Mathematical Models in Economics and Finance

Topic 1 – Proportional representation and Apportionment Schemes（公平及公正的分派）

1.1 General issues of apportionment of legislature seats

1.2 Quota Method of the Greatest Remainder (Hamilton's method) and paradoxes

1.3 Geometric characterization and apportionment simplex

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1.5 Huntington's family: Pairwise comparison of inequity

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1.7 Cumulative voting and proportional representation

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1.1 General issues of apportionment of legislature seats

To apportion is to distribute by right measure, to set off in just parts, to assign in due and proper proportion.

- Distributing available personnel or other resources in "integral parts" (integer programming):
  - distributing seats in a legislature based on populations or votes

- Apparently, some obvious process for *rounding fractions* or some optimal schemes for minimizing certain natural *measure of inequality* would fail. Each scheme may possess certain "flaws" or embarrassing "paradoxes" (反論, opposite to common sense or the truth).

Apportionment of US house seats based on states' populations

- $a_i =$ number of Representatives apportioned to the i^{th} state,

  $p_i =$ population in the i^{th} state, $i = 1, 2, \cdots, S$.

  The Constitution requires $a_i \geq 1$ and $p_i/a_i > 30,000$, where the current House size $= 435^*$ (fixed after New Mexico and Arizona became states in 1912).

$$
\begin{aligned}
& \text{Current number of constituents per Representative} \\
\approx\ & 300 \text{ million}/435 \gg 30,000
\end{aligned}
$$

* In 1959, Alaska and Hawaii were admitted to the Union, each receiving one seat, thus temporarily raising the House to 437. The apportionment based on the census of 1960 reverted to a House size of 435.

Statement of the Problem of Apportionment of House Seats

$h =$ number of congressional seats; $P =$ total US population $= \sum_{i=1}^{S} p_i$; the i^{th} state is entitled to $q_i = h\left(\frac{p_i}{P}\right)$ representatives.

Difficulty: the eligible quota $q_i = \frac{hp_i}{P}$ is in general not an integer. In simple terms, a_i is some form of *integer rounding* to q_i. Define $\overline{\lambda} = P/h =$ average number of constituents per Representative, then $q_i = p_i/\overline{\lambda}$. The (almost) continuous population weight p_i/P is approximated by the rational proportion a_i/h.

An apportionment solution is a function f, which assigns an apportionment vector $\boldsymbol{a}$ to any population vector $\boldsymbol{p}$ and fixed house size h. One usually talks about an apportionment method $M = M(\boldsymbol{p}, h)$, which is a non-empty set of apportionment solutions. Ties may occur, so the solution to $\boldsymbol{a}$ may not be unique.

Numbers of seats for the geographical constituency areas

District	*Number of seats*	*Estimated population (as on 30 June 2012)*	*% of deviation from resulting number*
Hong Kong Island	7	1,295,800	−9.77%
Kowloon West	5	1,081,700	+5.45%
Kowloon East	5	1,062,800	+3.61%
New Territories West	9	2,045,500	+10.78%
New Territories East	9	1,694,900	−8.21%

Related problem Apportionment of legislature seats to political parties based on the votes received by the parties.

Inconsistencies in apportionment based on either the *district* or *state-wide* criterion.

2004 Connecticut congressional elections − District criterion

District	1st	2nd	3rd	4th	5th	Total	Seats
Republican	$73,273$	$\mathbf{165,558}$	$68,810$	$\mathbf{149,891}$	$\mathbf{165,440}$	$622,972$	3
Democratic	$\mathbf{197,964}$	$139,987$	$\mathbf{199,652}$	$136,481$	$105,505$	$779,589$	2

We pick the winner in each district. The Democratic Party receives only 2 seats though the Party receives more votes ($779,589$) state-wide. This is a real life current example where 田忌賽馬 is put into practice.

If the state-wide criteria is used, then the Republican Party with only $\dfrac{622,972}{779,589 + 622,972} \times 100\% = 44.42\%$ of votes should receive only 2 seats.

This appears to be contradicting the principle: parties should share the seats according to their total votes in each state. How can we resolve the inconsistencies?

Gerrymandering

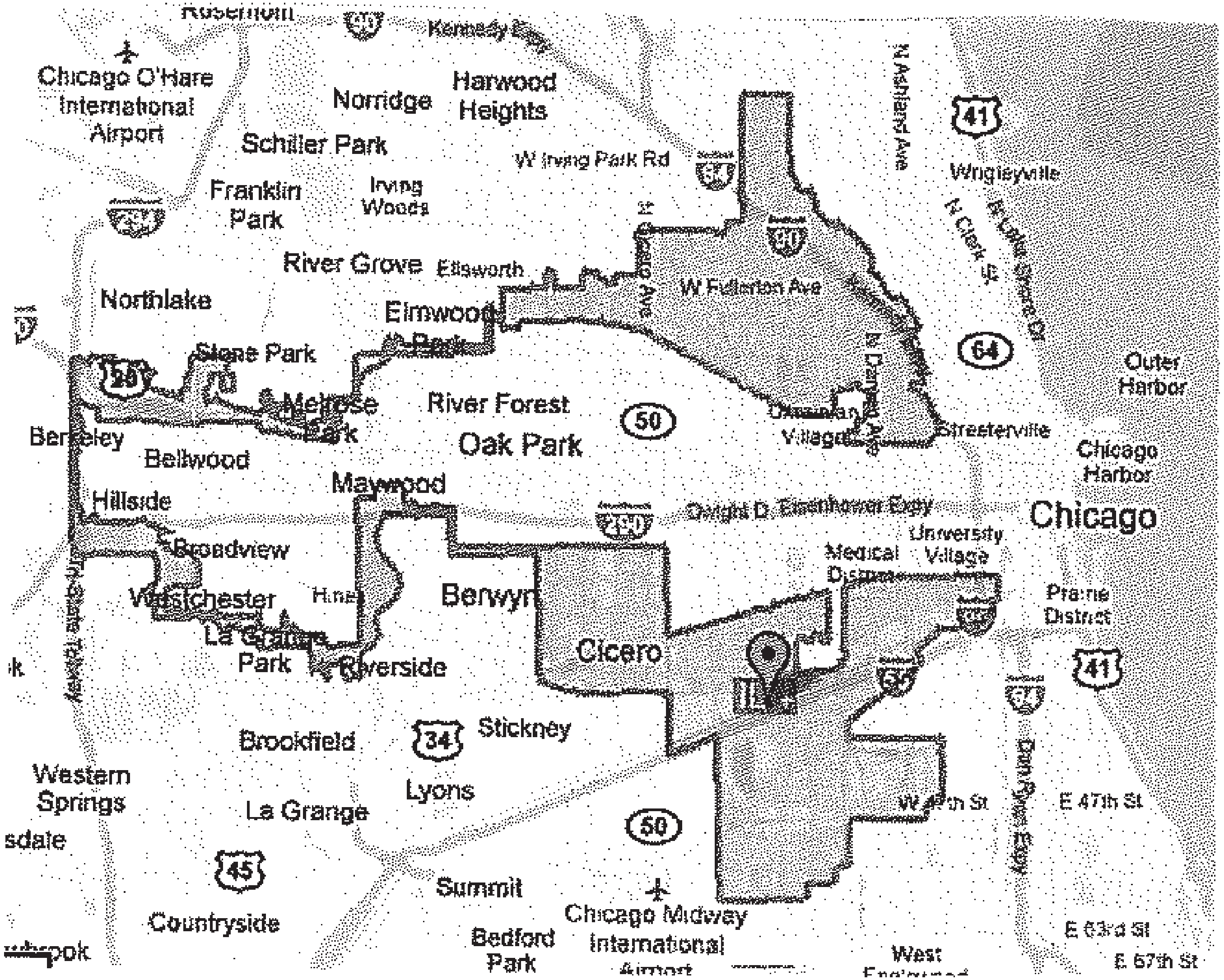
The practice of dividing a geographic area into electoral districts, often of highly irregular shape, to give one political party an unfair advantage by diluting the opposition's strength.

For example, Texas had redistributed following the census of 2000, but in the state elections of 2002, the Republicans took control of the state government and decided to redistribute once again. Both parties determine districts to maximize their advantage whenever they have the power to do so.

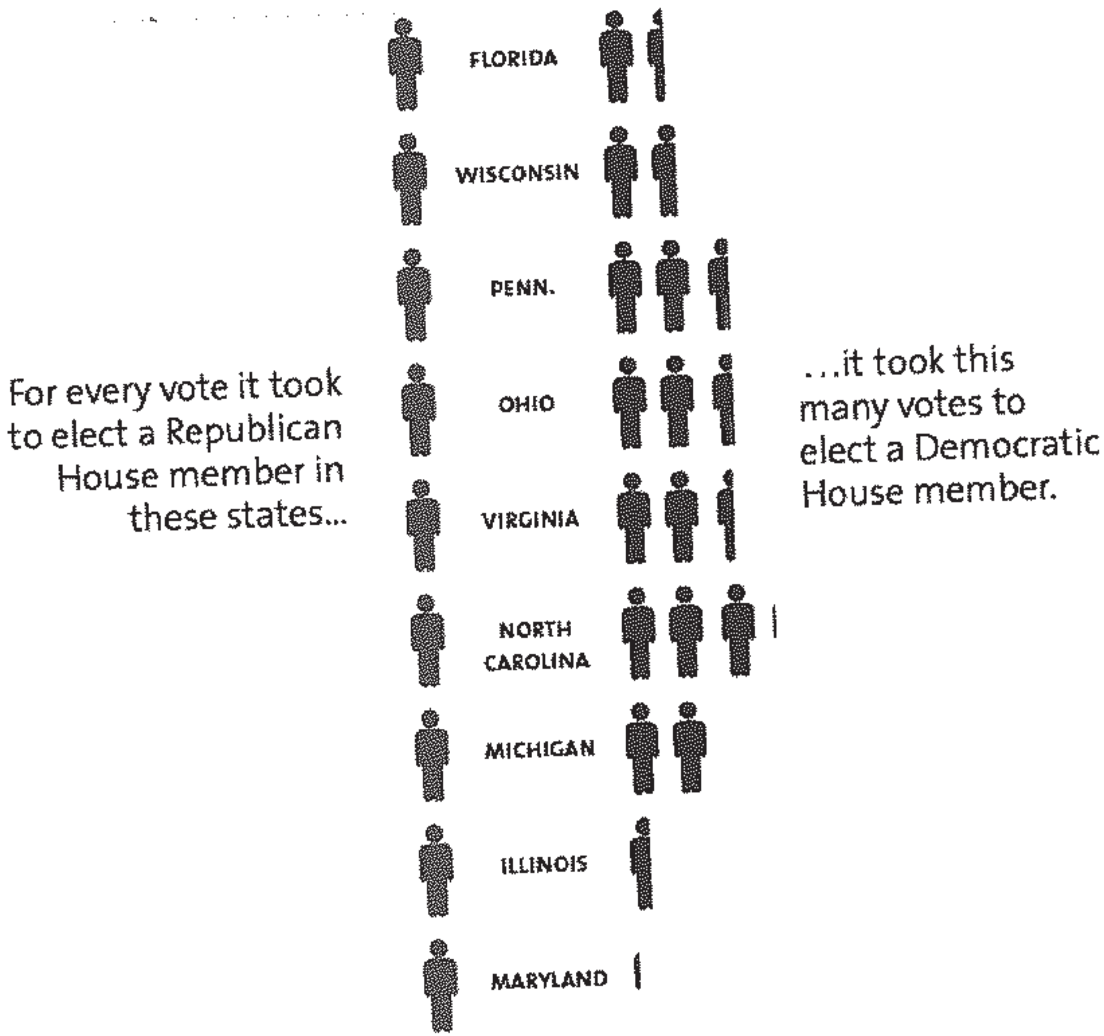
In 2012, the 234-201 House seats majority goes to the Republicans though the Democrats have a slight edge in the popular vote for House seats, 48.8%-47.6%.

Measure to resolve gerrymandering Allocation to district winners is designed such that it also depends on the state wide popularity vote.

Illinois Congressional District 4: Worst Example of Gerrymandering



Republicans *Democrats*



- In Florida, Democrats won nearly half the house race votes but fills about a third of the states' congressional seats.

Issues addressed in apportionment schemes

1. Find an operational method for interpreting the mandate of proportional representation (with reference to population or votes).

2. Identify the desirable properties that any fair method ought to observe. Not to produce paradoxes.

- The "best" method is unresolvable since there is no one method that satisfies all reasonable criteria and produce no paradoxes – Balinski-Young Impossibility Theorem.

- Intense debate surrounding the basis of population counts: How to count Federal employees living outside the US? Should we count illegal immigrants and permanent residents?

1.2 Quota Method of the Greatest Remainder (Hamilton's method) and paradoxes

After assigning at least one seat to each state, every state is then assigned its lower quota. This is possible provided that

$$h \geq \sum_{i=1}^{S} \max(1, \lfloor q_i \rfloor), \qquad (i)$$

a condition which holds in general. Next, we order the remainders $q_i - \lfloor q_i \rfloor$, and allocate seats to the states having the largest fractional remainders in sequential order.

- By its construction, the Hamilton method satisfies the quota property: $\lfloor q_i \rfloor \leq a_i < \lfloor q_i \rfloor + 1$.

- Recall that $h = \sum_{i=1}^{S} q_i$, thus $h \geq \sum_{i=1}^{S} \lfloor q_i \rfloor$; so condition (i) is not satisfied only when there are too many states with very small population that are rounded up to one seat based on the minimum requirement.

Constrained integer programming problem

We minimize $\sum_{i=1}^{S}(a_i - q_i)^2$

subject to $\sum_{i=1}^{S} a_i = h \quad \text{and} \quad a_i \geq 1, \quad i = 1, \cdots, S.$

It seeks for integer allocations a_i that are never less than unity and staying as close as possible (in some measure) to the fair shares q_i. The "inequity" is measured by the totality of $(a_i - q_i)^2$ summed among all states.

- Actually, in a more generalized setting, Hamilton's method minimizes

$$\sum_{i=1}^{S} |a_i - q_i|^{\alpha}, \quad \alpha \geq 1.$$

This amounts to a norm-minimizing approach.

- Any state which has been assigned the lower quota $\lfloor q_i \rfloor$ already will not be assigned a new seat until all other states have been assigned the lower quota. This is because the states that have been assigned the lower quota would have value of $q_i - a_i$ smaller than those states that have not.

- Provided $h \geq \sum_{i=1}^{S} \max(1, \lfloor q_i \rfloor)$, each state would receive at least $\max(1, \lfloor q_i \rfloor)$ seats.

Remark

Due to the minimum requirement that $a_i \geq 1$, it may be possible that not all states are assigned seats with number that is guaranteed to be at least the lower quota.

