1. \( \Leftarrow \) part: The trading strategy \( \mathcal{H} \) with \( V_0 < 0 \) and \( V_1(\omega) \geq 0, \forall \omega \in \Omega \), dominates the zero-holding trading strategy \( \mathcal{H} = (0 \ 0 \ \cdots \ 0)^T \). The zero-holding strategy gives \( \hat{V}_1(\omega) = V_0 < 0 \), so \( V_1(\omega) > \hat{V}_1(\omega) \) for all \( \omega \in \Omega \). Thus, \( \mathcal{H} \) dominates \( \mathcal{H} \).

\( \Rightarrow \) part: Existence of a dominant trading strategy means there exists a trading strategy \( \mathcal{H} = (h_1 \ \cdots \ h_M)^T \) such that \( V_0 = 0 \) and \( V_1(\omega) > 0, \forall \omega \in \Omega \). Let \( G_{\min}^* = \min_{\omega} G^*(\omega) = \min_{\omega} \sum_{m=1}^{M} h_m \Delta S_m^* \). Since \( G^*(\omega) = V_1^* - V_0^* > 0 \), we have \( G_{\min}^* > 0 \). Consider the new trading strategy with

\[
\hat{h}_m = h_m \text{ for } m = 1, \cdots, M, \\
\hat{h}_0 = -G_{\min}^* - \sum_{m=1}^{M} h_m S_m^*(0).
\]

Now, \( \hat{V}_0^* = \hat{h}_0 + \sum_{m=1}^{M} \hat{h}_m S_m^*(0) = -G_{\min}^* < 0 \); while

\[
\hat{V}_1^*(\omega) = \hat{h}_0 + \sum_{m=1}^{M} \hat{h}_m S_m^*(1; \omega) \\
= -G_{\min}^* + \sum_{m=1}^{M} h_m \Delta S_m^*(\omega) \geq 0,
\]

by virtue of the definition of \( G_{\min}^* \). Thus, \( \hat{\mathcal{H}} = (\hat{h}_1 \ \cdots \ \hat{h}_M)^T \) is a trading strategy that gives \( \hat{V}_0 < 0, \hat{V}_1(\omega) \geq 0, \forall \omega \in \Omega \).

2. For the given securities model, we have the discounted terminal payoff matrix:

\[
S(1; \Omega) = \begin{pmatrix}
1.1 & 1.1 \\
1.1 & 2.2 \\
1.1 & 3.3
\end{pmatrix}
\]

and initial price vector \( S(0) = (1 \ 4) \).

(a) With \( h_0 = 4, h_1 = -1 \), we obtain

\[
V_0 = \begin{pmatrix} 1 & 4 \\ -1 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} = 0 \\
V_1(\omega) = S(1; \Omega) \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3.3 \\ 2.2 \\ 1.1 \end{pmatrix} > 0, \quad V_1^*(\omega) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.
\]
Thus \( \begin{pmatrix} 4 \\ -1 \end{pmatrix} \) is a dominant trading strategy.

(b) \( G^* = V_1^* - V_0^* = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \).

(c) We shall use the result in Question 1. Now, \( G_{\min}^* = \min_{\omega} G^*(\omega) = 1 \) so that

\[
\hat{h}_0 = -1 - (-1)(4) = 3. \quad \text{Take } \hat{h} = \begin{pmatrix} 3 \\ -1 \end{pmatrix},
\]

then

\[
\hat{V}_0 = (1 \ 4) \begin{pmatrix} 3 \\ -1 \end{pmatrix} = -1 < 0
\]

\[
\hat{V}_1 = S(1;\Omega) \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2.2 \\ 1.1 \\ 0 \end{pmatrix} \geq 0.
\]

Thus \( \hat{H} \) is a trading strategy that starts with negative wealth \( \hat{V}_0 \) and ends with non-negative wealth \( \hat{V}_1 \) for sure.

