = B(P_4) = \frac{1}{6}$.

(c) $B(P_1) = \frac{5}{12}, B(P_2) = B(P_3) = \frac{1}{4}$ and $B(P_4) = \frac{1}{12}$.

(d) $B(P_1) = \frac{1}{2}$ and $B(P_2) = B(P_3) = B(P_4) = \frac{1}{6}$.

(e) $B(P_1) = B(P_2) = B(P_3) = \frac{1}{3}$ and $B(P_4) = 0$.

When we impose $w_3 = w_4$, then cases (c) and (e) are impossible.

Remarks

1. It is impossible to achieve

$$B_1 > B_2 > B_3 = B_4$$

   as required by the Editorial Committee.

2. The best weighted voting system for PAIN is case (c), which can be achieved by $[13; 8, 6, 6, 1]$.

3. For any 4-player weighted voting system with no player having veto power, there are at most one dummy player in the system. One can show easily that if there exist more than one dummy in the system, then at least one player will have veto power.

Proof of the Theorem

1. In the absence of veto power, any 4-player coalition is a winning coalition that does not yield any critical instances. Suppose any one of the players is critical, then that player would have veto power.

2. All the 3-player coalitions are winning coalitions. Otherwise, the missing player in that 3-player coalition has veto power. These 4 three-player coalitions are: $\{P_2, P_3, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_2, P_3\}$. Since all 3-player coalitions are winning, any single-player coalition (complement of the respective 3-player coalition) is losing.

3. In any 2-player winning coalition, both players must be critical since we do not permit single-player winning coalitions.

The possible 2-player winning coalitions are

$$\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_3, P_4\}.$$

However, if $\{P_1, P_2\}$ wins then $\{P_3, P_4\}$ must lose;
similarly, if $\{P_1, P_3\}$ wins then $\{P_2, P_4\}$ must lose;
lastly, if $\{P_1, P_4\}$ wins then $\{P_2, P_3\}$ must lose; or otherwise.

Therefore, there are two cases of having three 2-player winning coalitions:

$$\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\} \text{ or } \{P_1, P_2\}, \{P_1, P_3\}, \{P_2, P_3\}.$$

Various possible number of 2-player winning coalitions

The possible number of two-player winning coalitions can be 0,1,2,3. Each of the first 3 cases of having 0,1 or 2 two-player coalition generates one power distribution. The 2 remaining power distributions are generated when the number of two-player coalitions is 3.

Case (i) There is no 2-player winning coalition.

Every player is critical in each of the four 3-player winning coalitions. This is because dropping any one player turns a winning three-player winning coalition into losing (as there is no two-player winning coalition). Total number of critical instances = 12, and we have equal share of power among the 4 players.

$$B(P_1) = B(P_2) = B(P_3) = B(P_4) = \frac{1}{4}.$$

Case (ii) There is only one 2-player winning coalition, which must be $\{P_1, P_2\}$. Therefore, P_1 and P_2 are both critical in this coalition and also in the coalitions $\{P_1, P_2, P_3\}$ and $\{P_1, P_2, P_4\}$.

- On the other hand, P_3 and P_4 are not critical in $\{P_1, P_2, P_3\}$ and $\{P_1, P_2, P_4\}$ but are critical in $\{P_1, P_3, P_4\}$ and $\{P_2, P_3, P_4\}$.

- Also, P_1 is critical in $\{P_1, P_3, P_4\}$ and P_2 is critical in $\{P_2, P_3, P_4\}$.

- As a summary, we have

  4 critical instances for each of P_1 and P_2

  2 critical instances for each of P_3 and P_4.

  Therefore, $B(P_1) = B(P_2) = \frac{1}{3}, B(P_3) = B(P_4) = \frac{1}{6}$.

Case (iii) If there are two winning 2-player coalitions, then they must be $\{P_1, P_2\}$ and $\{P_1, P_3\}$. This can only occur if $w_3 > w_4$. Otherwise, if $w_3 = w_4$, then $\{P_1, P_3\}$ winning would imply $\{P_1, P_4\}$ winning as well.

- P_1 is critical in each of the three 3-player coalitions containing P_1, yielding 5 critical instances for P_1.
- P_2 is critical in $\{P_1, P_2, P_4\}, \{P_2, P_3, P_4\}$ and $\{P_1, P_2\}$.
- P_3 is critical in $\{P_1, P_3, P_4\}, \{P_2, P_3, P_4\}$ and $\{P_1, P_3\}$.
- P_4 is critical in $\{P_2, P_3, P_4\}$.

