Mathematical Models in Economics and Finance

Solution to Homework Three

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1. $\Leftarrow$ part: The trading strategy $\mathcal{H}$ with $V_0 < 0$ and $V_1(\omega) \geq 0, \forall \omega \in \Omega$, dominates the zero-holding trading strategy $\widehat{\mathcal{H}} = (0 \quad 0 \quad \cdots \quad 0)^T$. The zero-holding strategy gives $\widehat{V}_1(\omega) = V_0 < 0$, so $V_1(\omega) > \widehat{V}_1(\omega)$ for all $\omega \in \Omega$. Thus, $\mathcal{H}$ dominates $\widehat{\mathcal{H}}$.

   $\Rightarrow$ part: Existence of a dominant trading strategy means there exists a trading strategy $\mathcal{H} = (h_1 \quad \cdots \quad h_M)^T$ such that $V_0 = 0$ and $V_1(\omega) > 0, \forall \omega \in \Omega$. Let $G^*_{min} = \min_{\omega} G^*(\omega) = \min_{\omega} \sum_{m=1}^{M} h_m \Delta S^*_m$. Since $G^*(\omega) = V^*_1 - V^*_0 > 0$, we have $G^*_{min} > 0$. Consider the new trading strategy with

$$\begin{aligned}
\widehat{h}_m &= h_m \quad \text{for } m = 1, \cdots, M, \\
\widehat{h}_0 &= -G^*_{min} - \sum_{m=1}^{M} h_m S^*_m(0).
\end{aligned}$$

   Now, $\widehat{V}^*_0 = \widehat{h}_0 + \sum_{m=1}^{M} \widehat{h}_m S^*_m(0) = -G^*_{min} < 0$; while

$$\begin{aligned}
\widehat{V}^*_1(\omega) &= \widehat{h}_0 + \sum_{m=1}^{M} \widehat{h}_m S^*_m(1;\omega) \\
&= -G^*_{min} + \sum_{m=1}^{M} h_m \Delta S^*_m(\omega) \geq 0,
\end{aligned}$$

   by virtue of the definition of G^*_{min}. Thus, $\widehat{\mathcal{H}} = (\widehat{h}_1 \quad \cdots \quad \widehat{h}_M)^T$ is a trading strategy that gives $\widehat{V}_0 < 0, \widehat{V}_1(\omega) \geq 0, \forall \omega \in \Omega$.

2. For the given securities model, we have the discounted terminal payoff matrix: $S(1;\Omega) = \begin{pmatrix} 1.1 & 1.1 \\ 1.1 & 2.2 \\ 1.1 & 3.3 \end{pmatrix}$ and initial price vector $\boldsymbol{S}(0) = (1 \quad 4)$.

   (a) With $h_0 = 4, h_1 = -1$, we obtain

$$\begin{aligned}
V_0 &= (1 \quad 4) \begin{pmatrix} 4 \\ -1 \end{pmatrix} = 0 \\
V_1(\omega) &= S(1;\Omega) \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3.3 \\ 2.2 \\ 1.1 \end{pmatrix} > \mathbf{0}, \quad V^*_1(\omega) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.
\end{aligned}$$

Thus $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ is a dominant trading strategy.

(b) $G^* = V_1^* - V_0^* = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

(c) We shall use the result in Question 1. Now, $G^*_{min} = \min_\omega G^*(\omega) = 1$ so that $\widehat{h}_0 = -1 - (-1)(4) = 3$. Take $\widehat{\mathcal{H}} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, then

$$\begin{aligned}
\widehat{V}_0 &= (1 \quad 4)\begin{pmatrix} 3 \\ -1 \end{pmatrix} = -1 < 0 \\
\widehat{V}_1 &= S(1;\Omega)\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2.2 \\ 1.1 \\ 0 \end{pmatrix} \geq \mathbf{0}.
\end{aligned}$$

Thus $\widehat{\mathcal{H}}$ is a trading strategy that starts with negative wealth $\widehat{V}_0$ and ends with non-negative wealth $\widehat{V}_1$ for sure.