3. (a) If the law of one price does not hold, then there exist two trading strategies \( h \) and \( h' \) such that

\[ S^*(1)h = S^*(1)h' \text{ but } S^*(0)h > S^*(0)h'. \]

For any payoff \( x \) in the asset span, it can be expressed as \( x = S^*(1)\hat{h} \) for some \( \hat{h} \). Using the relation \( S^*(1)h = S^*(1)h' \), we have

\[
x = S^*(1)\hat{h} + kS^*(1)h - kS^*(1)h' = S^*(1)[\hat{h} + k(h - h')], \quad \text{for any value of } k.
\]

The initial price of the portfolio that generates \( x \) is given by

\[
S^*(0)\hat{h} + k[S^*(0)h - S^*(0)h'], \quad \text{for any value of } k.
\]

As \( S^*(0)h - S^*(0)h' \neq 0 \), the initial price of the portfolio with payoff \( x \) can assume any value.

(b) Uniqueness of the price of any security in the asset span is equivalent to satisfaction of law of one price. Consider the securities model

\[
S^*(1) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad S^*(0) = \begin{pmatrix} 1 & 4 & 2 \\ \frac{3}{4} & 2 & 3 \end{pmatrix}.
\]

The state prices \( (\pi_1 \  \pi_2 \  \pi_3) \) can be found by solving

\[
(\pi_1 \  \pi_2 \  \pi_3) \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 \\ \frac{3}{4} & 2 & 3 \end{pmatrix}.
\]
giving \( (\pi_1 \pi_2 \pi_3) = \left(\frac{1}{3} - \frac{1}{3} 1\right) \). It can be shown that by taking the portfolio \( h = \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix} \), we have
\[
V_0^* = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix} = 0
\]
while
\[
V_1^* = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} > 0.
\]
This indicates that \( h = \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix} \) represents an arbitrage opportunity. Indeed, \( V_0^* \) and \( V_1^* \) are related by
\[
0 = V_0^* = (\pi_1 \pi_2 \pi_3)V_1^* = \left(\frac{1}{3} - \frac{1}{3} 1\right)\begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} = 0.
\]

4. The state prices \( (\pi_1 \pi_2 \pi_3) \) are found by solving
\[
\begin{pmatrix} 1 & 3 & 2 \end{pmatrix} = (\pi_1 \pi_2 \pi_3) \begin{pmatrix} 1 & 6 & 3 \\ 1 & 2 & 2 \\ 1 & 12 & 6 \end{pmatrix}.
\]
The solution is found to be: \( (\pi_1 \pi_2 \pi_3) = \begin{pmatrix} 2 & \frac{1}{2} & -\frac{1}{6} \end{pmatrix} \). The state prices are \( \pi_i, i = 1, 2, 3 \). Positivity of the state prices is not observed so the securities model admits arbitrage opportunity. To find an arbitrage opportunity (for simplicity, we take \( h_1 = 0 \)), we seek for \((h_0 \ h_2)^T\) such that
\[
V_0^* = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} h_0 \\ h_2 \end{pmatrix} = h_0 + 2h_2 = 0
\]
while
\[
V_1(\omega) = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} h_0 \\ h_2 \end{pmatrix} = \begin{pmatrix} h_0 + 3h_2 \\ h_0 + 2h_2 \\ h_0 + 6h_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\]
with at least one strict inequality. A possible arbitrage portfolio is \((h_0 \ h_2)^T = (-2 \ 1)^T\). We short sell 2 units of the risk free asset, long hold one unit of the second risky asset and zero unit of the first risky asset (since \( h_1 = 0 \)). The resulting discounted payoff of the portfolio is given by
\[
V_1^*(\omega) = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.
\]
5. Let \( x_1 \) and \( x_2 \) be two discounted terminal payoff vectors in the asset span \( S \). This would imply that there exist \( h_1, h_2 \) such that \( x_i = S^*(1)h_i \) for \( i = 1, 2 \). By the law of one price, the pricing functional is given by \( F(x_i) = S^*(0)h_i \) for \( i = 1, 2 \). For any scalars \( \alpha_1 \) and \( \alpha_2 \), we consider
\[
\alpha_1 F(x_1) + \alpha_2 F(x_2) = \alpha_1 S^*(0)h_1 + \alpha_2 S^*(0)h_2
\]
while
\[
S^*(1)(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 x_1 + \alpha_2 x_2 \in S.
\]
Knowing that \( \alpha_1 x_1 + \alpha_2 x_2 \in S \), \( F(\alpha_1 x_1 + \alpha_2 x_2) = S^*(0)(\alpha_1 h_1 + \alpha_2 h_2) \) as deduced from the relation: \( \alpha_1 x_1 + \alpha_2 x_2 = S^*(1)(\alpha_1 h_1 + \alpha_2 h_2) \). We then have
\[
F(\alpha_1 x_1 + \alpha_2 x_2) = S^*(0)(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 F(x_1) + \alpha_2 F(x_2).
\]
This proves the linearity of the pricing functional.