There are 12 critical instances in total. Therefore,

$$B(P_1) = \frac{5}{12}, B(P_2) = B(P_3) = \frac{1}{4} \quad \text{and} \quad B(P_4) = \frac{1}{12}.$$

Case (iv) There are two cases involving three winning 2-player coalitions.

(a) $\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}$ winning

- Each of P_2, P_3, P_4 yields one critical instance in these 2-player winning coalitions while P_1 yields three critical instances.
- P_1 is critical in the coalitions $\{P_1, P_2, P_3\}, \{P_1, P_2, P_4\}, \{P_1, P_3, P_4\}$ while P_2, P_3 and P_4 are critical in $\{P_2, P_3, P_4\}$. Therefore,

$$B(P_1) = \frac{1}{2}, B(P_2) = B(P_3) = B(P_4) = \frac{1}{6}.$$

(b) We may have $\{P_1, P_2\}, \{P_1, P_3\}$ and $\{P_2, P_3\}$ winning. This occurs only when $w_3 > w_4$.

- Now P_4 is never a critical player since removing P_4 from any 3-player coalition leaves one of these three winning 2-player coalitions. Therefore, $B(P_4) = 0$.
- Removing any player from $\{P_1, P_2, P_3\}$ still leaves a winning coalition, but in the other three 3-player coalitions containing P_4, the other two players are critical.
- Here, we have a total of 12 critical instance, with each of P_1, P_2, P_3 being critical 4 times. As a result, we obtain

$$B(P_1) = B(P_2) = B(P_3) = \frac{1}{3}.$$

Remarks

1. With a complete enumeration of the power distributions feasible for weighted voting systems of size n, can one efficiently generate a complete list of feasible power distributions for size $n+1$ weighted voting system?

2. If a certain power distribution is desired, can one efficiently construct a weighted voting system that comes closest to the ideals?

Strict hierarchy of power $B(P_1) > B(P_2) > \cdots > B(P_n)$.

1. In the case of 4-player weighted voting system, there always exist at least two players with the same Banzhaf power index.

2. When $n = 5$, we have $[9; 5, 4, 3, 2, 1]$ which yields Banzhaf power distribution: $\left(\frac{9}{25}, \frac{7}{25}, \frac{5}{25}, \frac{3}{25}, \frac{1}{25}\right)$. Note that $[8; 5, 4, 3, 2, 1]$ does not induce the above power distribution. This is the only power distribution, of the 35 possibilities in the 5-player case, with strict hierarchy of power.

3. When $n = 6, [11; 6, 5, 4, 3, 2, 1]$ yields $\left(\frac{9}{28}, \frac{7}{28}, \frac{5}{28}, \frac{3}{28}, \frac{3}{28}, \frac{1}{28}\right)$. However, $[15; 9, 7, 4, 3, 2, 1]$ yields the power distribution with strict hierarchy

$$\left(\frac{5}{12}, \frac{3}{16}, \frac{1}{6}, \frac{1}{8}, \frac{1}{16}, \frac{1}{24}\right)$$

as desired.

2.3 Incomparability and desirability

We always consider *monotone* yes-no voting system — winning coalitions remain winning if new voters join them.

Let x and y be two players, how to formalize the following intuitive notions:

> "x and y have equal power"
> "x and y have the same amount of influence"
> "x and y are equally desirable in terms of the formation of a winning coalition"

The bottom line is to see whether x and y are both critical to turn the coalition Z from losing to winning upon joining the coalition.

Definition

Suppose x and y are two voters in a yes–no voting system. Then we shall say that x and y are equally desirable (or, the desirability of x and y is equal, or the same), denoted $x \approx y$, if and only if the following holds:

For every coalition Z containing neither x nor y,
the result of x joining Z is a winning coalition
if and only if
the result of y joining Z is a winning coalition,

We say: "x and y are equivalent" when $x \approx y$. In other words, x and y are equally desirable with reference to the formation of a winning coalition.

Example

Consider $[51; 1, 49, 50]$ with 3 players A, B and C. The winning coalitions are

$$\{A, C\}, \{B, C\} \quad \text{and} \quad \{A, B, C\}.$$

1. $A \approx B$

   Test for all coalitions that do not contain A and B. The only coalitions containing neither A nor B are the empty coalition (Z_1) and the coalition consisting of C alone (Z_2). The result of A joining Z_1 is the same as the result of B joining Z_1 (a losing coalition). The result of A joining Z_2 is the same as the result of B joining Z_2 (a winning coalition).