3. (a) If the law of one price does not hold, then there exist two trading strategies $\boldsymbol{h}$ and $\boldsymbol{h}'$ such that

$$S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}' \text{ but } \boldsymbol{S}^*(0)\boldsymbol{h} > \boldsymbol{S}^*(0)\boldsymbol{h}'.$$

For any payoff $\boldsymbol{x}$ in the asset span, it can be expressed as $\boldsymbol{x} = S^*(1)\widehat{\boldsymbol{h}}$ for some $\widehat{\boldsymbol{h}}$. Using the relation $S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}'$, we have

$$\begin{aligned}
\boldsymbol{x} &= S^*(1)\widehat{\boldsymbol{h}} + kS^*(1)\boldsymbol{h} - kS^*(1)\boldsymbol{h}' \\
&= S^*(1)[\widehat{\boldsymbol{h}} + k(\boldsymbol{h} - \boldsymbol{h}')], \quad \text{for any value of } k.
\end{aligned}$$

The initial price of the portfolio that generates $\boldsymbol{x}$ is given by

$$\boldsymbol{S}^*(0)\widehat{\boldsymbol{h}} + k[\boldsymbol{S}^*(0)\boldsymbol{h} - \boldsymbol{S}^*(0)\boldsymbol{h}'], \quad \text{for any value of } k.$$

As $\boldsymbol{S}^*(0)\boldsymbol{h} - \boldsymbol{S}^*(0)\boldsymbol{h}' \neq 0$, the initial price of the portfolio with payoff $\boldsymbol{x}$ can assume any value.

(b) Uniqueness of the price of any security in the asset span is equivalent to satisfaction of law of one price. Consider the securities model

$$S^*(1) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } \boldsymbol{S}^*(0) = \begin{pmatrix} 1 & \frac{4}{3} & \frac{2}{3} \end{pmatrix}.$$

The state prices $(\pi_1 \quad \pi_2 \quad \pi_3)$ can be found by solving

$$(\pi_1 \quad \pi_2 \quad \pi_3)\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{4}{3} & \frac{2}{3} \end{pmatrix}.$$

giving $(\pi_1 \quad \pi_2 \quad \pi_3) = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}$. It can be shown that by taking the portfolio $\boldsymbol{h} = \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix}$, we have

$$V_0^* = \begin{pmatrix} 1 & \frac{4}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix} = 0$$

while

$$V_1^* = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} > \mathbf{0}.$$

This indicates that $\boldsymbol{h} = \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix}$ represents an arbitrage opportunity. Indeed, V_0^* and V_1^* are related by

$$0 = V_0^* = (\pi_1 \quad \pi_2 \quad \pi_3) V_1^* = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} = 0.$$

4. The state prices $(\pi_1 \quad \pi_2 \quad \pi_3)$ are found by solving

$$(1 \quad 3 \quad 2) = (\pi_1 \quad \pi_2 \quad \pi_3) \begin{pmatrix} 1 & 6 & 3 \\ 1 & 2 & 2 \\ 1 & 12 & 6 \end{pmatrix}.$$

The solution is found to be: $(\pi_1 \quad \pi_2 \quad \pi_3) = \begin{pmatrix} \frac{2}{3} & \frac{1}{2} & -\frac{1}{6} \end{pmatrix}$. The state prices are $\pi_i, i = 1, 2, 3$. Positivity of the state prices is not observed so the securities model admits arbitrage opportunity. To find an arbitrage opportunity (for simplicity, we take $h_1 = 0$), we seek for $(h_0 \quad h_2)^T$ such that

$$V_0^* = (1 \quad 2) \begin{pmatrix} h_0 \\ h_2 \end{pmatrix} = h_0 + 2h_2 = 0$$

while

$$V_1^*(\omega) = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} h_0 \\ h_2 \end{pmatrix} = \begin{pmatrix} h_0 + 3h_2 \\ h_0 + 2h_2 \\ h_0 + 6h_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

with at least one strict inequality. A possible arbitrage portfolio is $(h_0 \quad h_2)^T = (-2 \quad 1)^T$. We short sell 2 units of the risk free asset, long hold one unit of the second risky asset and zero unit of the first risky asset (since $h_1 = 0$). The resulting discounted payoff of the portfolio is given by

$$V_1^*(\omega) = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.$$

5. Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be two discounted terminal payoff vectors in the asset span $\mathcal{S}$. This would imply that there exist $\boldsymbol{h}_1, \boldsymbol{h}_2$ such that $\boldsymbol{x}_i = S^*(1)\boldsymbol{h}_i$ for $i = 1, 2$. By the law of one price, the pricing functional is given by $F(\boldsymbol{x}_i) = \boldsymbol{S}^*(0)\boldsymbol{h}_i$ for $i = 1, 2$. For any scalars α_1 and α_2, we consider

$$\begin{aligned} \alpha_1 F(\boldsymbol{x}_1) + \alpha_2 F(\boldsymbol{x}_2) &= \alpha_1 \boldsymbol{S}^*(0)\boldsymbol{h}_1 + \alpha_2 \boldsymbol{S}^*(0)\boldsymbol{h}_2 \\ &= \boldsymbol{S}^*(0)(\alpha_1 \boldsymbol{h}_1 + \alpha_2 \boldsymbol{h}_2) \end{aligned}$$

while

$$\begin{aligned} S^*(1)(\alpha_1 \boldsymbol{h}_1 + \alpha_2 \boldsymbol{h}_2) &= \alpha_1 S^*(1)\boldsymbol{h}_1 + \alpha_2 S^*(1)\boldsymbol{h}_2 \\ &= \alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 \in \mathcal{S}. \end{aligned}$$