- Provided that condition (i) is satisfied, all states will be assigned with seats equal to their lower quota or at least one seat. The remaining seats are assigned according to the ranking order of the fractional remainders. Once the upper quota has been assigned to a particular state, no further seat will be assigned. Combining these observations, the quota property is satisfied.

- Why does the Hamilton apportionment procedure minimize the sum of inequity as measured by $\sum_{i=1}^{S}(a_i - q_i)^2$? This is because after each seat assignment, the largest magnitude of reduction is achieved when compared to other methods of apportionment.

Loss of House Monotone Property

State	Population	25 seats exact quota	26 seats exact quota	27 seats exact quota
A	9061	8.713 [9]	9.061 [9]	9.410 [9]
B	7179	6.903 [7]	7.179 [7]	7.455*[8]
C	5259	5.057 [5]	5.259 [5]	5.461*[6]
D	3319	3.191 [3]	**3.319***[4]	**3.447** [3]
E	1182	1.137 [1]	1.182 [1]	1.227 [1]
	26000	25	26	27

- The integers inside [ ] show the apportionments.
- When $h = 26$, State D is assigned an additional seat beyond the lower quota of 3. However, when $h = 27$, the extra seat is taken away since States B and C take the two additional seats beyond their lower quotas. State D suffers a drop from 4 seats to 3 seats when the total number of seats increases from 26 to 27.

Alabama Paradox (1882)

In 1882, the US Census Bureau supplied Congress with a table showing the apportionment produced by Hamilton's method for all sizes of the House between 275 and 350 seats. Using Hamilton's method, the state of Alabama would be entitled to 8 representatives in a House having 299 members, but in a House having 300 members it would only receive 7 representatives – loss of *house monotone property*.

- Alabama had an exact quota of 7.646 at 299 seats and 7.671 at 300 seats, while Texas and Illinois increased their quotas from 9.640 and 18.640 to 9.682 and 18.702, respectively.
- At $h = 300$, Hamilton's method gave Texas and Illinois each an additional representative. Since only one new seat was added, Alabama was forced to lose one seat. Apparently, the more populous state has the larger increase in the remainder part. Thus, Hamilton's method favors the larger states.

House monotone property (Property H)

An apportionment method M is said to be house monotone if for every apportionment solution $f \in M$

$$f(\boldsymbol{p}, h) \leq f(\boldsymbol{p}, h+1).$$

That is, if the House increases its size, then no state will lose a former seat using the same method M.

A method observes house monotone property if the method awards extra seats to states when h increases, rather than computing a general redistribution of the seats.

Why does Hamilton's method not observe the House monotone property?

The rule of assignment of the additional seat may alter the existing allocations. With an increase of one extra seat, the quota $q_i = h\frac{p_i}{P}$ becomes $\widehat{q}_i = (h+1)\frac{p_i}{P}$. The increase in the quota is p_i/P, which differs across the different states (a larger increase for the more populous states). It is possible that a less populous state that is originally over-rounded becomes under-rounded.

- When the number of states is 2, Alabama paradox will not occur. When a state is favorable (rounded up) at h, it will not be rounded down to the floor value of the original quota at the new house size $h+1$ since the increase in the quota of the other state is always less than one.

New States Paradox

If a new state enters, bringing in its complement of new seats [that is, the number it should receive under the apportionment method in use], a given state may lose representation to another even though there is no change in either of their population.

Example

In 1907, Oklahoma was added as a new state with 5 new seats to house (386 to 391). Maine's apportionment went up (3 to 4) while New York's went down (38 to 37). This is due to the change in priority order of assigning the surplus seats based on the fractional remainders.

Consider an apportionment of h seats among 3 states, we ask "If $\boldsymbol{p} = (p_1 \quad p_2 \quad p_3)$ apportions h seats to $\boldsymbol{a} = (a_1 \quad a_2 \quad a_3)$, is it possible that the population $\boldsymbol{p}' = (p_1 \quad p_2)$ apportions $h - a_3$ seats to $\boldsymbol{a}' = (a_1 + 1 \quad a_2 - 1)$?

Example

Consider the Hamiltonian apportionment of 4 seats to 2 states whose populations are 623 and 377. Now suppose a new state with population 200 joins the union and the house size is increased to 5.

- Earlier case, $\boldsymbol{q} = (2.49 \quad 1.51)$ so states 1 and 2 each receives 2 seats.

- After addition of a new state, $\boldsymbol{q} = (2.60 \quad 1.57 \quad 0.83)$ and state 2 has lost a seat to state 1 since the new apportionment is $(3 \quad 1 \quad 1)$.

Population monotonicity

Suppose the population (quota) of a state changes due to redrawing of state boundaries or actual migration of population. Given the fixed values of h and S, if a state's quota increases, then its apportionment does not decrease.

Failure of the population monotone property in Hamilton's method

Suppose a state R_ℓ decreases in population and the excess population is distributed to one state called "lucky" in class D (rounding down) with a larger share of the excess population and another state called "misfortune" in class U with a smaller share. After the redistribution, it is possible that R_ℓ remains in class U, while state "lucky" moves up to class U but state "misfortune" goes down to class D.

Example $h = 32, \quad \boldsymbol{q} = (2.34 \quad 4.88 \quad 8.12 \quad 7.30 \quad 9.36)$

with $\qquad \boldsymbol{a} = (2 \quad 5 \quad 8 \quad 7 \quad 10).$

Population migration from State B to State A and State E lead to

$$\begin{aligned} \boldsymbol{q}_{new} &= (2.42 \quad 4.78 \quad 8.12 \quad 7.30 \quad 9.38) \\ \boldsymbol{a}_{new} &= (3 \quad 5 \quad 8 \quad 7 \quad 9). \end{aligned}$$

State A has a larger share of the migrated population compared to State E, where

$$\begin{aligned} q_A: &\quad 2.34 \to 2.42 \\ q_E: &\quad 9.36 \to 9.38 \\ q_B: &\quad 4.88 \to 4.78. \end{aligned}$$

What has happened to State E? The quota of State E increases but its apportionment decreases.

Quota property (Property Q)

An apportionment method M is said to satisfy the quota property if for every apportionment solution f in M, and any $\boldsymbol{p}$ and h, the resulting apportionment $\boldsymbol{a} = f(\boldsymbol{p}, h)$ satisfies

$$\lfloor q_i \rfloor \leq a_i \leq \lceil q_i \rceil \quad \text{for all } i.$$

Hamilton's method satisfies the Quota Property by its construction. By virtue of the Quota Property, it is not impossible for any state to lose more than one seat when the house size is increased by one.

Balinski-Young Impossibility Theorem

Any apportionment method that does not violate the quota rule must produce paradoxes, and any apportionment method that does not produce paradoxes must violate the quota rule.

Lower quota property

M satisfies lower quota if for every $\boldsymbol{p}, h$ and $f \in M$,

$$\boldsymbol{a} \geq \lfloor \boldsymbol{q} \rfloor.$$

Upper quota property

M satisfies upper quota if for every $\boldsymbol{p}, h$ and $f \in M$,

$$\boldsymbol{a} \leq \lceil \boldsymbol{q} \rceil.$$

Relatively well-rounded

If $a_i > q_i + \frac{1}{2}$ (rounded up even when the fractional remainder is less than 0.5), State i is over-rounded, if $a_j < q_j - \frac{1}{2}$ (rounded down even when the fractional remainder is larger than 0.5), State j is under-rounded. If there exists no pair of States i and j with a_i over-rounded and a_j under-rounded, then $\boldsymbol{a}$ is *relatively well-rounded*.

Desirable properties in Hamilton's Method

1. Binary fairness (pairwise switching)

   One cannot switch a seat from any state i to any other state j and reduce the sum: $|a_i - q_i| + |a_j - q_j|$.

   Hamilton's method, which minimizes $\sum_{i=1}^{S} |a_i - q_i|$, does satisfy "binary fairness".

   Proof: Two classes of states:

   Class U with $a_i = \lceil q_i \rceil$ (rounding up; favorable)

   Class D with $a_j = \lfloor q_j \rfloor$ (rounding down; unfavorable)

   Write the fractional remainders as $R_i = q_i - \lfloor q_i \rfloor$ and $R_j = q_j - \lfloor q_j \rfloor$, where

   $$1 > R_i \geq 0 \quad \text{and} \quad 1 > R_j \geq 0.$$

(i) A switch of one seat between two states falling within the same class increases $|a_i - q_i| + |a_j - q_j|$.

As an illustration, suppose both States i and j fall in class D with

$$|a_i - q_i| = R_i \quad \text{and} \quad |a_j - q_j| = R_j.$$

Since $|1 + a_i - q_i| = 1 - R_i$ and $|a_j - 1 - q_j| = 1 + R_j$, so that

$$|1 + a_i - q_i| + |a_j - 1 - q_j| = 2 + R_j - R_i > R_i + R_j.$$

(ii) Obviously, inequity increases when a seat is switched from a state in class D to another state in class U. A switch of one seat from one state in class U to another state in class D also increases $|a_i - q_i| + |a_j - q_j|$.

Original sum $= R_D + (1 - R_U)$ while the new sum $= 1 - R_D + R_U$. Since $R_U > R_D$, so the switching increases $|a_i - q_i| + |a_j - q_j|$.

2. Hamilton's method has the mini-max property: $\min_a \max_i |a_i - q_i|$.

The worst discrepancy between a_i and q_i among all states is measured by $\max_i |a_i - q_i|$. Among all apportionment methods, Hamilton's method minimizes $\max_i |a_i - q_i|$.

Proof: Arrange the remainders of the states accordingly

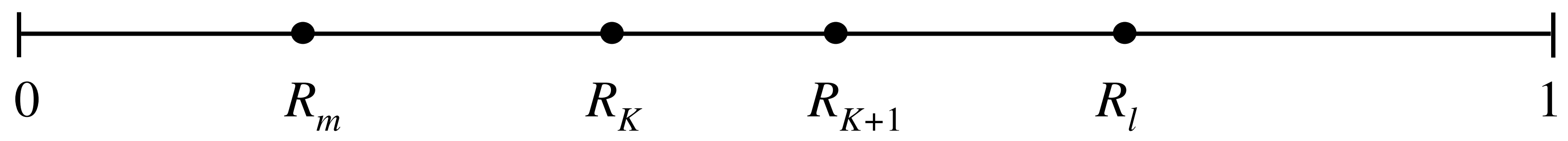
$$\underbrace{R_1 < \cdots < R_K}_{\text{Class D}} < \underbrace{R_{K+1} < \cdots < R_S}_{\text{Class U}}$$

When Hamilton's method is used, assuming no minimum requirement, the apportionment observes the quota property. We then have

$$\max_i |a_i - q_i| = \max(R_K, 1 - R_{K+1}).$$

Consider an alternative apportionment where there exists State ℓ with $R_\ell \geq R_{K+1}$ but it ends up in Class D (rounded down instead of rounded up), then there must exists another state (say, State m) with $R_m \leq R_K$ that ends up in Class U. Let $\widehat{a}_i$ and $\widehat{a}_m$ denote the new apportionments of the respective states.

Now, $\widehat{a}_m - q_m = 1 - R_m$ and $\widehat{a}_\ell - q_\ell = R_\ell$. Further, since $1 - R_m > 1 - R_{K+1}$ and $R_\ell > R_K$, so the new apportionment would have an increase in $\max_i |a_i - q_i|$.



Remarks

1. The objective function (inequity measure) in the minimization procedure under Hamilton's apportionment can be extended to the ℓ_p-norm, where

$$\|\boldsymbol{a} - \boldsymbol{q}\|_p = \left[\sum_{i=1}^{S} |a_i - q_i|^p\right]^{1/p}, \ p \geq 1.$$

   The minimax property can be shown to remain valid under the choice of any ℓ_p-norm. The special cases of ℓ_1-norm and ℓ_∞-norm correspond to $\sum_{i=1}^{S} |a_i - q_i|$ and $\max_i |a_i - q_i|$, respectively.

2. Suppose $\sum_{i=1}^{S} |a_i - q_i|$ is minimized under Hamilton's apportionment, then the switching of a seat among any pair of states would not reduce the sum: $|a_i - q_i| + |a_j - q_j|$.

3. By applying the mini-max property under ℓ_1-norm, we can conclude that an apportionment solution satisfies the binary fairness property if and only if it is a Hamilton apportionment solution.

Summary of Hamilton's method

Assuming no minimum requirement:

- Every state is assigned at least its lower quota. Order the fractional remainders. Assign the extra seats to those states with larger values of fractional remainder.

- Minimize $\sum_{i=1}^{S}(a_i - q_i)^2$ subject to $\sum_{i=1}^{S} a_i = h$.

- Satisfying the quota property: each q_i is either rounded up or rounded down to give a_i.

- Binary fairness

- $\min_a \max_i |a_i - q_i|$

Paradoxes House Monotone; New State Paradox; Population Monotone

History of Hamilton's method in US House apportionment

- The first apportionment occurred in 1794, based on the population figures* from the first national census in 1790. Congress needed to allocate exactly 105 seats in the House of Representatives to the 15 states.

- Hamilton's method was approved by Congress in 1791, but the bill was vetoed by President George Washington (first use of presidential veto).

- Washington's home state, Virginia, was one of the losers in the method, receiving 18 seats despite a standard quota of 18.310.

- The Jefferson apportionment method was eventually adopted and gave Virginia 19 seats.

*The population figures did not fully include the number of slaves and native Americans who lived in the U.S. in 1790.

- Jefferson's method is a divisor method, which may not satisfy the quota property. The year 1832 was the end of Jefferson's method. If Jefferson's method has continued to be used, every apportionment of the House since 1852 would have violated quota. In 1832, Jefferson's method gave New York 40 seats in the House even though its standard quota was only 38.59.

- Websters' method, another but improved divisor method (regarded as the best approximation method by modern day experts), was used for the apportionment of 1842. The method may violate quota, but the chance is very slim. If Webster's method has been used consistently from the first apportionment of the House in 1794 to the most recent reapportionment in 2002, it would still have yet to produce a quota violation.

- The very possibility of violating quota lead Congress leery of Webster's method. In 1850, Congressman Samuel Vinton proposed what be thought was a brand new method (actually identical to Hamilton's method). In 1852, Congress passed a law adopting Vinton's method.

- *Compromise adopted in 1852*

  In 1852, and future years, Congress would increase the total number of seats in the House to a number for which Hamilton's and Webster's method would yield identical apportionment.

- A major deficiency in Hamilton's method is the loss of *House Monotone* property. Such paradox occurred in 1882 and 1902. In 1882, US Congress opted to go with a House size of 325 seats to avoid the Alabama paradox. Another similar case occurred in 1902 (final death blow to Hamilton's method) lead Congress to adopt Webster's method with a total House size of 386 seats.

1.3 Geometric characterization and apportionment simplex

When the number of states $S = 3$, we are able to perform geometric characterization of the apportionment problem in the $\mathbb{R}^3$-plane.