2. A and C are not equivalent

   Neither belongs to $Z = \{B\}$, but A joining Z yields $\{A, B\}$ which is losing while C joining Z yields $\{B, C\}$ which is winning with 51 votes.

Definition ‘

For two voters x and y in a yes–no voting system, we say that the *desirability of x and y is incomparable* (不可比較), denoted by

$$x|y,$$

if and only if there are coalitions Z and Z', neither one of which contains x or y, such that the following hold:

1. the result of x joining Z is a winning coalition, but the result of y joining Z is a losing coalition, and

2. the result of y joining Z' is a winning coalition, but the result of x joining Z' is a losing coalition.

x is critical but not y for coalition Z while y is critical but not x for another coalition Z'.

Example

In the US federal system, a House Representative and a Senator are incomparable. The Vice President and a Senator are not incomparable. Since the Vice President is the tie breaker, it is not possible that a coalition with the Vice President is critical while with a senator is not.

Question Which yes–no voting systems will have incomparable voters?

Proposition

For any yes–no voting system, the following are equivalent:

1. There exist voters x and y whose desirability is incomparable.

2. The system fails to be swap robust.

Proof

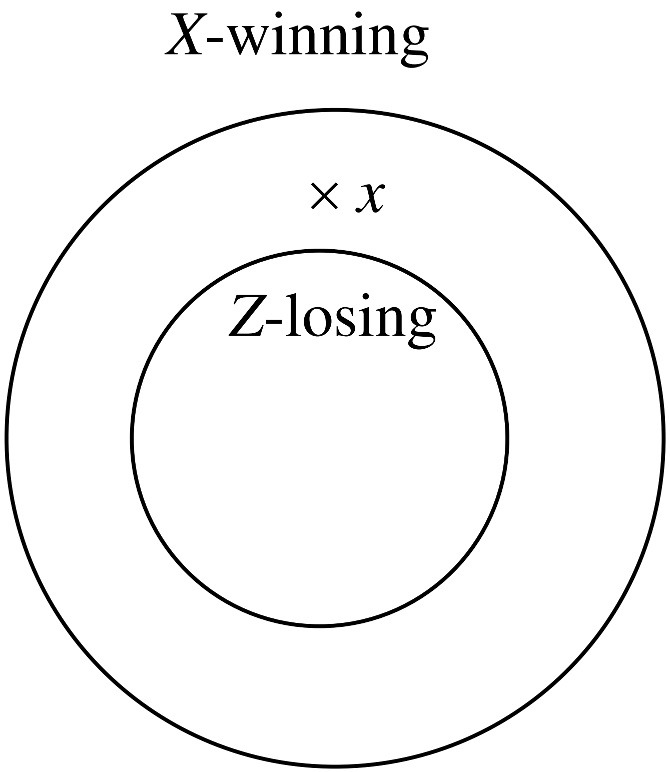
$1 \Rightarrow 2$

- Assume that the desirability of x and y is incomparable, and let Z and Z' be coalitions such that:

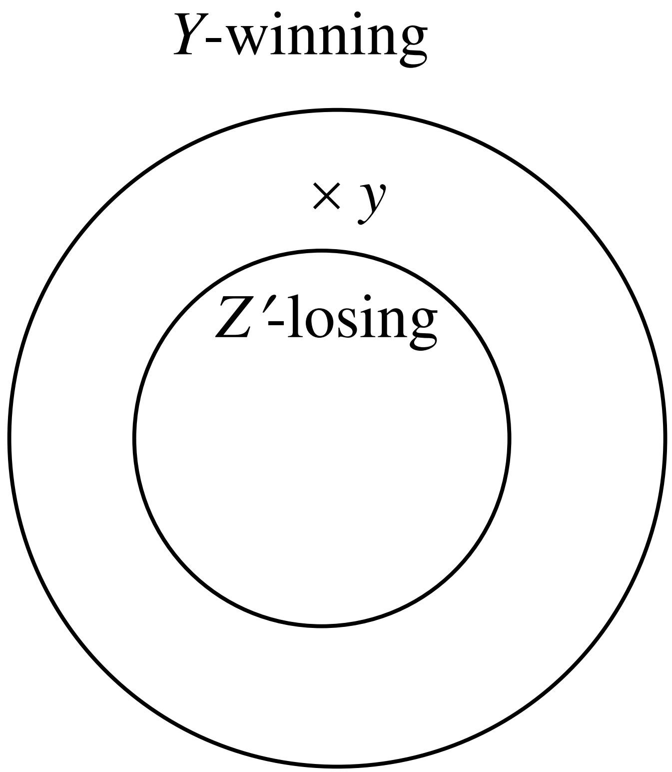
  Z with x added is winning;
  Z with y added is losing;
  Z' with y added is winning; and
  Z' with x added is losing.