Knowing that $\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 \in \mathcal{S}, F(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2)$ is given by $\boldsymbol{S}^*(0)(\alpha_1 \boldsymbol{h}_1 + \alpha_2 \boldsymbol{h}_2)$ as deduced from the relation: $\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 = S^*(1)(\alpha_1 \boldsymbol{h}_1 + \alpha_2 \boldsymbol{h}_2)$. We then have

$$F(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2) = \boldsymbol{S}^*(0)(\alpha_1 \boldsymbol{h}_1 + \alpha_2 \boldsymbol{h}_2) = \alpha_1 F(\boldsymbol{x}_1) + \alpha_2 F(\boldsymbol{x}_2).$$

This proves the linearity of the pricing functional.

6. Consider $S^*(1) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and $\boldsymbol{S}^*(0) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Since the three rows of $S^*(1)$ are independent, so that the row space of $S^*(1)$ spans the whole $\mathbb{R}^3$. Hence, $\boldsymbol{S}^*(0)$ is sure to lie in the row space of $S^*(1)$. Therefore, we can conclude that the law of one price holds for the given securities model. However, we observe that $(-1 \quad 1 \quad 1)^T$ dominates the trading strategy $(0 \quad 0 \quad 0)^T$ as $V_0^* = \boldsymbol{S}^*(0)(-1 \quad 1 \quad 1)^T = 0$ and

$$V_1^* = S^*(1) \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} > \boldsymbol{0}.$$

7. Let $\boldsymbol{q} = (q(\omega_1) \quad q(\omega_2) \quad q(\omega_3))$. Since the initial bet is one dollar, we have to solve

$$\boldsymbol{q}S(1;\Omega) = (1 \quad 1 \quad 1),$$

giving

$$q(\omega_i) = \frac{1}{d_i + 1} > 0 \quad \text{for} \quad i = 1, 2, 3. \tag{1}$$

We also have to observe $\sum_{i=1}^{3} q(\omega_i) = 1$, that is,

$$\sum_{i=1}^{3} \frac{1}{d_i + 1} = 1. \tag{2}$$

Eqs. (1) and (2) state the required conditions for the existence of a risk neutral probability measure for the betting game. An example would be $d_1 = 1, d_2 = 3$ and $d_3 = 3$. The betting game pays out \$2 if ω_1 occurs and \$4 if either ω_2 or ω_3 occurs.

8. Note that the last two columns are seen to be

$$\begin{pmatrix}3\\4\\6\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}2\\3\\5\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}4\\5\\7\end{pmatrix} = 2\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}2\\3\\5\end{pmatrix}.$$

The rank of $\widehat{S}(1;\Omega)$ is 2. We also observe that

$$\begin{aligned}\boldsymbol{S}_2^*(1;\Omega) &= \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_1^*(1;\Omega) \text{ while } S_2(0) \neq S_0(0) + S_1(0);\\ \boldsymbol{S}_3^*(1;\Omega) &= \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_2^*(1;\Omega) \text{ while } S_3(0) \neq S_0(0) + S_2(0).\end{aligned}$$

Hence, the law of one price does not hold. In fact, $\widehat{S}^*(0) = (1 \quad 3 \quad 5 \quad 9)$ does not lie in the row space of $\widehat{S}^*(1;\Omega)$. This is equivalent to saying that solution to the linear system

$$\widehat{\boldsymbol{S}}^*(0) = \boldsymbol{q}\widehat{S}^*(1;\Omega)$$

does not exist.