For a given total population P, there is a *population simplex* represented by

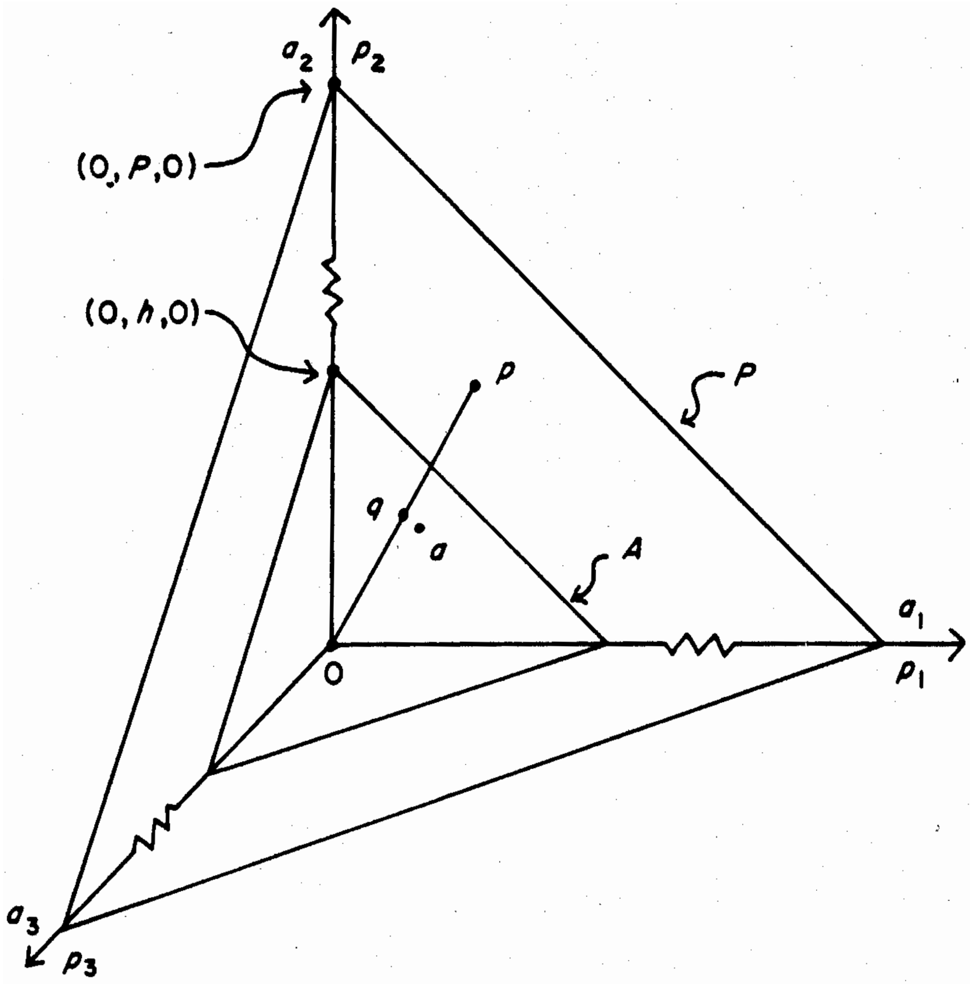
$$\mathbb{P} = \{(p_1, p_2, p_3) : p_1 + p_2 + p_3 = P, \quad p_1, p_2 \text{ and } p_3 \text{ are integers}\},$$

where (p_1, p_2, p_3) are the integer points on an inclined equilateral triangle with vertices $(P, 0, 0), (0, P, 0)$ and $(0, 0, P)$.

For any house size h, there is an *apportionment simplex* represented by

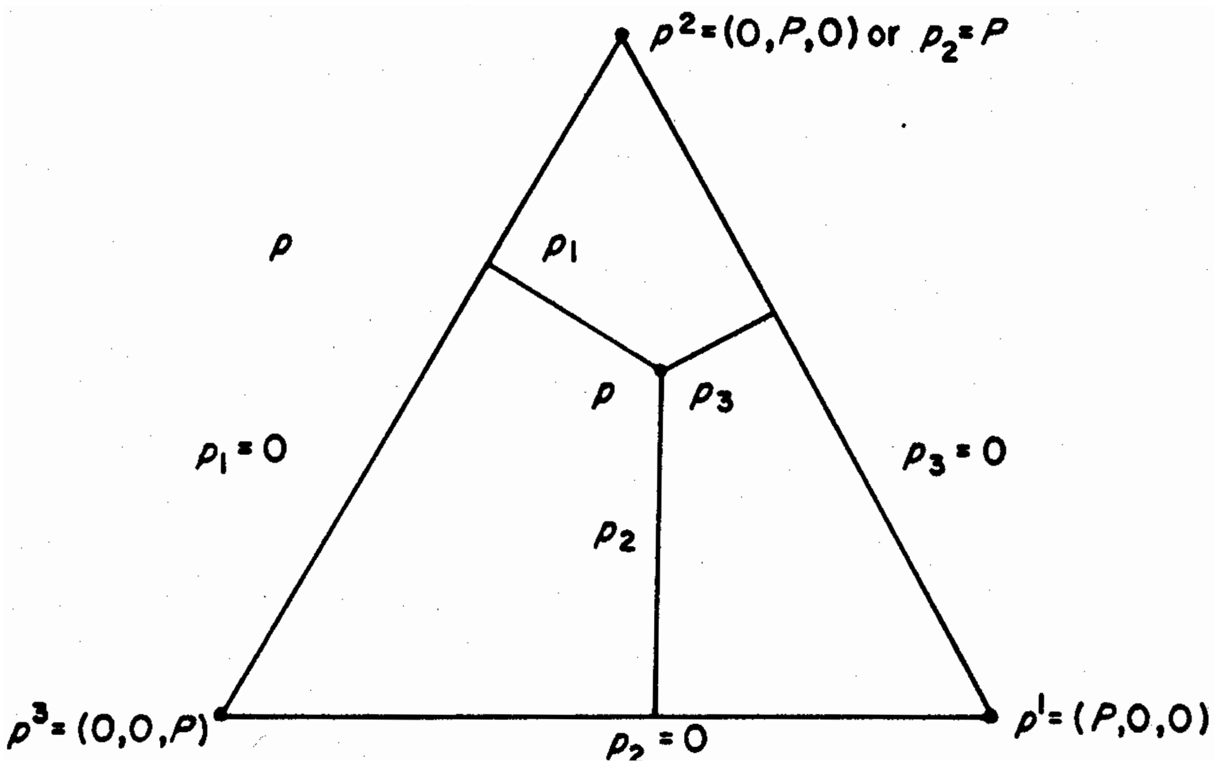
$$\mathcal{A} = \{(a_1, a_2, a_3) : a_1 + a_2 + a_3 = h, \quad a_1, a_2 \text{ and } a_3 \text{ are integers}\}.$$

The point $\boldsymbol{q}$ is the point of intersection of the line OP on the plane $\mathcal{A}$.



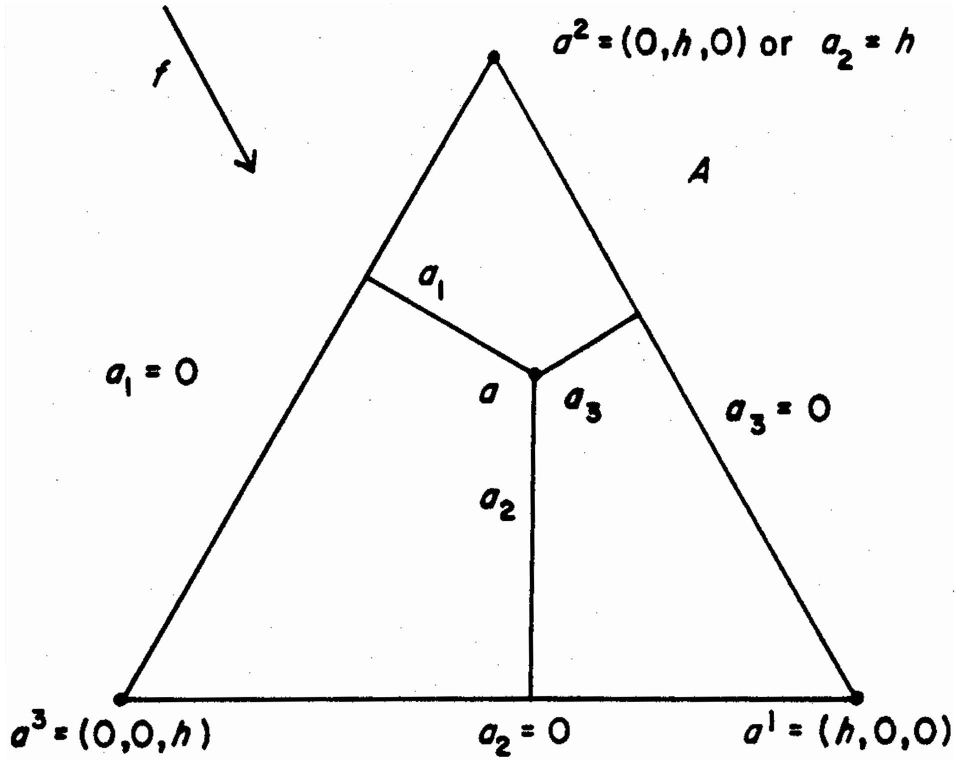
The Apportionment Problem for $S = 3$. $\mathcal{A}$ is the plane of apportionment while $\mathcal{P}$ is the plane of population. Both $\boldsymbol{q}$ and $\boldsymbol{a}$ lie on $\mathcal{A}$. We find $\boldsymbol{a}$ that is closest to $\boldsymbol{q}$ based on certain criterion of minimizing the inequity measure.

The population vector $\boldsymbol{p}$ intersects the apportionment plane $\mathcal{A}$ at the quota vector $\boldsymbol{q}$. The apportionment problem is to choose an *integer valued* apportionment vector $\boldsymbol{a}$ on $\mathcal{A}$ which is in some sense "close" to $\boldsymbol{q}$.



The left edge lies in the p_2-p_3 plane with $p_1 = 0$. The distance from (p_1, p_2, p_3) to the p_2-p_3 plane is p_1.

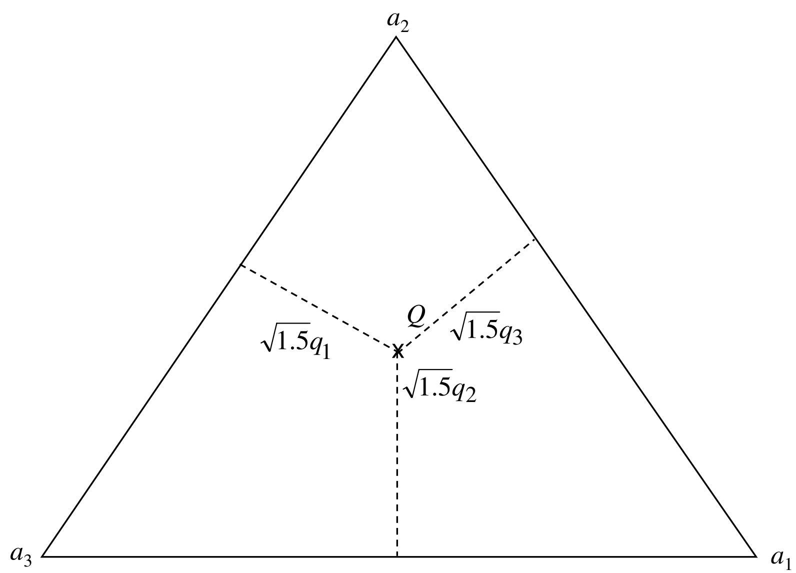
Apportionment function



The apportionment function $f = f(\boldsymbol{p}, h)$ partitions into regions about each integer vector $\boldsymbol{a} \in \mathcal{A}$ such that if $\boldsymbol{q}$ falls into such a region, then it is rounded to the corresponding $\boldsymbol{a}$.

How to locate the quota point q on the plane?

Recall $q_1 + q_2 + q_3 = h =$house size. The distance from the vertex $(h, 0, 0)$ to the opposite edge is $\sqrt{\left(\sqrt{2}h\right)^2 - \left(\frac{h}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}}h$. The quota vector $\boldsymbol{q}$ has 3 coordinates q_1, q_2 and q_3, where $\sqrt{3/2}q_i$ is the perpendicular distance from the point Q (representing the vector $\boldsymbol{q}$) to the edge opposite to the point a_i.



The vertex $a_1(h, 0, 0)$ lies on the a_1-axis while the opposite edge lies in the $a_2 - a_3$ plane.

Hamilton's apportionment

- When $S = 3$, Hamilton's method effectively divides the plane into regular hexagons around the points representing possible apportionment vectors (except for those apportionment vectors whose ruling regions are truncated by an edge).

- Non-uniqueness of solution for $\boldsymbol{a}$ occurs when $\boldsymbol{q}$ lies on an edge of these regular polygons. A separate rule is needed to break ties.

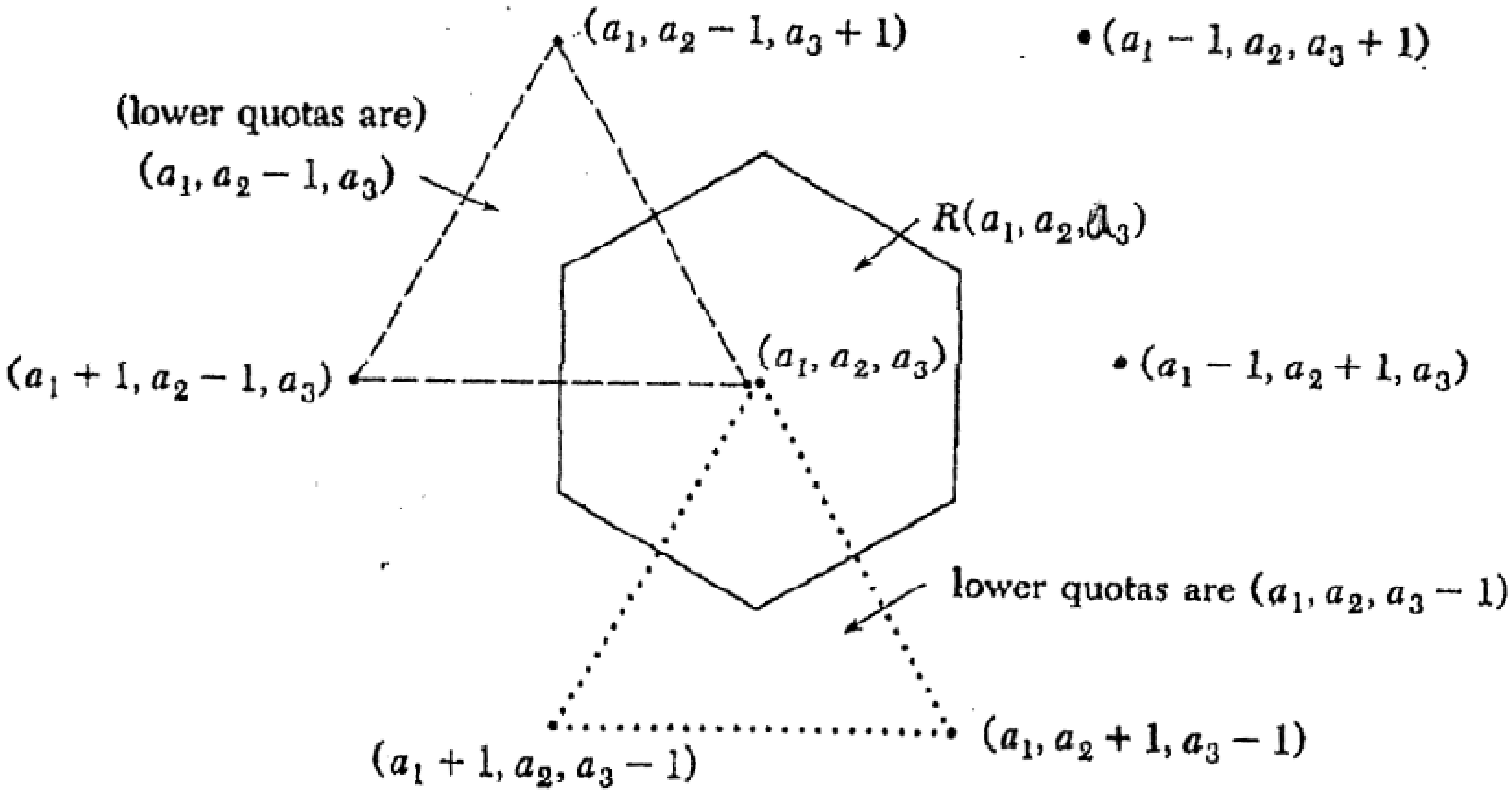
- When the house size increases, the sizes of the hexagons decrease.