- To see that the system is not swap robust, let X be the result of adding x to the coalition Z, and let Y be the result of adding y to the coalition Z'. Both X and Y are winning, but the one-for-one swap of x for y renders both coalitions losing.

Both Z and Z' do not contain x and y.



$Z \cup \{y\}$ remains to be losing



$Z' \cup \{x\}$ remains to be losing.

$2 \Rightarrow 1$

Assume that the system is not swap robust. Then we can choose winning coalitions X and Y with x in X but not in Y, and y in Y but not in X, such that both coalitions become losing if x is swapped for y. Let Z be the result of deleting x from the coalition X, and let Z' be the result of deleting y from the coalition Y. Then

Z with x added is X, and this is winning;
Z with y added is losing;
Z' with y added is Y, and this is winning; and
Z' with x added is losing.

This shows that the desirability of x and y is incomparable and completes the proof.

Corollary In a weighted voting system, we do not have voters whose desirability is incomparable. Say, suppose x's weight is heavier than y, while it may be possible that in one coalition x turns a coalition into winning but y does not, but it is not possible that the situation reverses in another coalition.

Proposition

Suppose the two players A and B are equally desirable in a yes-no voting system, then their Shapley-Shubik index are the same.

Proof

We compare the number of pivotal orderings of A and B. For any ordering that A pivots, we have the following two possibilities:

(i) B enters later than the pivotal position held by A

Let Z be the collection of players that have entered before A. Obviously, Z is losing and it does not contain A and B. Since $Z \cup \{A\}$ is winning, A and B are equally desirable, so $Z \cup \{B\}$ is also winning. Since Z is a losing coalition, so B is pivotal in the same ordering with A being swapped by B.

(ii) B has entered prior to the pivotal position held by A

Let Z' be the collection of players that have entered before A but excluding B in a given A-pivoted ordering. Note that $Z' \cup \{B\}$ is losing and so does $Z' \cup \{A\}$ since A and B are equally desirable. However, $Z' \cup \{B\} \cup \{A\}$ is winning, so B is pivotal in the ordering obtained by swapping A and B in this A-pivoted ordering.

In both cases, we observe that all A-pivoted orderings become B-pivoted orderings once A and B are swapped in position. Hence, A and B have the same number of pivotal orderings. As a result, $\phi_A = \phi_B$.

Remarks

1. Similar argument can be used to show that the Banzhaf indexes are the same for two players that are equally desirable in a yes-no voting game.

2. The converse statement is not true in general. Consider a voting system with 4 players A_1, A_2, A_3 and A_4, where the passage of a bill requires the support of (i) at least one of A_1 and A_2, (ii) at least one of A_3 and A_4. It is easy to show that A_1 and A_3 are incomparable since $\{A_2\} \cup \{A_1\}$ loses but $\{A_2\} \cup \{A_3\}$ wins. However, the 4 players share equal power, where

$$\phi_{A_1} = \phi_{A_2} = \phi_{A_3} = \phi_{A_4} = \frac{1}{4}.$$

Proposition

The relation of equal desirability is an equivalence relation on the set of voters in a yes–no voting system. That is, the following statements hold:

1. The relation is reflexive: if $x = y$ (that is, if x and y are literally the same voter), then x and y are equally desirable.