Next, we check whether $\begin{pmatrix}6\\8\\12\end{pmatrix}$ is attainable. We ask whether solution to the following linear system

$$\begin{pmatrix}1 & 2 & 3 & 4\\1 & 3 & 4 & 5\\1 & 5 & 6 & 7\end{pmatrix}\begin{pmatrix}h_0\\h_1\\h_2\\h_3\end{pmatrix} = \begin{pmatrix}6\\8\\12\end{pmatrix}$$

exists. The Gaussian elimination procedure gives

$$\left(\begin{array}{cccc|c}1 & 2 & 3 & 4 & 6\\1 & 3 & 4 & 5 & 8\\1 & 5 & 6 & 7 & 12\end{array}\right) \to \left(\begin{array}{cccc|c}1 & 2 & 3 & 4 & 6\\0 & 1 & 1 & 1 & 2\\0 & 3 & 3 & 3 & 6\end{array}\right)$$
$$\to \left(\begin{array}{cccc|c}1 & 2 & 3 & 4 & 6\\0 & 1 & 1 & 1 & 2\\0 & 0 & 0 & 0 & 0\end{array}\right)$$
$$\to \left(\begin{array}{cccc|c}1 & 0 & 1 & 2 & 2\\0 & 1 & 1 & 1 & 2\\0 & 0 & 0 & 0 & 0\end{array}\right).$$

The set of all possible trading strategies that generate the payoff is seen to be

$$\begin{pmatrix}h_0\\h_1\\h_2\\h_3\end{pmatrix} = \begin{pmatrix}2 - h_2 - 2h_3\\2 - h_2 - h_3\\h_2\\h_3\end{pmatrix} \quad \text{for any values of } h_2, h_3 \in \mathbb{R}.$$

Thus, $\begin{pmatrix}6\\8\\2\end{pmatrix}$ lies in the asset span. For example, we take $h_2 = h_3 = 1$ so that $h_1 = 0$ and $h_0 = -1$, giving the following replicating strategy:

$$\begin{pmatrix}6\\8\\12\end{pmatrix} = -\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}3\\4\\6\end{pmatrix} + \begin{pmatrix}4\\5\\7\end{pmatrix}.$$

Note that $\widehat{S}^*(0)\begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} = 8 + h_2 + 4h_3$. The cost of the replicating portfolio is dependent on h_2 and h_3. This verifies that the Law of One Price does not hold in this securities model. There are infinitely many possible prices for this contingent claim.

9. From $\begin{cases} 1 = \Pi_u R + \Pi_d R \\ S = \Pi_u uS + \Pi_d\, dS \end{cases}$, the state prices Π_u and Π_d can be expressed in terms of u, d and R:

$$\Pi_u = \frac{R-d}{u-d}\frac{1}{R} \quad \text{and} \quad \Pi_d = \frac{u-R}{u-d}\frac{1}{R}.$$

The call value under the binomial model is given by

$$c \;=\; \Pi_u c_u + \Pi_d c_d = \frac{\frac{R-d}{u-d}c_u + \frac{u-R}{u-d}c_d}{R} = \frac{pc_u + (1-p)c_d}{R},$$

where $p = \dfrac{R-d}{u-d}$.

10. We test whether a risk neutral measure $\boldsymbol{Q} = (Q_1 \quad Q_2 \quad Q_3)$ exists for the given securities model. This is done by solving

$$(Q_1 \quad Q_2 \quad Q_3)\begin{pmatrix} 1 & 4 & 5 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} = (1 \quad 2 \quad 3).$$

We obtain the set of risk neutral measures R, as characterized by $(Q_1 \quad Q_2 \quad Q_3) = (\lambda \quad 1-3\lambda \quad 2\lambda)$, $0 < \lambda < \dfrac{1}{3}$. For $Y^* = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$, we have $E_Q[Y^*] = (\lambda \quad 1-3\lambda \quad 2\lambda)\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 4 + \lambda$. We deduce that

$$\begin{aligned} V_+ &= \sup\{E_Q[Y^*] : Q \in R\} = 4 + \frac{1}{3} = \frac{13}{3} \\ V_- &= \inf\{E_Q[Y^*] : Q \in R\} = 4. \end{aligned}$$

Hence, in order to avoid arbitrage, the range of reasonable initial price is $\left[4, \quad 4\frac{1}{3}\right]$.

11. For the securities model, it is easy to check that the set of risk neutral measures is characterized by

$$(Q_1 \quad Q_2 \quad Q_3) = (\alpha \quad 1-2\alpha \quad \alpha),\ 0 < \alpha < \frac{1}{2}.$$

Consider $E_Q[Y^*] = (Q_1 \quad Q_2 \quad Q_3) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \Big/ S_0(1) = \dfrac{\alpha(y_1 - 2y_2 + y_3) + y_2}{S_0(1)}$, which is independent of α if and only if $y_1 - 2y_2 + y_3 = 0$. Since attainability of a contingent claim is equivalent to uniqueness of risk neutral price, so the necessary and sufficient condition for Y to be attainable is $y_1 - 2y_2 + y_3 = 0$.