Explanation of the regular hexagonal shape

Given three states and h seats, the population $q = (q_1, q_2, q_3)$ apportions to $a = (a_1, a_2, a_3)$ if either each $q_i = a_i$ or if any one of the following six conditions hold:

lower quota is	*and*
$(a_1, a_2 - 1, a_3)$	$q_2 - (a_2 - 1) > \max(q_1 - a_1, q_3 - a_3)$
$(a_1 - 1, a_2 - 1, a_3)$	$q_3 - a_3 < \min\{q_1 - (a_1 - 1), q_2 - (a_2 - 1)\}$
$(a_1 - 1, a_2, a_3)$	$q_1 - (a_1 - 1) > \max\{q_2 - a_2, q_3 - a_3\}$
$(a_1 - 1, a_2, a_3 - 1)$	$q_2 - a_2 < \min\{q_1 - (a_1 - 1), q_3 - (a_3 - 1)\}$
$(a_1, a_2, a_3 - 1)$	$q_3 - (a_3 - 1) > \max\{q_1 - a_1, q_2 - a_2\}$
$(a_1, a_2 - 1, a_3 - 1)$	$q_1 - a_1 < \min\{q_2 - (a_2 - 1), q_3 - (a_3 - 1)\}$

- The first case corresponds to rounding down in State 1 and State 3 while rounding up in State 2. This occurs when the fractional remainder of State 2 is the largest among the 3 fractional remainders.



$(a_1, a_2 - 1, a_3 + 1)$
$(a_1 - 1, a_2, a_3 + 1)$
(lower quotas are)
$(a_1, a_2 - 1, a_3)$
$R(a_1, a_2, a_3)$
$(a_1 + 1, a_2 - 1, a_3)$
(a_1, a_2, a_3)
$(a_1 - 1, a_2 + 1, a_3)$
lower quotas are $(a_1, a_2, a_3 - 1)$
$(a_1, a_2 + 1, a_3 - 1)$
$(a_1 + 1, a_2, a_3 - 1)$

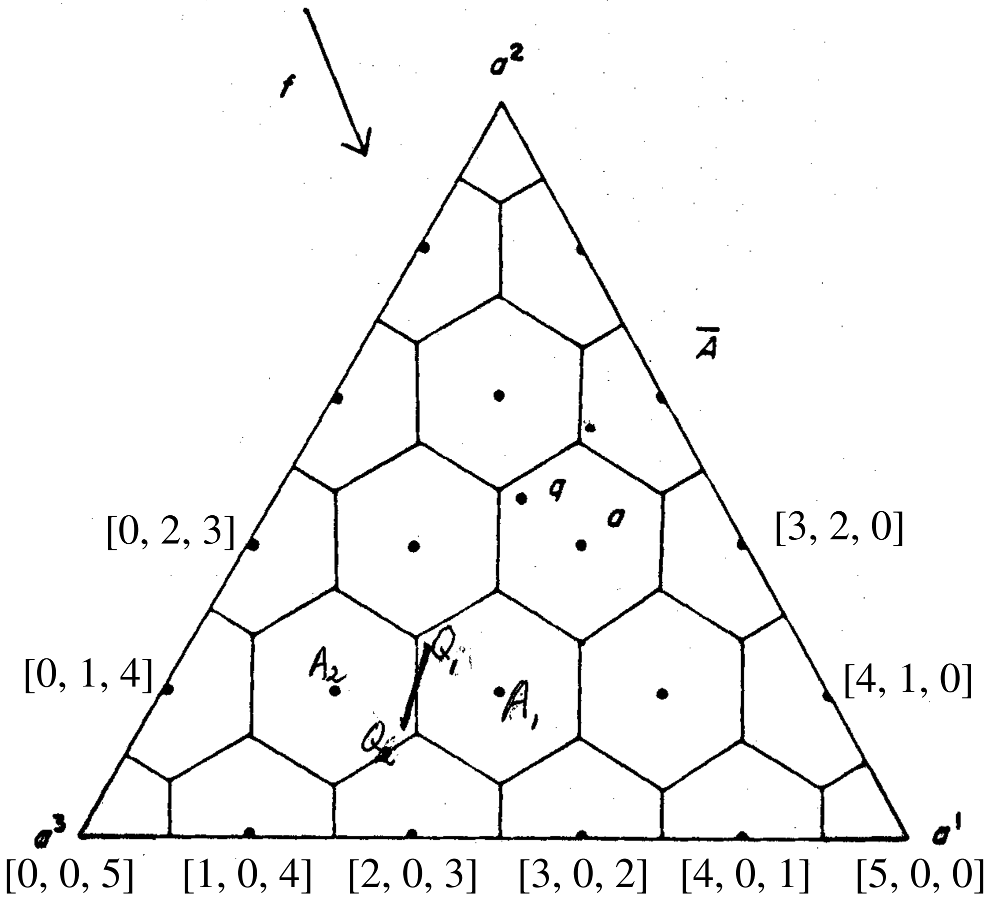
Hexagonal region formed by the intersection of 6 perpendicular bisectors

- The dashed triangle indicates the region in which lower quotas are $(a_1, a_2 - 1, a_3)$; the boundaries of $R_{(a_1,a_2,a_3)}$ within the triangle are the perpendicular bisectors of the line segments joining (a_1, a_2, a_3) with $(a_1, a_2 - 1, a_3 + 1)$ and $(a_1 + 1, a_2 - 1, a_3)$, corresponding to the inequalities $q_2 - (a_2 - 1) > q_3 - a_3$ and $q_2 - (a_2 - 1) > q_1 - a_1$, respectively.

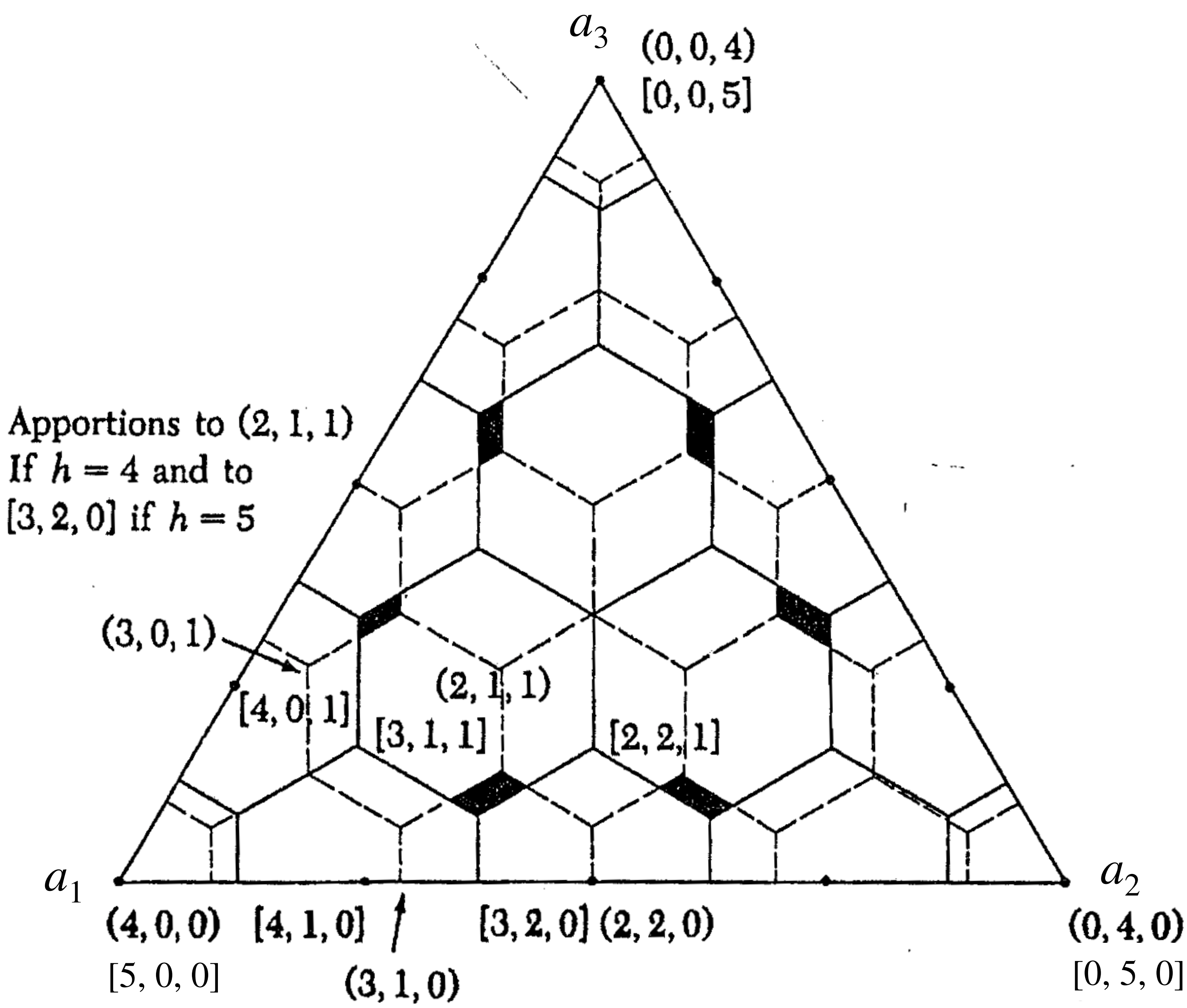
- Similarly, the dotted triangle represents the region in which lower quotas are $(a_1, a_2, a_3 - 1)$.

- The apportionment region R_a is the region formed by bisecting the line segment joining $\boldsymbol{a}$ to each of its neighbors.

Violation of population monotonicity



Hamilton's Method for $S = 3$ and $h = 5$. Compared to Q_1, Q_2 may have a larger value of the first component (further away from the edge opposite to a_1) but it lies in the hexagon $A_2[1,1,3]$ whose first component is smaller than that of $A_1[2,1,2]$.



a_3
(0, 0, 4)
[0, 0, 5]
Apportions to (2, 1, 1)
If $h = 4$ and to
[3, 2, 0] if $h = 5$
(3, 0, 1)
[4, 0, 1]
(2, 1, 1)
[3, 1, 1]
[2, 2, 1]
a_1
a_2
(4, 0, 0)
[5, 0, 0]
[4, 1, 0]
(3, 1, 0)
[3, 2, 0]
(2, 2, 0)
(0, 4, 0)
[0, 5, 0]

Alabama paradox

Hamilton's apportionment diagram for $S = 3, h = 5$ (dotted lines and apportionments in square brackets) is overlaid on Hamilton apportionment diagram for $S = 3, h = 4$ (solid lines and round brackets), with a few apportionments labeled. Populations in the shaded regions are susceptible to the Alabama Paradox. Consider the lowest left shaded region, it lies in $(2, 1, 1)$ and $[3, 2, 0]$ so that the last state loses one seat when the house size increases from $h = 4$ to $h = 5$.

Another notion of the Population Paradox

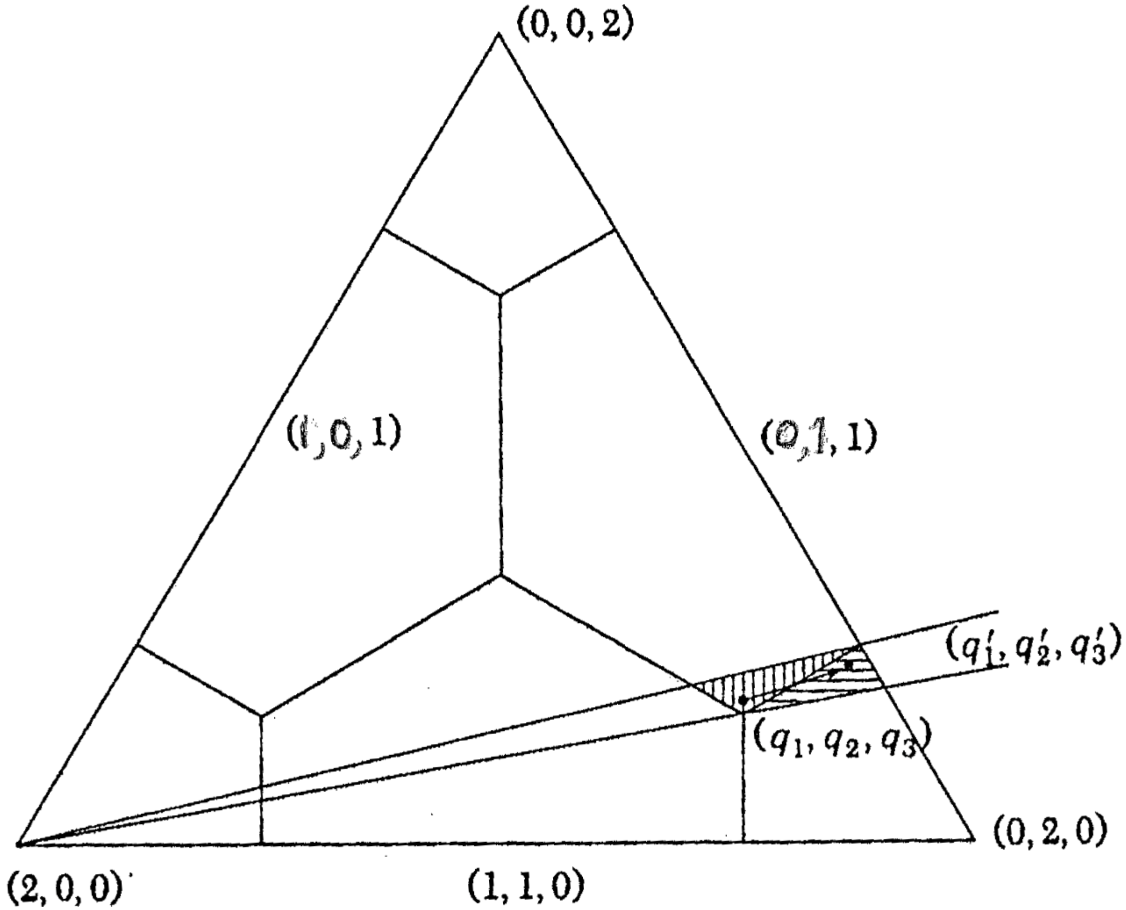
Fix house size h and number of states S but let populations increase (as reflected from census data on two different dates). State i may lose a seat to state j even if state i's population is growing at a faster rate than state j's. If the initial population is p and after some time the population is p', the statement "state i's population is growing faster than state j's" means that

$$\frac{p_i'}{p_i} > \frac{p_j'}{p_j}$$

or, equivalently,

$$\frac{q_i'}{q_j'} > \frac{q_i}{q_j}.$$

Thus, a population increase can cause state i to lose a seat to state j if and only if simultaneously $\boldsymbol{q}$ lies in the domain of $\boldsymbol{a} = (\cdots, a_i, \cdots, a_j, \cdots)$ while $\boldsymbol{q}'$ lies in that of $\boldsymbol{a}' = (\cdots, a_i - 1, \cdots, a_j + 1, \cdots)$, with the inequality above satisfied.