2. The relation is symmetric: if x and y are equally desirable, then y and x are equally desirable.

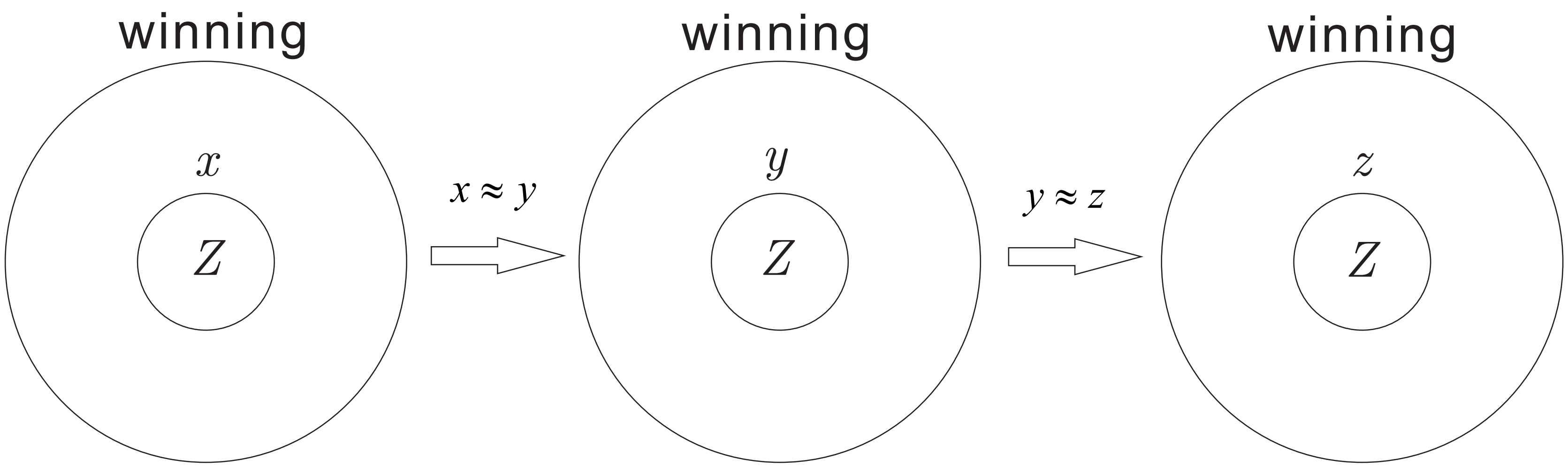
3. The relation is transitive: if x and y are equally desirable and y and z are equally desirable, then x and z are equally desirable.

Proof (transitivity)

Assume that Z is an arbitrary coalition containing neither x nor z. We must show that the result of x joining Z is a winning coalition if and only if the result of z joining Z is a winning coalition.

(i) $y \notin Z$

- Since $x \approx y$ and neither x nor y belongs to Z, the result of x joining Z is a winning coalition if and only if the result of y joining Z is a winning coalition.
- Since $y \approx z$ and neither y nor z belongs to Z, the result of y joining Z is a winning coalition if and only if the result of z joining Z is a winning coalition.
- The result of x joining Z is a winning coalition if and only if the result of z joining Z is a winning coalition, as desired.



(ii) $z \in Z$

- Let A denote the coalition resulting from y leaving Z, so $Z = A \cup \{y\}$. Assume that $Z \cup \{x\}$ is a winning coalition. We want to show that $Z \cup \{z\}$ is also a winning coalition. Now,