6. Consider \( S^*(1) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \) and \( S^*(0) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \).

Since the three rows of \( S^*(1) \) are independent, so that the row space of \( S^*(1) \) spans the whole \( \mathbb{R}^3 \). Hence, \( S^*(0) \) is sure to lie in the row space of \( S^*(1) \). Therefore, we can conclude that the law of one price holds for the given securities model. However, we observe that \( -1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) dominates the trading strategy \( 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) as \( V_0^* = S^*(0)(-1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) = 0 \) and
\[
V_1^* = S^*(1) \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} > 0.
\]

7. Let \( q = (q(\omega_1) \quad q(\omega_2) \quad q(\omega_3)) \). Since the initial bet is one dollar, we have to solve
\[
qS(1; \Omega) = (1 \quad 1 \quad 1),
\]
giving
\[
q(\omega_i) = \frac{1}{d_i + 1} > 0 \quad \text{for} \quad i = 1, 2, 3. \quad (1)
\]
We also have to observe \( \sum_{i=1}^{3} q(\omega_i) = 1 \), that is,
\[
\sum_{i=1}^{3} \frac{1}{d_i + 1} = 1. \quad (2)
\]
Eqs. (1) and (2) state the required conditions for the existence of a risk neutral probability measure for the betting game. An example would be \( d_1 = 1, d_2 = 3 \) and \( d_3 = 3 \). The betting game pays out $2 if \( \omega_1 \) occurs and $4 if either \( \omega_2 \) or \( \omega_3 \) occurs.
8. Note that the last two columns are seen to be
\[
\begin{pmatrix}
3 \\
4 \\
6
\end{pmatrix}
= \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} + \begin{pmatrix}
2 \\
3 \\
5
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
4 \\
5 \\
7
\end{pmatrix}
= 2 \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} + \begin{pmatrix}
2 \\
3 \\
5
\end{pmatrix}.
\]

The rank of \(\hat{S}(1; \Omega)\) is 2. We also observe that
\[
\begin{align*}
S^*_2(1; \Omega) &= S^*_0(1; \Omega) + S^*_1(1; \Omega) \quad \text{while} \quad S^*_2(0) \neq S^*_0(0) + S^*_1(0); \\
S^*_3(1; \Omega) &= S^*_0(1; \Omega) + S^*_2(1; \Omega) \quad \text{while} \quad S^*_3(0) \neq S^*_0(0) + S^*_2(0).
\end{align*}
\]

Hence, the law of one price does not hold. In fact, \(\hat{S}^*(0) = (1 \ 3 \ 5 \ 9)\) does not lie in the row space of \(\hat{S}^*(1; \Omega)\). This is equivalent to saying that solution to the linear system
\[
\hat{S}^*(0) = q \hat{S}^*(1; \Omega)
\]
does not exist.

Next, we check whether \(\begin{pmatrix}6 \\ 8 \\ 12\end{pmatrix}\) is attainable. We ask whether solution to the following linear system
\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 3 & 4 & 5 \\
1 & 5 & 6 & 7
\end{pmatrix}
\begin{pmatrix}
h_0 \\
h_1 \\
h_2 \\
h_3
\end{pmatrix}
= \begin{pmatrix}6 \\ 8 \\ 12\end{pmatrix}
\]
exists. The Gaussian elimination procedure gives
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 6 \\
1 & 3 & 4 & 5 & 8 \\
1 & 5 & 6 & 7 & 12
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 & 4 & 6 \\
0 & 1 & 1 & 1 & 2 \\
0 & 3 & 3 & 3 & 6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 & 4 & 6 \\
0 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 1 & 2 & 2 \\
0 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The set of all possible trading strategies that generate the payoff is seen to be
\[
\begin{pmatrix}
h_0 \\
h_1 \\
h_2 \\
h_3
\end{pmatrix}
= \begin{pmatrix}2 - h_2 - 2h_3 \\
2 - h_2 - h_3 \\
h_2 \\
h_3
\end{pmatrix}
\quad \text{for any values of } h_2, h_3 \in \mathbb{R}.
\]