Any line through the vertex $(2, 0, 0)$ represents points with constant proportion of q_3/q_2. The Population Paradox is revealed when a change in population from (q_1, q_2, q_3) [lying in the region: $(0, 1, 1)$] to (q_1', q_2', q_3') [lying in the region: $(0, 2, 0)$] causes state 3 to lose a seat to state 2 even though $q_3'/q_3 > q_2'/q_2$. Here, $S = 3$ and $h = 2$.

Numerical example

- Suppose $S = 3, h = 3$, and the populations at some time t_1 are $420, 455$, and 125, respectively. At a later time t_2, the populations are $430, 520$, and 150.

- All states have experienced growth, and the fastest-growing state is S_3, where $\frac{150}{125} = 1.2 > \frac{520}{455} > \frac{430}{420}$.

- However, $q_{t_1} = (1.26, 1.36, 0.38)$, which results in a Hamiltonian apportionment of $(1, 1, 1)$, while $q_{t_2} = (1.17, 1.42, 0.41)$, which apportions to $(1, 2, 0)$. State 3 loses its seat to the more slowly growing state 2.

1.4 Divisor methods

Based on the idea of an *ideal district size* or common divisor, a divisor λ is specified, where λ is an approximation to the theoretical population size per seat $\overline{\lambda} = P/h$. Some rounding of the numbers p_i/λ are used to determine a_i, whose sum equals h. This class of methods are called the divisor methods.

Jefferson's method (used by US Congress from 1794 through 1832)

Let $\lfloor\lfloor x \rfloor\rfloor$ be the greatest integer less than x if x is non-integer, and otherwise be equal to x or $x-1$. For example, $\lfloor\lfloor 4 \rfloor\rfloor$ can be equal to 4 or 3.

For a given $h, \overline{\lambda} =$ average size $= \sum_{i=1}^{S} p_i/h$, choose $\lambda\,(\leq \overline{\lambda})$ such that $\sum_{i=1}^{S} \left\lfloor\left\lfloor \frac{p_i}{\lambda} \right\rfloor\right\rfloor = h$ has a solution.

To meet the requirement of giving at least one representative to each state, we take $a_i = \max\left(1, \left\lfloor\left\lfloor \frac{p_i}{\lambda} \right\rfloor\right\rfloor\right)$, where λ is a positive number chosen so that $\sum_{i=1}^{S} a_i = h$. Here, λ is a quantity that is close to $\overline{\lambda} =$ average population represented by a single representative.

Here, $\overline{\lambda} = \dfrac{p_1 + \cdots + p_S}{h}$ and $q_i = \dfrac{p_i}{\overline{\lambda}}$.

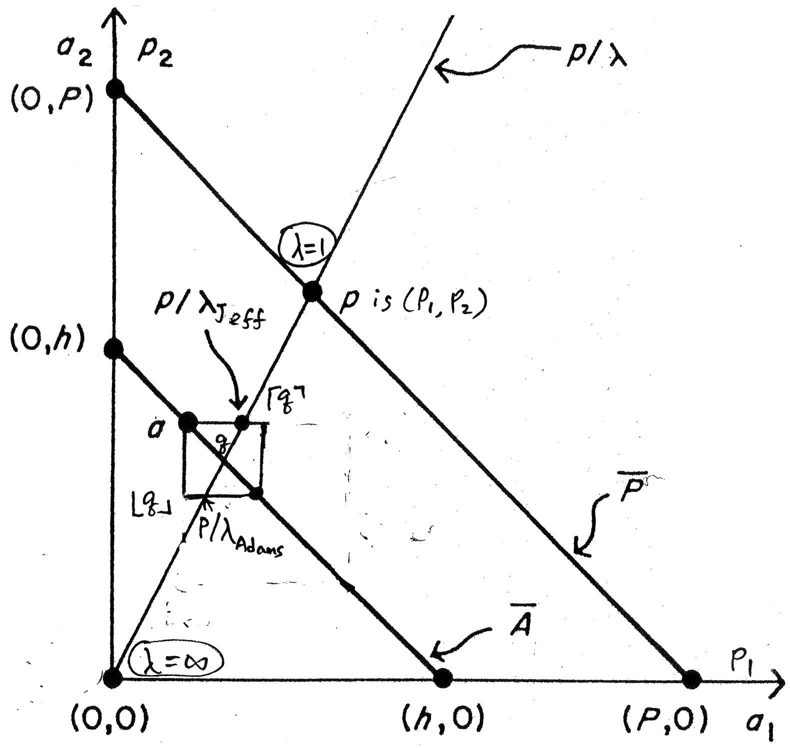
- Jefferson's method favors the larger states, like Virginia (Virginians had the strongest influence in early US history). The method was challenged due to its violation of the quota property, which was then replaced by another divisor method (Webster's method) in 1842.

- Jefferson's method can be viewed as a particular "rounding" procedure. Choose a common divisor λ, and for each state compute p_i/λ and *round down* to the nearest integer.

- In the unlikely event of a tie (不分勝負), one obtains

$$\sum_{i=1}^{S} \lfloor \frac{p_i}{\lambda} \rfloor = h' > h \text{ (or } < h)$$

for all λ. When λ increases gradually , it reaches some threshold value λ_0 at which the above sum just obtains the first value $h' > h$, and for which two or more of the terms p_i/λ_0 are integer valued. One must use some ad hoc rule to decide which states ($h' - h$ in total) must lose a seat so that $a_i = \frac{p_i}{\lambda_0} - 1$ for those states.



Jefferson Method for $S = 2$.

- $\boldsymbol{q} = (q_1 \quad q_2)$ lies on the line $\overline{A}$. Apportionment solutions must be points on $\overline{A}$ with integer coordinates.

- λ_{Jeff} is the approximation to $\bar{\lambda}$ based on the Jefferson method, where

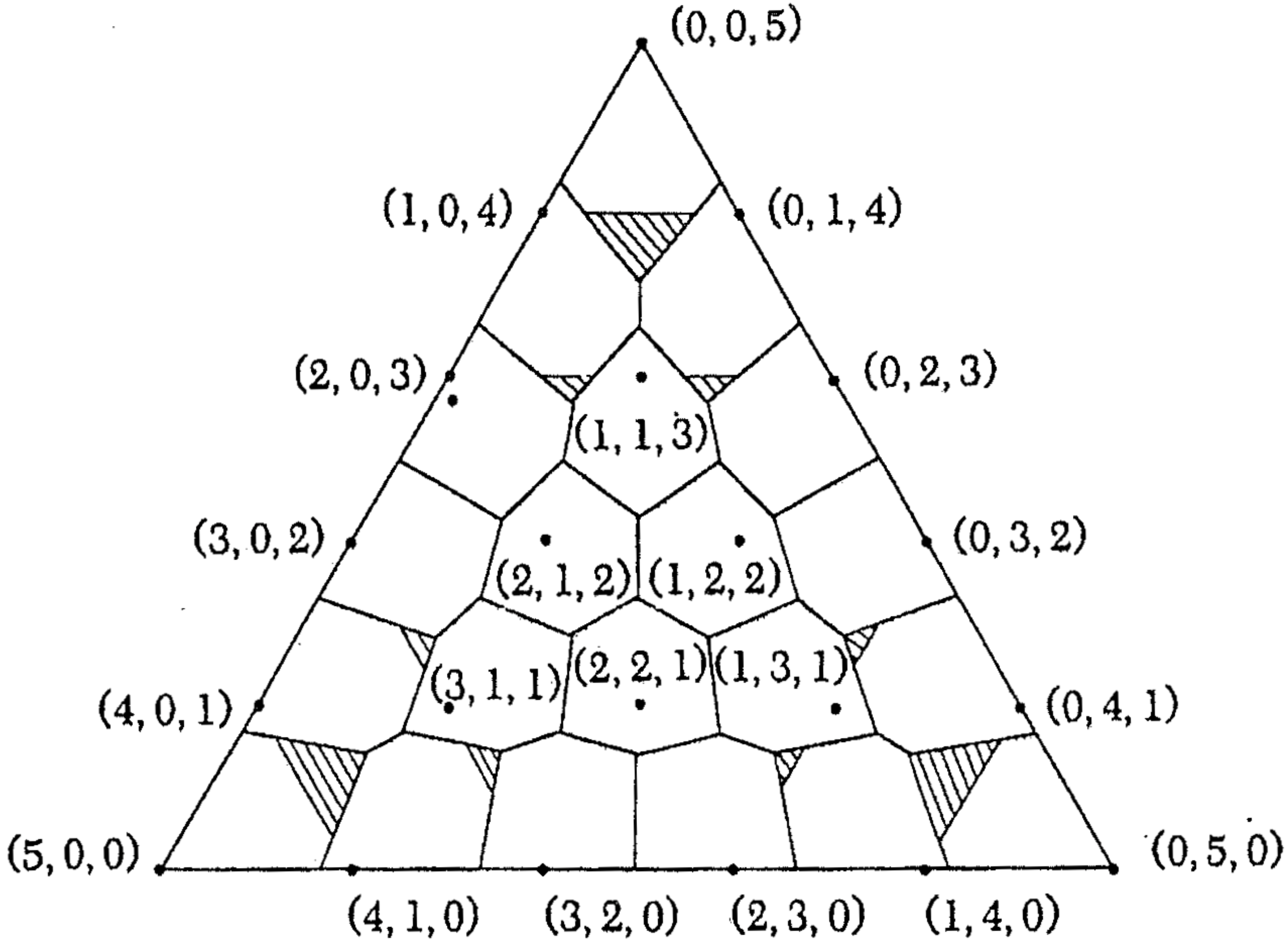
$$\left\lfloor\left\lfloor \frac{p_1}{\lambda_{Jeff}} \right\rfloor\right\rfloor + \left\lfloor\left\lfloor \frac{p_2}{\lambda_{jeff}} \right\rfloor\right\rfloor = h.$$

- The quota vector $\boldsymbol{q}$ contained in Box B lies on the line $\overline{A}$. The upper left corner and the lower right corner of B are possible apportionment points (whose coordinates are all integer-valued) which lie on $\overline{A}$. If the upper left (lower right) corner is chosen, then the apportionment favors State 2 (State 1). The quota point corresponds to the case where λ equals $\bar{\lambda}$, where $\bar{\lambda}$ is the average population per representative. We increase λ gradually until at $\lambda = \lambda_0$, $\boldsymbol{p}/\lambda_0$ hits the upper side of B (favoring state 2 which has a larger population). In this case, p_1/λ_0 is rounded down to a_1 while $p_2/\lambda_0 = a_2$.

- $\displaystyle\sum_{i=1}^{S} \lfloor \frac{p_i}{\lambda} \rfloor$ is a non-increasing step function of λ as we move along the ray P/λ from P (corresponding to $\lambda = 1$) to 0 (corresponding to $\lambda = \infty$). Normally, $\sum_{i=1}^{S} \lfloor \frac{p_i}{\lambda} \rfloor$ drops its value by one as λ increases gradually. When the step decrease is 2 or more, it may occur that there is no solution to $\sum_{i=1}^{S} \lfloor \frac{p_i}{\lambda} \rfloor = h$ for some h.

- When State 1 is the less populous state (as shown in the figure), the apportionment solution at the left top corner is chosen, thus favoring the more populous State. However, when State 1 is taken to be the more populous state (slope of P/λ is now less than one), the apportionment point chosen will be at the right bottom corner, again favoring the more populous state.

- The more populous state is favored over the less populous state in Jefferson's apportionment. For example, in 1794 apportionment in which $h = 105$, Virginia with $q = 18.310$ was rewarded with 19 seats while Delaware with $q = 1.613$ was given only one seat.



Apportionment diagram for Jefferson's method, $S = 3, h = 5$. Populations in the shaded regions apportion in violation of the upper quota property. At the top of the figure, the shaded region is apportioned to $(0, 0, 5)$ even though $q_3 < 4$.

Adams Method

Alternatively, one might consider finding apportionments by *rounding up*. Let $\lceil\lceil x \rceil\rceil$ be the smallest integer greater than x if x is not an integer, and otherwise equal to x or $x+1$. Choose $\lambda\,(\geq \overline{\lambda})$ such that

$$\sum_{i=1}^{S} \lceil\lceil p_i/\lambda \rceil\rceil = h$$

can be obtained, then apportionment for h can be found by taking

$$a_i = \lceil\lceil p_i/\lambda \rceil\rceil$$

satisfying $\sum_{i=1}^{S} a_i = h$. This is called the Adams method. Since all quota values are rounded up, the Adams method guarantees at least one seat for every state. The Adams method favors smaller state (just a mirror image of the Jefferson method).