$$Z \cup \{x\} = A \cup \{x\} \cup \{y\}.$$

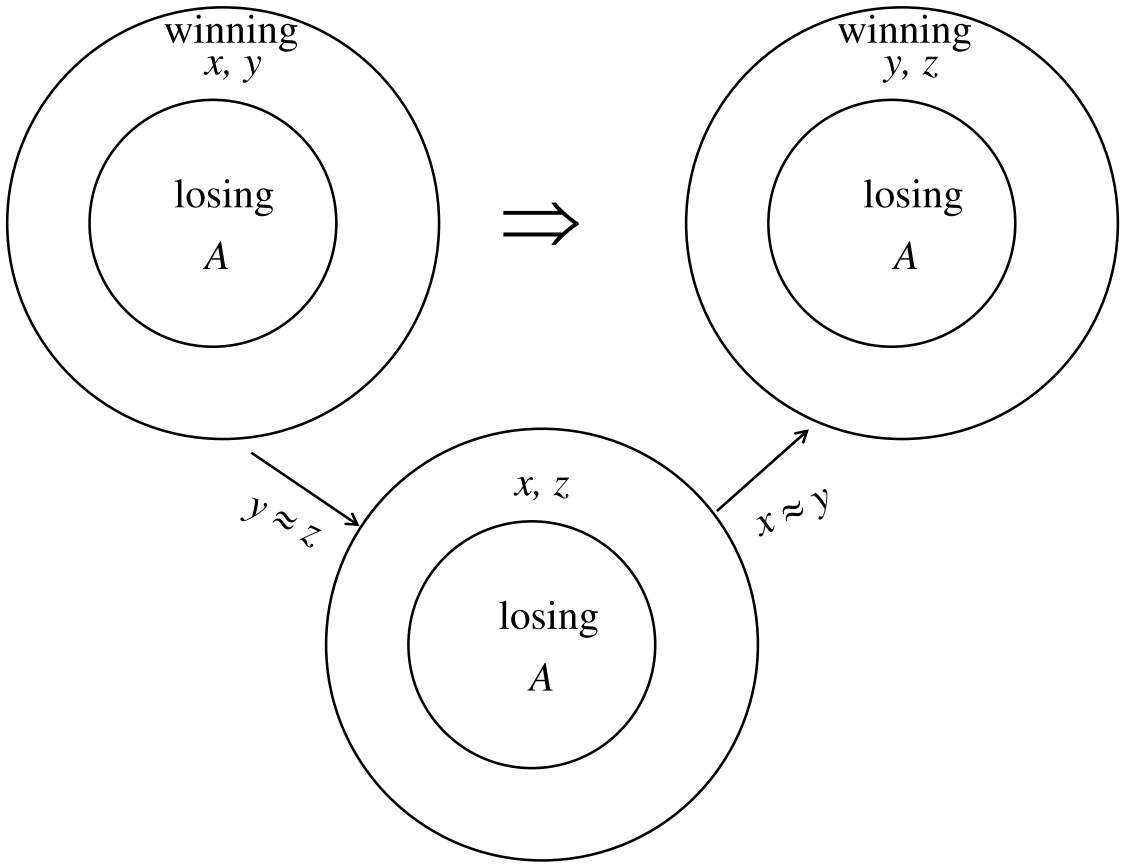
- Let $Z' = A \cup \{x\}$. Thus $Z' \cup \{y\}$ is a winning coalition. Since $y \approx z$ and neither y nor z belongs to Z', we know that $Z' \cup \{z\}$ is also a winning coalition. But $Z' \cup \{z\} = A \cup \{z\} \cup \{x\}$.
- Let $Z'' = A \cup \{z\}$. Thus $Z'' \cup \{x\}$ is a winning coalition. Since $x \approx y$ and neither x nor y belongs to Z'', we know that $Z'' \cup \{y\}$ is also a winning coalition. But $Z'' \cup \{y\} = A \cup \{z\} \cup \{y\} = Z \cup \{z\}$. Thus, $Z \cup \{z\}$ is a winning coalition as desired.
- A completely analogous argument would show that if $Z \cup \{z\}$ is a winning coalition, then so is $Z \cup \{x\}$. This completes the proof.

A does not contain x, y and z

$$Z = A \cup \{y\}, \quad Z' = A \cup \{x\}, \quad Z'' = A \cup \{z\}.$$

Assume that $Z \cup \{x\} = A \cup \{x, y\}$ is winning, we want to show that

$$Z \cup \{z\} = A \cup \{y, z\} \text{ is also winning.}$$



Proposition

For any two voters x and y in a weighted voting system, the following are equivalent:

1. x and y are equally desirable.

2. There exists an assignment of weights to the voters and a quota that realize the system and that give x and y the same weight.

3. There are two different ways to assign weights to the voters and two (perhaps equal) quotas such that both realize the system, but in one of the two weightings, x has more weight than y and, in the other weighting, y has more weight than x.

Proof

1. $\boxed{1 \Rightarrow 2}$

   Assume that x and y are equally desirable and there is a weighting and quota that realize the system. Let $w(x), w(y)$ and q denote the weight of x, weight of y and quota, respectively. [$w(Z) =$ total weight of the coalition Z].

   We try to construct a new weighting w_n such that with the same quota

   $$w_n(x) = w_n(y).$$

   - The new weighting is obtained by keeping the weight of every voter except x and y the same and setting both $w_n(x)$ and $w_n(y)$ equal to the average of $w(x)$ and $w(y)$.

Assuming Z is a coalition, to show that this new weighting still realizes the same system, it is necessary to show that Z is winning in the new weighting if and only if Z is winning in the old weighting.