Thus, \(\begin{pmatrix}6 \\ 8 \\ 2\end{pmatrix}\) lies in the asset span. For example, we take \(h_2 = h_3 = 1\) so that \(h_1 = 0\) and \(h_0 = -1\), giving the following replicating strategy:
\[
\begin{pmatrix}6 \\ 8 \\ 12\end{pmatrix}
= - \begin{pmatrix}1 \\ 1 \\ 1\end{pmatrix} + \begin{pmatrix}3 \\ 4 \\ 5\end{pmatrix} + \begin{pmatrix}4 \\ 6 \\ 7\end{pmatrix}.
\]
Note that \( \hat{S}^*(0) \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} = 8 + h_2 + 4h_3 \). The cost of the replicating portfolio is dependent on \( h_2 \) and \( h_3 \). This verifies that the Law of One Price does not hold in this securities model. There are infinitely many possible prices for this contingent claim.

9. From \( \left\{ 1 = \Pi_u R + \Pi_d R , S = \Pi_u uS + \Pi_d dS \right\} \), the state prices \( \Pi_u \) and \( \Pi_d \) can be expressed in terms of \( u, d \) and \( R \):

\[
\Pi_u = \frac{R - d}{u - d} \frac{1}{R} \quad \text{and} \quad \Pi_d = \frac{u - R}{u - d} \frac{1}{R}.
\]

The call value under the binomial model is given by

\[
c = \Pi_u c_u + \Pi_d c_d = \frac{\frac{R-d}{u-d} c_u + \frac{u-R}{u-d} c_d}{R} = \frac{p c_u + (1 - p) c_d}{R},
\]

where \( p = \frac{R - d}{u - d} \).

10. We test whether a risk neutral measure \( Q = (Q_1 \ Q_2 \ Q_3) \) exists for the given securities model. This is done by solving

\[
(Q_1 \ Q_2 \ Q_3) \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} = (1 \ 2 \ 3).
\]

We obtain the set of risk neutral measures \( R \), as characterized by \( (Q_1 \ Q_2 \ Q_3) = (\lambda \ 1 - 3\lambda \ 2\lambda), 0 < \lambda < \frac{1}{3} \). For \( Y^* = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \), we have \( E_Q[Y^*] = (\lambda \ 1 - 3\lambda \ 2\lambda) \left( \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \right) = 4 + \lambda \). We deduce that

\[
V_+ = \sup \{ E_Q[Y^*] : Q \in R \} = 4 + \frac{1}{3} = \frac{13}{3}
\]

\[
V_- = \inf \{ E_Q[Y^*] : Q \in R \} = 4.
\]

Hence, in order to avoid arbitrage, the range of reasonable initial price is \( [4, \ 4 + \frac{1}{3}] \).

11. For the securities model, it is easy to check that the set of risk neutral measures is characterized by

\[
(Q_1 \ Q_2 \ Q_3) = (\alpha \ 1 - 2\alpha \ \alpha), 0 < \alpha < \frac{1}{2}.
\]
Consider $E_Q[Y^*] = (Q_1 \quad Q_2 \quad Q_3) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} / S_0(1) = \frac{\alpha(y_1 - 2y_2 + y_3) + y_2}{S_0(1)}$, which is independent of $\alpha$ if and only if $y_1 - 2y_2 + y_3 = 0$. Since attainability of a contingent claim is equivalent to uniqueness of risk neutral price, so the necessary and sufficient condition for $Y$ to be attainable is $y_1 - 2y_2 + y_3 = 0$. 