Lemma on the Jefferson apportionment

Given $\boldsymbol{p}$ and $h, \boldsymbol{a}(a_1 \cdots a_S)$ is a Jefferson apportionment for h if and only if

$$\max_i \frac{p_i}{a_i + 1} \leq \min_i \frac{p_i}{a_i}. \qquad (A)$$

Proof

By definition, $a_i = \lfloor\lfloor p_i/\lambda \rfloor\rfloor$ so that

$$a_i + 1 \geq \frac{p_i}{\lambda} \geq a_i \Leftrightarrow \frac{p_i}{a_i + 1} \leq \lambda \leq \frac{p_i}{a_i} \text{ for all } i,$$

(if $a_i = 0, p_i/a_i = \infty$). Equivalently,

$$\max_i \frac{p_i}{a_i + 1} \leq \min_i \frac{p_i}{a_i}.$$

Interpretation of the Lemma

Recall that the smaller value of p_i/a_i (= population size represented by each seat) the better for that state. Alternatively, a state is better off than another state if $\dfrac{p_i}{a_i} < \dfrac{p_j}{a_j}$.

- To any state k, assignment of an additional seat would make it to become the best off state among all states since

$$\frac{p_k}{a_k^{new}} = \frac{p_k}{a_k + 1} \leq \max_i \frac{p_i}{a_i + 1} \leq \min_i \frac{p_i}{a_i} \leq \min_{i \neq k} \frac{p_i}{a_i}.$$

- Though there may be inequity among states as measured by their shares of p_i/a_i, the "unfairness" is limited to less than one seat (the assignment of one extra seat makes that state to become the best off).

- Jefferson apportionment satisfies the lower quota property. Suppose not, there exists $\boldsymbol{a}$ for h such that $a_i < \lfloor q_i \rfloor$ or $a_i \leq q_i - 1$. For some state $j \neq i$, we have $a_j > q_j$. Recall $q_i = p_i/\overline{\lambda}$ and $q_j = p_j/\overline{\lambda}$ so that
$$\frac{p_j}{a_j} < \overline{\lambda} \leq \frac{p_i}{a_i + 1},$$
a contradiction to the Lemma. However, it does not satisfy the upper quota property (historical apportionment in 1832, where New York State was awarded 40 seats with quota of 38.59 only).

- In a similar manner, the Adams method satisfies
$$\max_i \frac{p_i}{a_i} \leq \min_i \frac{p_i}{a_i - 1} \text{ for } a_i \geq 1.$$
Based on this inequality, it can be shown that it satisfies the upper quota property. Similarly, the Adams method does not satisfy the lower quota property.

Recursive scheme of Jefferson's apportionment

The set of Jefferson solutions is the set of all solutions $\boldsymbol{f}$ obtained recursively as follows:

(i) $\boldsymbol{f}(\boldsymbol{p}, 0) = \boldsymbol{0}$;

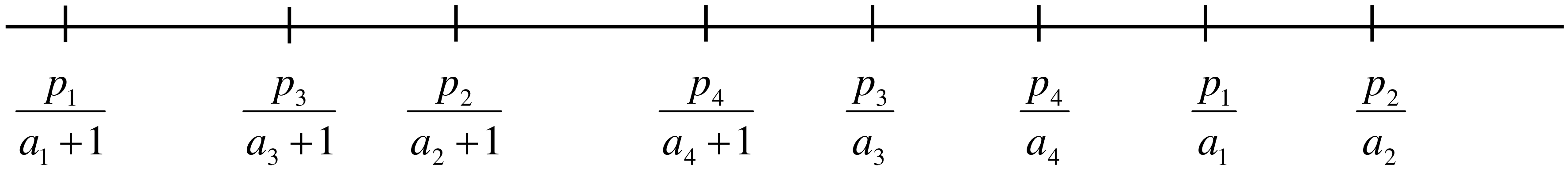
(ii) if $a_i = f_i(\boldsymbol{p}, h)$ is an apportionment for h, let k be some state for which $\dfrac{p_k}{a_k + 1} = \max_i \dfrac{p_i}{a_i + 1}$, then

$$f_k(\boldsymbol{p}, h+1) = a_k + 1 \quad \text{and} \quad f_i(\boldsymbol{p}, h+1) = a_i \text{ for } i \neq k.$$

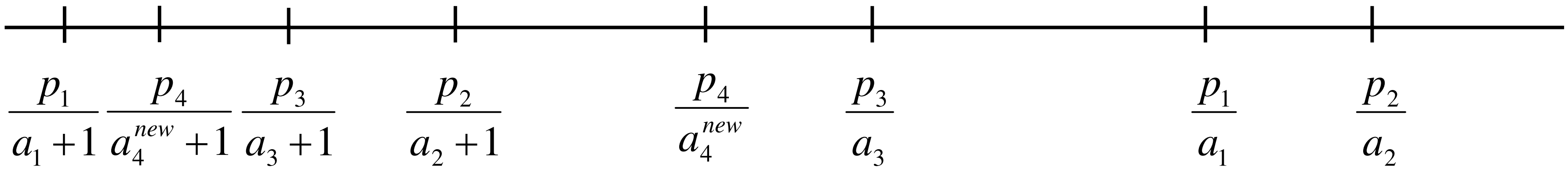
Remark

The above algorithm dictates how the additional seat is distributed while other allocations remain the same. Hence, house monotone property of the Jefferson apportionment is automatically observed.

Consider the case $S = 4$, we rank $\dfrac{p_i}{a_i + 1}, i = 1, 2, 3, 4.$



Since $\dfrac{p_i}{a_i + 1}$ is maximized at $i = 4$, we assign the extra seat to State 4. Now, $a_4^{new} = a_4^{old} + 1$.



After one seat has been assigned to State 4, $\frac{p_i}{a_i+1}$ is maximized at $i = 2$. Next, we assign the extra seat to State 2.

Justification of the recursive scheme

Given $\boldsymbol{p}, \boldsymbol{f}(\boldsymbol{p}, 0) = \mathbf{0}$ satisfies ineq. (A). Suppose we have shown that any solution up through h obtained via the recursive scheme satisfies ineq. (A), then giving one more seat to some state k that maximizes $\frac{p_i}{a_i + 1}$ would result in an apportionment also satisfying ineq. (A).

Conversely, suppose $\boldsymbol{f}$ is a Jefferson solution that is not obtained via the recursive scheme. There is a solution $\boldsymbol{g}$ obtained via the scheme and an house size h such that $\boldsymbol{g}^h = \boldsymbol{f}^h$ but for some $\boldsymbol{p}, \boldsymbol{g}(\boldsymbol{p}, h+1) \neq \boldsymbol{f}(\boldsymbol{p}, h+1)$. Then $\boldsymbol{q}$ must accord the $(h+1)^{\text{st}}$ seat to some state ℓ such that

$$\frac{p_\ell}{a_\ell + 1} < \max_i \frac{p_i}{a_i + 1} = \frac{p_k}{a_k + 1}.$$

With $a_\ell^{new} = a_\ell + 1$, this new allocation leads to $\frac{p_\ell}{a_\ell^{new}} < \frac{p_k}{a_k + 1}$, which violates ineq. (A). Hence a contradiction.

Webster's method (first adopted in 1842, replacing Jefferson's method but later replaced by Hill's method in 1942)

For any real number z, whose fractional part is not $\frac{1}{2}$, let $[z]$ be the integer closest to z. If the fractional part of z is $\frac{1}{2}$, then $[z]$ has two possible values.

The Webster Method is

$$f(\boldsymbol{p}, h) = \{\boldsymbol{a} : a_i = [p_i/\lambda], \sum_{i=1}^{S} a_i = h \text{ for some positive } \lambda\}.$$

It can be shown that λ satisfies

$$\max_{a_i \geq 0} \frac{p_i}{a_i + \frac{1}{2}} \leq \lambda \leq \min_{a_i > 0} \frac{p_i}{a_i - \frac{1}{2}}.$$

This is obvious from the property that

$$a_i + \frac{1}{2} \geq \frac{p_i}{\lambda} \geq a_i - \frac{1}{2} \text{ for all } i.$$

The special case $a_i = 0$ has to be ruled out in the right side inequality since $a_i - \dfrac{1}{2}$ becomes negative when $a_i = 0$.

Violation of upper quota

1. Violation of the upper quota by both Jefferson's and Webster's Methods

State i	$p_i = 100q_i$	$\lfloor q_i \rfloor$	$\lceil q_i \rceil$	Ham	Jeff	Web
1	8785	87	88	88	90	90
2	126	1	2	2	1	1
3	125	1	2	2	1	1
4	124	1	2	1	1	1
5	123	1	2	1	1	1
6	122	1	2	1	1	1
7	121	1	2	1	1	1
8	120	1	2	1	1	1
9	119	1	2	1	1	1
10	118	1	2	1	1	1
11	117	1	2	1	1	1
$\sum$	$10,000$	97	108	100	100	100

2. Violation of the lower quota by Webster's Method

State i	$p_i = 100q_i$	$\lfloor q_i \rfloor$	$\lceil q_i \rceil$	Ham	Jeff	Web
1	9215	92	93	92	95	90
2	159	1	2	2	1	2
3	158	1	2	2	1	2
4	157	1	2	2	1	2
5	156	1	2	1	1	2
6	155	1	2	1	1	2
$\sum$	$10,000$	97	103	100	100	100

The 100$^{\text{th}}$ seat is allocated to state 6 under Webster's apportionment since $102.23 = \dfrac{9215}{89.5} < \dfrac{155}{1.5} = 103.3$

Relatively well-rounded property

Webster's method can never produce an apportionment that rounds up for q_i for a state i with $q_i - \lfloor q_i \rfloor < 0.5$ while rounding down q_j for a state j with $q_j - \lfloor q_j \rfloor > 0.5$.

Integer programming formulation of Webster's Method

Recall that $\frac{a_i}{p_i}$ gives the per capital representation of state $i, i = 1, \cdots, S$; and the ideal per capital representation is h/P. Consider the sum of squared difference of $\frac{a_i}{p_i}$ to $\frac{h}{P}$ weighted by p_i

$$\overline{s} = \sum_{i=1}^{S} p_i \left(\frac{a_i}{p_i} - \frac{h}{P} \right)^2 = \sum_{i=1}^{S} \frac{a_i^2}{p_i} - \frac{h^2}{P}.$$

Webster's method: minimizes $\overline{s}$ subject to $\sum_{i=1}^{S} a_i = h$.

Proof

Suppose $\boldsymbol{a}$ is a Webster apportionment solution, then it satisfies the property:

$$\max_{a_i \geq 0} \frac{p_i}{a_i + \frac{1}{2}} \leq \lambda \leq \min_{a_i > 0} \frac{p_i}{a_i - \frac{1}{2}}.$$

It suffices to show that if an optimal choice has been made under the Webster scheme, then an interchange of a single seat between any 2 states r and s cannot reduce $\overline{s}$.

We prove by contradiction. Suppose such an interchange is possible in reducing $\overline{s}$, where $a_r > 0$ and $a_s \geq 0$, then this implies that (all other allocations are kept the same)

$$\frac{(a_r - 1)^2}{p_r} + \frac{(a_s + 1)^2}{p_s} < \frac{a_r^2}{p_r} + \frac{a_s^2}{p_s}$$

$$\Leftrightarrow \quad \frac{p_r}{a_r - \frac{1}{2}} < \frac{p_s}{a_s + \frac{1}{2}}.$$

This is an obvious violation to the above property. Therefore, the Webster apportionment solution $\boldsymbol{a}$ minimize $\overline{s}$ subject to $\sum_{i=1}^{S} a_i = h$.

Generalized formulation of the divisor method

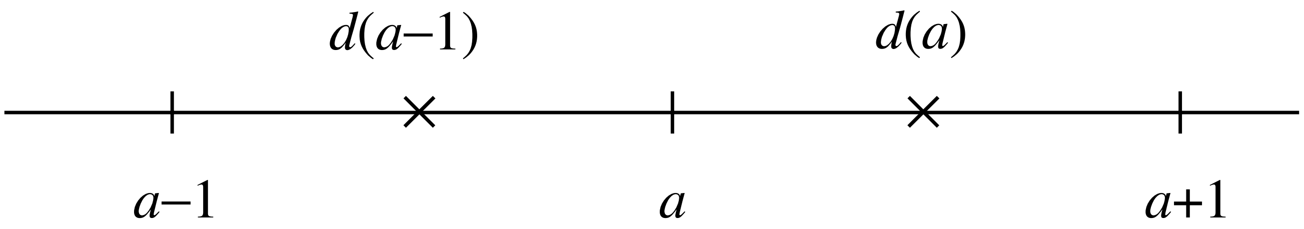
Any rounding procedure can be described by specifying a dividing point $d(a)$ in each interval $[a, a+1]$ for each non-negative integer a.

Any monotone increasing $d(a)$ defined for all integers $a \geq 0$ and satisfying

$$a \leq d(a) \leq a+1$$

is called a divisor criterion.

For any positive real number z, a d-rounding of z (denoted by $[z]_d$) is an integer a such that $d(a-1) \leq z \leq d(a)$. This is unique unless $z = d(a)$, in which case it takes on either a or $a+1$.



- For example, Webster's $d(a) = a + \frac{1}{2}$. Suppose z lies in $(2.5, 3.5)$, it is rounded to 3. When $z = 3.5$, it can be either rounded to 3 or 4.

- Also, Jefferson's $d(a) = a + 1$ (Greatest Divisor Method) while Adams' $d(a) = a$ (Smallest Divisor Method). For Jefferson's method, if $a < z < a + 1$, then $[z]_d = a$. When $z = a + 1$, then $[z]_d$ can be either a or $a + 1$. For example, when $z = 3.8$, then $d(2) \leq z \leq d(3) = 4$, so $[3.8]_d = 3$; when $z = 4 = 3 + 1$, then $a = 3$ and $[4]_d = 3$ or 4.

The divisor method based on d is

$$M(\boldsymbol{p}, h) = \left\{ \boldsymbol{a} : a_i = [p_i/\lambda]_d \quad \text{and} \quad \sum_{i=1}^{S} a_i = h \text{ for some positive } \lambda \right\}.$$

In terms of the min-max inequality:

$$M(\boldsymbol{p}, h) = \left\{ \boldsymbol{a} : \min_{a_i > 0} \frac{p_i}{d(a_i - 1)} \geq \max_{a_j \geq 0} \frac{p_j}{d(a_j)}, \quad \sum_{i=1}^{S} a_i = h \right\}.$$

This is a consequence of $d(a_i - 1) \leq \dfrac{p_i}{\lambda} \leq d(a_i)$. We exclude $a_i = 0$ in the left inequality since $d(a_i - 1)$ is in general negative when $a_i = 0$.

The divisor method M based on d may be defined recursively as:

(i) $M(\boldsymbol{p}, 0) = \mathbf{0}$,

(ii) if $\boldsymbol{a} \in M(\boldsymbol{p}, h)$ and k satisfies

$$\frac{p_k}{d(a_k)} = \max_i \frac{p_i}{d(a_i)},$$

then $\boldsymbol{b} \in M(\boldsymbol{p}, h+1)$, with $b_k = a_k + 1$ and $b_i = a_i$ for $i \neq k$.