Case 1 Neither x nor y belong to Z

$w(Z) = w_n(Z)$ and so $w_n(Z) \geq q$ if and only if Z is winning.

Case 2 Both x and y belong to Z

$$\begin{aligned} w_n(Z) &= w_n(Z - \{x\} - \{y\}) + w_n(x) + w_n(y) \\ &= w(Z - \{x\} - \{y\}) + \frac{w(x) + w(y)}{2} \times 2 \\ &= w(Z). \end{aligned}$$

Case 3 x belongs to Z but y does not belong to Z

Let $Z' = Z - \{x\}$ so that $w_n(Z') = w(Z')$. Consider

$$\begin{aligned} w_n(Z) &= w_n(Z') + w_n(x) = w_n(Z') + \frac{w(x) + w(y)}{2} \\ &= \frac{w(Z') + w(x) + w(Z') + w(y)}{2} = \frac{w(Z) + w(Z - \{x\} \cup \{y\})}{2}. \end{aligned}$$

Since x and y are equally desirable, either both $Z = Z' \cup \{x\}$ and $Z - \{x\} \cup \{y\} = Z' \cup \{y\}$ are winning or both are losing.

If both are winning, then $w_n(Z) \geq \dfrac{q+q}{2} = q$.

If both are losing, then $w_n(Z) < \dfrac{q+q}{2} = q$.

2. $\boxed{2 \Rightarrow 3}$

We start with a weighting and quota where x and y have the same weight.

$$\begin{aligned} L_H &= \text{weight of the heaviest losing coalition} \\ W_L &= \text{weight of the lightest winning coalition} \end{aligned}$$

$$L_H < q \leq W_L.$$

Let q' be the average of L_H and q. It is seen that q' still works as a quota since

$$L_H < q' < W_L.$$

Let ϵ be any positive number such that

$$L_H + \epsilon < q' < W_L - \epsilon.$$

The system is unchanged if we either increase the weight of x by ϵ or decrease the weight of x by ϵ. There are 2 weightings that realize the system, one of which makes x heavier than y and the other of which makes y heavier than x.

3. $\boxed{3 \Rightarrow 1}$

Assume two weightings w and w', two quotas q and q', such that

(i) A coalition Z is winning if and only if $w(Z) \geq q$.

(ii) A coalition Z is winning if and only if $w'(Z) \geq q'$.

(iii) $w(x) > w(y)$. (iv) $w'(y) > w'(x)$.

We start with an arbitrary coalition Z containing neither x nor y. Suppose $Z \cup \{y\}$ is winning so that $w(Z \cup \{y\}) \geq q$. Since $w(x) > w(y)$, so $w(Z \cup \{x\}) \geq q$, thus $Z \cup \{x\}$ is winning. Similarly, $Z \cup \{x\}$ is winning $\Rightarrow Z \cup \{y\}$ is winning.

Example

Consider the yes–no voting system with 3 players whose winning coalitions are $\{A, C\}, \{B, C\}$ and $\{A, B, C\}$. The weighted voting system:

$$[51; 1, 49, 50] \quad \text{and} \quad [51; 49, 1, 50]$$

both realize the system. By the above result, we deduce that $A \approx B$.

How about two voters x and y whose desirability is neither equal nor incomparable? For any two voters x and y in a yes–no voting system, we say that x is more desirable than y, denoted by

$$x > y,$$

if and only if the following hold:

1. for every coalition Z containing neither x nor y, if $Z \cup \{y\}$ is winning then so is $Z \cup \{x\}$, and
2. there exists a coalition Z' containing neither x nor y such that $Z' \cup \{x\}$ is winning, but $Z' \cup \{y\}$ is losing.

Example

Consider the yes-no voting system with minority veto where the 9 voters are classified into the majority group of 6 voters $\{M_1, M_2, \ldots, M_6\}$ and the minority group of 3 voters $\{m_1, m_2, m_3\}$. The passage of a bill requires at least 5 votes from all voters and at least 1 vote from the minority voters.

(a) Any two members in the majority group are equally desirable (same result is applicable to the minority group).