Dean's method (Harmonic Mean Method)

The i^{th} state receives a_i seats where p_i/a_i is as close as possible to the common divisor λ when compared to $\dfrac{p_i}{a_i+1}$ and $\dfrac{p_i}{a_i-1}$. For all i, we have

$$\frac{p_i}{a_i} - \lambda \leq \lambda - \frac{p_i}{a_i+1} \quad \text{and} \quad \lambda - \frac{p_i}{a_i} \leq \frac{p_i}{a_i-1} - \lambda$$

which simplifies to

$$\frac{a_i + \frac{1}{2}}{a_i(a_i+1)} p_i \leq \lambda \leq \frac{a_i - \frac{1}{2}}{a_i(a_i-1)} p_i \text{ for all } i.$$

Define $d(a) = \dfrac{a(a+1)}{a+\frac{1}{2}} = \dfrac{1}{\frac{1}{2}\left(\frac{1}{a} + \frac{1}{a+1}\right)}$ (harmonic mean of consecutive integers a and $a+1$), then

$$\max_i \frac{p_i}{d(a_i)} \leq \lambda \leq \min_j \frac{p_j}{d(a_j-1)}.$$

Hill's method (Equal Proportions Methods)

- Besides the Harmonic Mean, where $\frac{1}{d(a)} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{a+1}\right)$ (Dean's method) and the Arithmetic Mean $d(a) = \frac{1}{2}(a+a+1)$ (Webster's method), the choice of the Geometric Mean $d(a) = \sqrt{a(a+1)}$ leads to the Equal Proportions method (also called Hill's method).

- For a population p_i and common divisor λ, suppose p_i/λ fall-s within $[a, a+1]$, then p_i/λ is rounded up to $a+1$ seats if $p_i/\lambda > d(a) = \sqrt{a(a+1)}$ and rounded down to a seats if $p_i/\lambda < d(a) = \sqrt{a(a+1)}$. If $p_i/\lambda = \sqrt{a(a+1)}$, the rounding is not unambiguously defined.

State	p_i	q_i	GR	SD	HM	EP	MF	GD
			a_i for Method					
1	91,490	91.490	92	88	89	90	93	94
2	1,660	1.660	2	2	2	2	2	1
3	1,460	1.460	2	2	2	2	1	1
4	1,450	1.450	1	2	2	2	1	1
5	1,440	1.440	1	2	2	2	1	1
6	1,400	1.400	1	2	2	1	1	1
7	1,100	1.100	1	2	1	1	1	1
Totals	100,000	100	100	100	100	100	100	100
Min λ				1,040	1,023	1,011	979	964
Max λ				1,051	1,033	1,018	989	973

Allocations for the six divisor methods with $S = 100$. The minimum and maximum integer values of λ which yield these allocations are also shown.

Geometric characterization of the divisor methods ($S = 3$)

Hexagonal regions on the plane: $q_1 + q_2 + q_3 = h$ *with* $\boldsymbol{a} = (r \quad s \quad t)$

Here, $S = 3$. We find the hexagonal region consisting of the quota vectors $(q_1 \quad q_2 \quad q_3)$ such that they give the same apportionment solution $\boldsymbol{a} = (r \quad s \quad t)$.

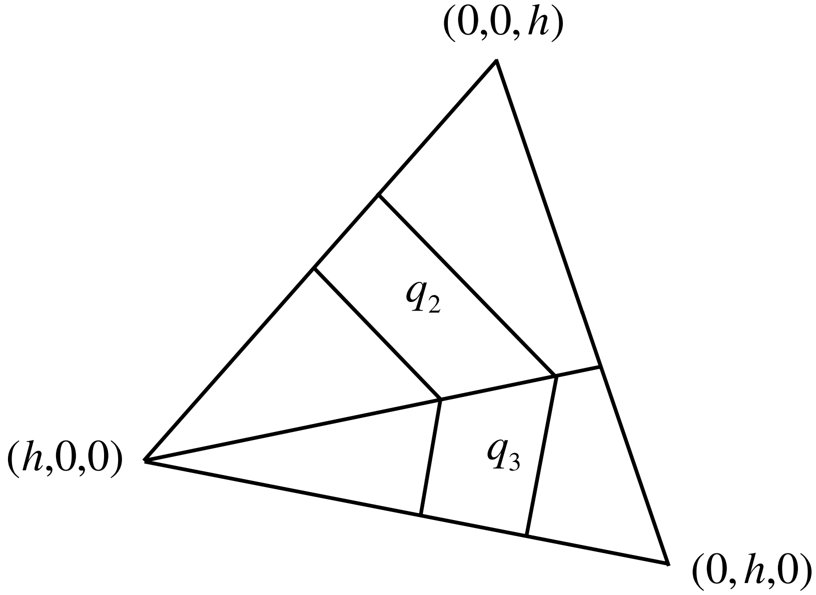
According to the divisor method, the apportionment vector $\boldsymbol{a} = (r \quad s \quad t)$ is resulted when the population vector $(p_1 \quad p_2 \quad p_3)$ satisfies

$$d(r-1) < \frac{p_1}{\lambda} < d(r), \quad d(s-1) < \frac{p_2}{\lambda} < d(s), \quad d(t-1) < \frac{p_3}{\lambda} < d(t),$$

where λ is the common divisor. We then deduce that

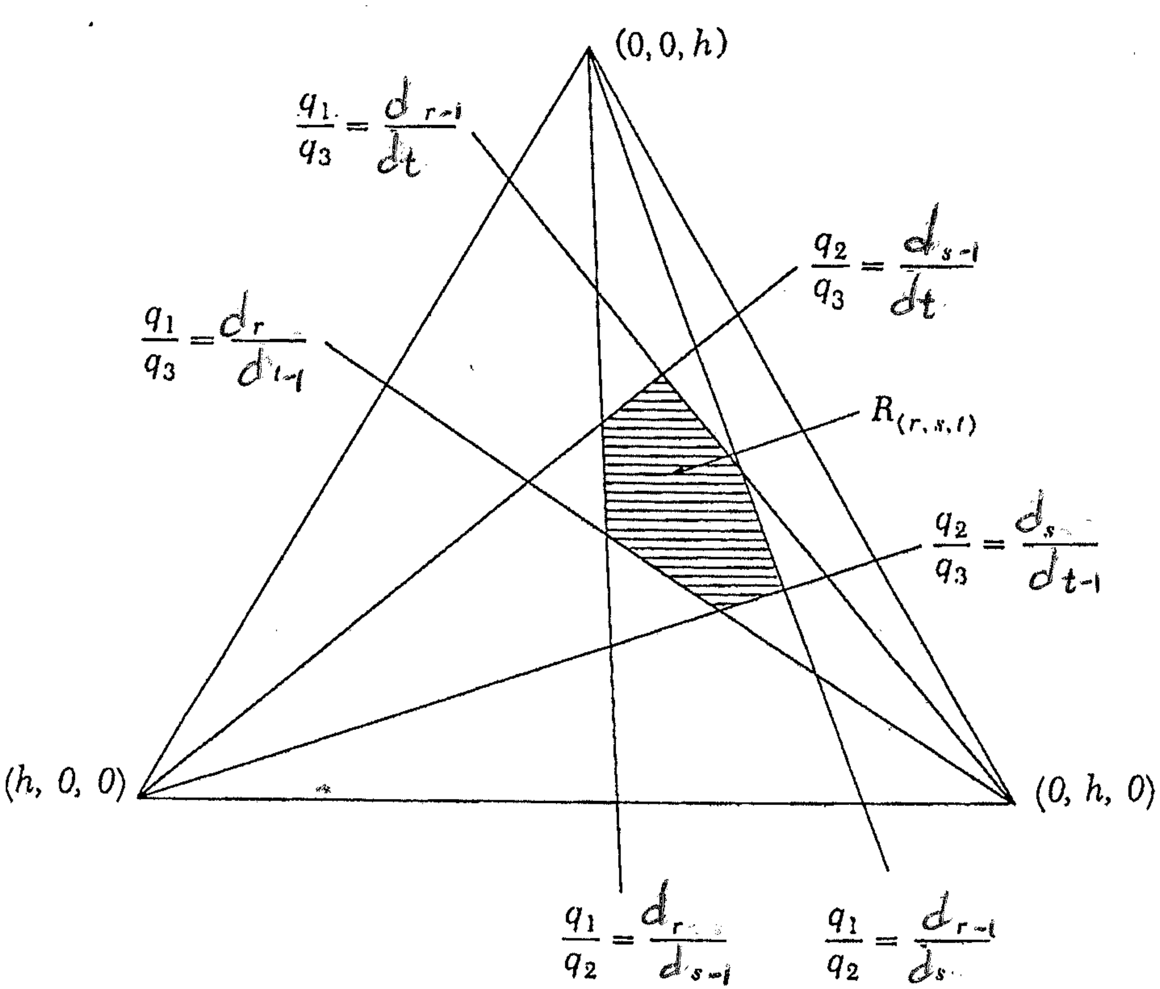
$$\frac{d(s-1)}{d(t)} < \frac{p_2}{p_3} < \frac{d(s)}{d(t-1)}, \quad \frac{d(r-1)}{d(s)} < \frac{p_1}{p_2} < \frac{d(r)}{d(s-1)}, \quad \frac{d(r-1)}{d(t)} < \frac{p_1}{p_3} < \frac{d(r)}{d(t-1)}.$$

Geometrically, a line on the plane: $q_1 + q_2 + q_3 = h$ through the point $(h, 0, 0)$ corresponds to $\frac{q_2}{q_3} =$ constant.

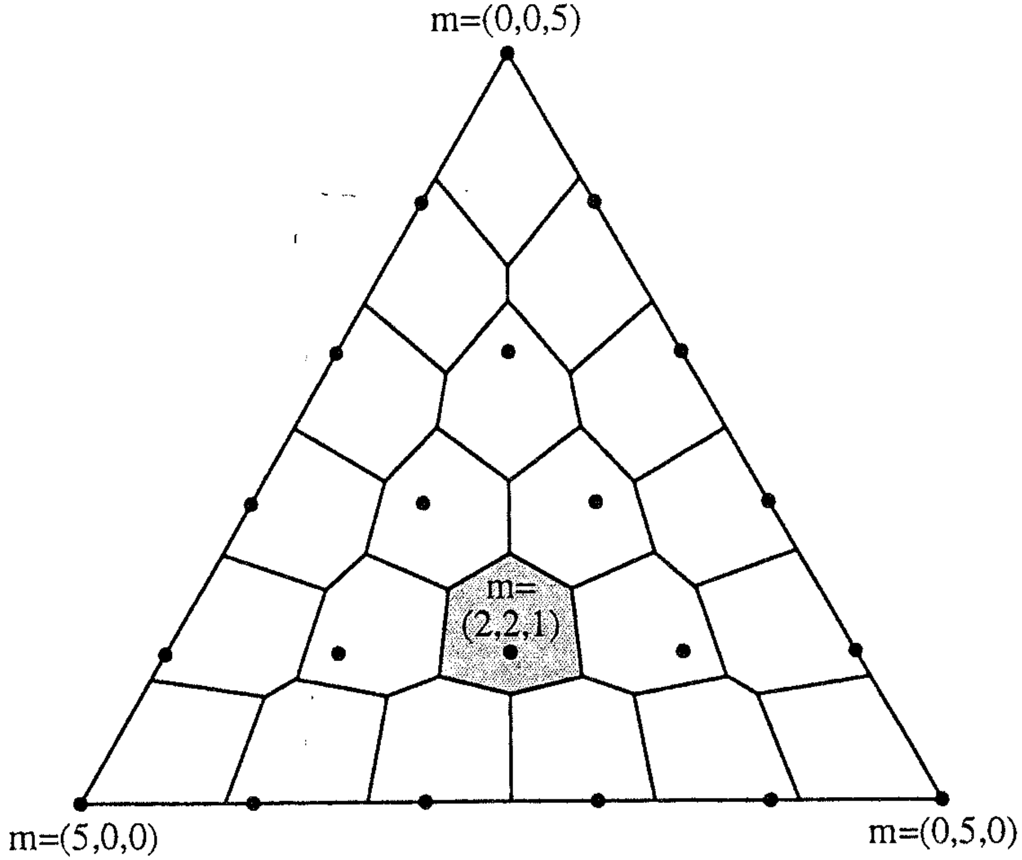


The bounding edges of the hexagon consisting quota vectors that give the apportionment vector $\boldsymbol{a} = (r \quad s \quad t)$ are given by

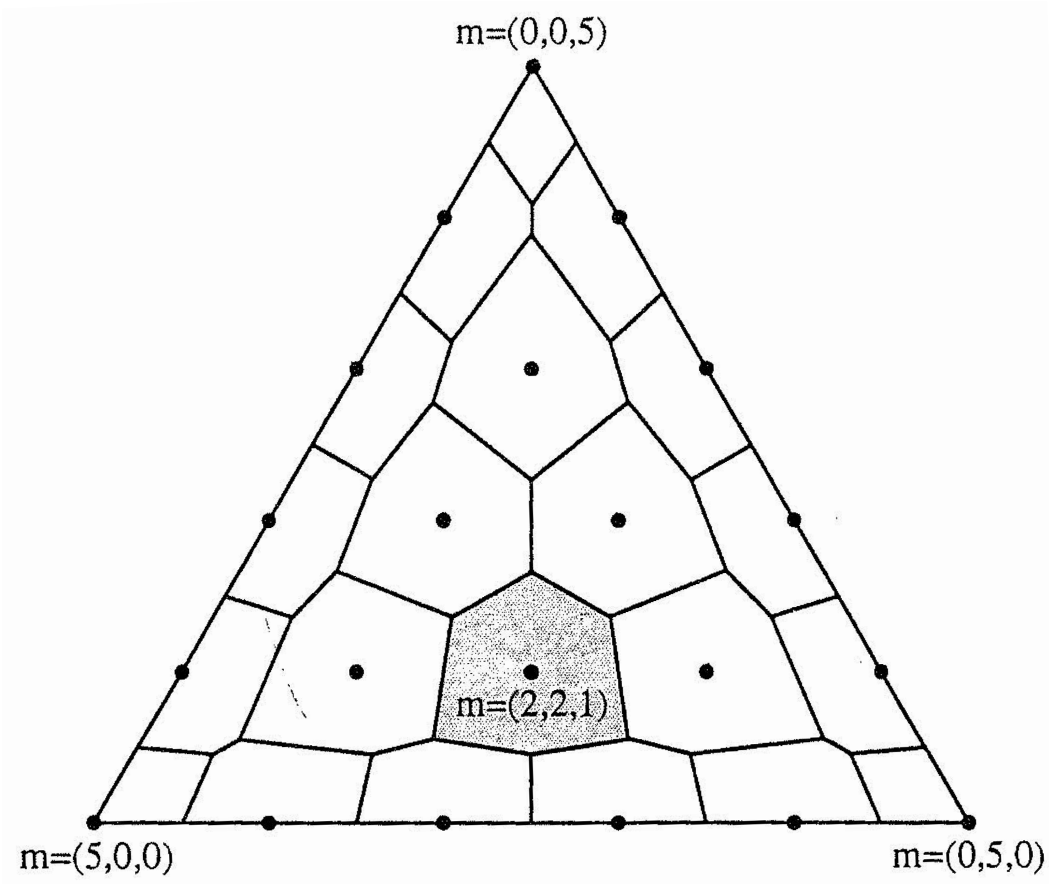
$$\frac{p_2}{p_3} = \frac{d(s-1)}{d(t)}, \frac{p_2}{p_3} = \frac{d(s)}{d(t-1)},$$
$$\frac{p_1}{p_2} = \frac{d(r)}{d(s-1)}, \frac{p_1}{p_2} = \frac{d(r-1)}{d(s)}, \frac{p_1}{p_3} = \frac{d(r)}{d(t-1)}, \frac{p_1}{p_3} = \frac{d(r-1)}{d(t)}.$$



A typical divisor method apportionment region and its boundaries for $S = 3$. Here, d_r denotes the rounding point for the apportionment r.