Let M_1 and M_2 be two members in the majority group. Consider any coalition Z without M_1 and M_2, we have

$$
\begin{aligned}
& Z \cup \{M_1\} \text{ is winning} \\
\Rightarrow\ & Z \text{ must contain at least 4 votes from } \{M_3, \ldots, M_6, m_1, m_2, m_3\} \\
& \text{and at least one minority vote} \\
\Rightarrow\ & Z \cup \{M_2\} \text{ contains at least 5 votes and at least one minority vote} \\
\Rightarrow\ & Z \cup \{M_2\} \text{ is winning}
\end{aligned}
$$

Following similar steps, we have $Z \cup \{M_2\}$ is winning $\Rightarrow Z \cup \{M_1\}$ is winning. Combining the results, we obtain

$$Z \cup \{M_1\} \text{ is winning } \iff Z \cup \{M_2\} \text{ is winning}$$

(b) For every Z that does not contain x and y, it is easy to establish: $Z \cup \{M_1\}$ is winning $\Rightarrow Z \cup \{m_1\}$ is winning.

This is because the head count requirement and minority veto requirement are satisfied.

However, it is straightforward to choose a coalition Z' that does not contain M_1 and m_1, where $Z' \cup \{m_1\}$ is winnning and $Z' \cup \{M_1\}$ is losing. One such example is

$$Z' = \{M_2, M_3, M_4, M_5\}.$$

Example

In the US federal system, x is a senator and y is the Vice President, then $x > y$. *Hint*: Consider coalition Z where the number of senators is 50 or above (with the President and one-half majority of Representatives) and coalition Z' where the number of senators is 66 (without the President and two-thirds majority of Representatives).

We write $x \geq y$ to mean either $x > y$ or $x \approx y$. The relation $\geq$ is called the *desirability relation on individuals*.

- The binary relation $\geq$ is called a *preordering* because it is transitive and reflexive.

- A preordering is said to be *linear* if for every pair x and y, one has either $x \geq y$ or $y \geq x$.

Definition A yes–no voting system is said to be *linear* if and only if there are no incomparable voters (equivalently, if the desirability relation on individuals is a linear preordering).

Propositions

1. A yes–no voting system is linear if and only if it is swap robust. This is a direct consequence of the property that non swap robust is equivalent to the existence of a pair of voters whose desirability is incomparable.

   Corollary Every weighted voting system is linear.

2. In a weighted voting system we have $x > y$ if and only if x has strictly more weight than y in every weighting that realizes the system.

- When $x > y$, it is not possible that x has strictly more weight than y in one weighting and reverse in the other weighting. Also, the possibility that x and y have equal weight in any weighting is ruled out.

Example

Consider the yes-no voting system of 4 players with the following winning coalitions:

$\{A, C\}, \{B, C\}, \{A, B, C\}, \{C, D\}, \{A, C, D\}, \{B, C, D\}, \{A, B, D\}, \{A, B, C, D\}$.

(a) A and B are "equally desirable".

Test for all coalitions that do not contain A and B:

$$Z_1 = \phi, Z_2 = \{C\}, Z_3 = \{D\}, Z_4 = \{C, D\}.$$

The result of A joining any of these $Z_i, i = 1, 2, 3, 4$, is the same as that of B joining Z_i.

(b) C is more desirable than D, denoted by $C > D$, since the following coalitions that do not contain C and D:

$$Z_1' = \phi, Z_2' = \{A\}, Z_3' = \{B\}, Z_4' = \{A, B\},$$

we have

$$Z_1' \bigcup \{C\} \text{ is losing and } Z_1' \bigcup \{D\} \text{ is losing}$$
$$Z_4' \bigcup \{C\} \text{ is winning and } Z_4' \bigcup \{D\} \text{ is winning}$$

while

$$Z_i' \bigcup \{C\} \text{ is winning but } Z_i' \bigcup \{D\} \text{ is losing, } i = 2, 3.$$

(c) The given yes-no voting system is a weighted voting system. Possible assignments of voters' weights and quota that realize the system are:

$$\{4; 1, 1, 3, 2\}, \{9; 1, 3, 8, 5\}, \{9; 3, 1, 8, 5\}, \{9; 2, 2, 8, 5\} \text{ etc.}$$

Remarks

1. The first and the fourth weighted voted assignments give the same weight to A and B.

2. In the second system, B has more weight than A; while in the third system, A has more weight than B. The fourth system is obtained by taking the weight of A and B to be equal to the average of the weights of A and B in the second or the third system.

3. Since C is more desirable than D, the weight of C in any assignment is always greater than that of D.

4. The given yes-no voting system is swap robust (since it is a weighted voting system), so there are no incomparable voters. Indeed

$$A \approx B < D < C.$$