Apportionment simplex that shows Jefferson's apportionment of $S = 3$ and $h = 5$. The cells adjacent to the edges have larger sizes indicate that Jefferson's apportionment favors larger states.



Apportionment simplex that shows Webster's apportionment of $S = 3$ and $h = 5$. The interior cells tend to have larger sizes when compared with those of Jefferson's apportionment.

Minimum and maximum apportionment requirements

In order that every state receives at least one representative, it is necessary to have $d(0) = 0$ (assuming $p_i/0 > p_j/0$ for $p_i > p_j$). While the Adams, Hill and Dean methods all satisfy this perperty, we need to modify the Webster $\left[d(a) = a + \frac{1}{2}\right]$ and Jefferson Method $[d(a) = a + 1]$ by setting $d(0) = 0$ in the special case $a = 0$.

A divisor method M based on d for problems with both minimum requirement $\boldsymbol{a}^{min}$ and maximum requirement $\boldsymbol{r}^{max}, \boldsymbol{r}^{min} \leq \boldsymbol{r}^{max}$, can be formulated as

$$\begin{aligned} M(\boldsymbol{p}, h) \;\; = \;\; & \Big\{ \boldsymbol{a} : a_i = mid\left(r_i^{\text{min}}, r_i^{max}, [p_i/\lambda]_d\right) \\ & \qquad \text{and } \sum_{i=1}^{S} a_i = h \text{ for some positive } \lambda \Big\}. \end{aligned}$$

Here, $mid(u, v, w)$ is the middle in value of the three numbers u, v and w.

Consistency (uniformity)

Let $\boldsymbol{a} = (\boldsymbol{a}^{S_1}, \boldsymbol{a}^{S_2}) = M(\boldsymbol{p}, h)$, where S_1 and S_2 are two subsets of S that partition S. An apportionment method is said to be uniform if $(\boldsymbol{a}^{S_1}, \boldsymbol{a}^{S_2}) = M(\boldsymbol{p}, h)$ would imply $\boldsymbol{a}^{S_1} = M(\boldsymbol{p}^{S_1}, \Sigma_{S_1} a_i)$. On the other hand, suppose $\tilde{\boldsymbol{a}}^{S_1} = M(\boldsymbol{p}^{S_1}, \Sigma_{S_1} a_i)$, then $(\tilde{\boldsymbol{a}}^{S_1}, \boldsymbol{a}^{S_2}) = M(\boldsymbol{p}, h)$.

This would mean

(i) If a method apportions $\boldsymbol{a}^{S_1}$ to the states in S_1 in the entire problem, then the same method applied to apportioning $h_{S_1} = \Sigma_{S_1} a_i$ seats among the states in S_1 with the same data in the subproblem will admit the same result.

(ii) If the method applied to this subproblem admits another solution, then the method applied to the entire problem also admits the corresponding alternative solution.

- Uniformity implies of a method that if one knows how any pair of states share any number of seats then the method is completely specified.

Example

Consider the Hamilton apportionment of 100 seats based on the following population data among 5 states.

State	Population	Quota	Number of seats
1	7368	29.578	30
2	1123	4.508	4
3	7532	30.236	30
4	3456	13.873	14
5	5431	21.802	22
total	24910	100	100

Consider the subproblem of assigning 64 seats among the first 3 states.

State	Population	Quota	Number of seats
1	7368	29.429	29
2	1123	4.485	5
3	7532	30.085	30
total	16023	64	64

Surprisingly, restricting the apportionment problem to a subset of all states does not yield the same seat assignment for the states involved in the subproblem: state 1 loses one seat to state 2.

- The New State Paradox occurs since the apportionment solution changes with the addition of 2 new states: state 4 and state 5.

- A consistent apportionment scheme would not admit the "New States" Paradox.

Balinski-Young Impossibility Theorem

- Divisor methods automatically satisfy the House Monotone Property.

- An apportionment method is uniform and population monotone if and only if it is a divisor method.

  The proof is highly technical.

- Divisor methods are known to produce violation of the quota property.

Conclusion It is impossible for an apportionment method that always satisfies quota and be incapable of producing paradoxes.

1.5 Huntington's family: Pairwise comparison of inequity

- Consider the ratio $p_i/a_i =$ average number of constituents per seat (district size) in state i, the ideal case would be when all p_i/a_i were the same for all states. Between any 2 states, there will always be certain inequity which gives one of the states a slight advantage over the other. For a population $\boldsymbol{p} = (p_1, p_2, \cdots, p_S)$ and an apportionment $(a_1, a_2, \cdots, a_S)$ for House size h, if $p_i/a_i > p_j/a_j$, then state j is "better off" than state i in terms of district size.

- How is the "amount of inequity" between 2 states measured? Some possible choices of measure of inequity are:

(i) $\left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right|$, (ii) $\left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right| \Big/ \min\left(\frac{p_i}{a_i}, \frac{p_j}{a_j}\right)$, (iii) $\left|\frac{a_i}{p_i} - \frac{a_j}{p_j}\right|$,

(iv) $\left|a_i - a_j\frac{p_i}{p_j}\right|$, (v) $\left|a_i\frac{p_j}{p_i} - a_j\right|$.

Huntington's rule

A transfer is made from the more favored state to the less favored state if this reduces this measure of inequity.

- An apportionment is stable in the sense that no inequity, computed according to the chosen measure, can be reduced by transferring one seat from a better off state to a less well off state.

Huntington considered 64 cases involving the relative and absolute differences and ratios involving the 4 parameters p_i, a_i, p_j, a_j for a pair of states i and j. He arrived at 5 different apportionment methods.

- Some schemes are "unworkable" in the sense that the pairwise comparison approach would not in general converge to an overall minimum – successive pairwise improvements could lead to cycling.

Hill's method (Method of Equal Proportions) revisited

Hill's method has been used to apportion the House since 1942.

Let $T_{ij}\left(\frac{p_i}{a_i}, \frac{p_j}{a_j}\right)$ be the relative difference between $\frac{p_i}{a_i}$ and $\frac{p_j}{a_j}$, defined by

$$T_{ij}\left(\frac{p_i}{a_i}, \frac{p_j}{a_j}\right) = \left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right| \Big/ \min\left(\frac{p_i}{a_i}, \frac{p_j}{a_j}\right).$$

The ideal situation is $T = 0$ for all pairs of i and j.

Lemma on Hill's method

Between two states i and j, we consider (i) $a_i + 1$ and a_j to be a better assignment than (ii) a_i and $a_j + 1$

if and only if $\dfrac{p_i}{\sqrt{a_i(a_i+1)}} > \dfrac{p_j}{\sqrt{a_j(a_j+1)}}$.

Remark

With an additional seat, should it be assigned to State i with a_i seats or State j with a_j seats? The decision factor is to compare

$$\frac{p_i}{\sqrt{a_i(a_i+1)}} \quad \text{and} \quad \frac{p_j}{\sqrt{a_j(a_j+1)}}.$$

The one with a higher rank index value $r(p,a) = \dfrac{p}{\sqrt{a(a+1)}}$ should receive the additional seat.

Proof

Suppose that when State i has $a_i + 1$ seats and State j has a_j seats, State i is the more favored state i.e.

$$\frac{p_j}{a_j} - \frac{p_i}{a_i + 1} > 0;$$

while when State i has a_i seats and State j has $a_j + 1$ seats, State j is the more favored state i.e.

$$\frac{p_i}{a_i} - \frac{p_j}{a_j + 1} > 0.$$

Should we transfer one seat in assignment (ii) from State j to State i so that assignment (i) is resulted?

Based on the Huntington rule and the given choice of inequity measure for the Hill methods, Assignment (i) is a better assignment than (ii) if and only if

$$T_{ij}\left(\frac{p_i}{a_i+1},\frac{p_j}{a_j}\right) < T_{ij}\left(\frac{p_i}{a_i},\frac{p_j}{a_j+1}\right)$$

$$\begin{aligned}
\Leftrightarrow\quad & \frac{p_j/a_j - p_i/(a_i+1)}{p_i/(a_i+1)} < \frac{p_i/a_i - p_j/(a_j+1)}{p_j/(a_j+1)} \\
\Leftrightarrow\quad & \frac{p_j(a_i+1) - p_i a_i}{p_i a_j} < \frac{p_i(a_j+1) - p_j a_i}{p_j a_i} \\
\Leftrightarrow\quad & \frac{p_j^2}{a_j(a_j+1)} < \frac{p_i^2}{a_i(a_i+1)}.
\end{aligned}$$

That is, the measure of inequity as quantified by T_{ij} of the Hill method is reduced.

Algorithm for Hill's method

Compute the quantities $\dfrac{p_i}{\sqrt{n(n+1)}}$ for all i starting with $n = 0$ and then assign the seats in turn to the largest such numbers.

Floodland	Galeland	Hailland	Snowland	Rainland
$\frac{9061}{\sqrt{1{\cdot}2}}$	$\frac{7179}{\sqrt{1{\cdot}2}}$	$\frac{5259}{\sqrt{1{\cdot}2}}$	$\frac{3319}{\sqrt{1{\cdot}2}}$	$\frac{1182}{\sqrt{1{\cdot}2}}$
$\frac{9061}{\sqrt{2{\cdot}3}}$	$\frac{7179}{\sqrt{2{\cdot}3}}$	$\frac{5259}{\sqrt{2{\cdot}3}}$	$\frac{3319}{\sqrt{2{\cdot}3}}$	$\frac{1182}{\sqrt{2{\cdot}3}}$
$\frac{9061}{\sqrt{3{\cdot}4}}$	$\frac{7179}{\sqrt{3{\cdot}4}}$	...	...	...
...	...	...	...	...

Five seats have already been allocated (one to each state)

Comparing (i) Floodland with 4 seats and Snowland with 1 seat, against (ii) Floodland with 3 seats and Snowland with 2 seats, since $9061/\sqrt{3\cdot 4} = 2616 > 3319/\sqrt{1\cdot 2} = 2347$, so assignment (i) is better than assignment (ii).

Floodland	Galeland	Hailland	Snowland	Rainland
6407 - 6	5076 - 7	3719 - 8	2347 - 12	836
3699 - 9	2931 - 10	2147 - 13	1355 - 20	483
2616 - 11	2072 - 14	1518 - 18	958 - 27	...
2026 - 15	1605 - 17	1176 - 23	742	...
1658 - 16	1311 - 21	960 - 26	...	...
1401 - 19	1108 - 24	811	...	...
1211 - 22	959	...	...	...
1070 - 25	846	...	...	...

Remarks on the rank index

- Since the ranking function $\dfrac{1}{\sqrt{n(n+1)}}$ equal ∞ for $n = 0$, this method automatically gives each state at least one seat if $h \geq S$, so the minimum requirement of at least one seat for each state is always satisfied.

- If a tie occurs between states with unequal populations (extremely unlikely), Huntington suggests that it be broken in favor of the larger state.

- It does not satisfy the quota property. Actually, it can violate both lower and upper quota.

- The Hungtinton approach to the apportionment makes use of "local" measures of inequity.

Violation of quota property

- Hill's method does not satisfy both the upper and lower quota property.

State	Population	Exact Quota	Allocation
A	9215	92.15	90
B	159	1.59	2
C	158	1.58	2
D	157	1.57	2
E	156	1.56	2
F	155	1.55	2
Totals	$10,000$	100	100

House monotone property

- By its construction, Hill's method is *house monotone*.

	Council Size		
	26	27	28
Floodland	10	10	11
Galeland	7	7	7
Hailland	5	5	5
Snowland	3	4	4
Rainland	1	1	1

Pairwise comparison using $\left|\frac{a_i}{p_i} - \frac{a_j}{p_j}\right|$*, Webster's method revisited*

Give to each state a number of seats so that no transfer of any seat can reduce the difference in per capita representation between those states. That is, supposing that State i is favored over State j, $\frac{p_j}{a_j} > \frac{p_i}{a_i}$, no transfer of seats will be made if

$$\frac{a_i}{p_i} - \frac{a_j}{p_j} \leq \frac{a_j + 1}{p_j} - \frac{a_i - 1}{p_i}$$

for all i and j. This simplifies to

$$\begin{aligned} a_i p_j - p_i a_j &\leq p_i(a_j + 1) - p_j(a_i - 1) \\ \frac{p_j}{a_j + \frac{1}{2}} &\leq \frac{p_i}{a_i - \frac{1}{2}}. \end{aligned}$$

We can deduce

$$\max_{all\ a_j} \frac{p_j}{a_j + \frac{1}{2}} \leq \min_{a_i > 0} \frac{p_i}{a_i - \frac{1}{2}} \text{ (same result as for Webster's Method).}$$

Five traditional divisor methods

Method	*Alternative name*	Divisor $d(a)$	Pairwise comparison $\left(\frac{a_i}{p_i} > \frac{a_j}{p_j}\right)$	Adoption by US Congress
Adams	Smallest divisors	a	$a_i - a_j \frac{p_i}{p_j}$	–
Dean	Harmonic means	$\frac{a(a+1)}{a+\frac{1}{2}}$	$\frac{p_j}{a_j} - \frac{p_i}{a_i}$	–
Hill	Equal propor- tions	$\sqrt{a(a+1)}$	$\frac{a_i/p_i}{a_j/p_j} - 1$	1942 to now
Webster	Major Fractions	$a + \frac{1}{2}$	$\frac{a_i}{p_i} - \frac{a_j}{p_j}$	1842; 1912; 1932*
Jefferson	Largest divisors	$a + 1$	$a_i \frac{p_j}{p_i} - a_j$	1794 to 1832

- 1922 – US Congress failed to reapportion the House at all after 1920 census.

  1932 – allocations based on Hill and Webster are identical.

- A National Academy of Sciences Committee issued a report in 1929. The report considered the 5 divisor methods and focused on the pairwise comparison tests. The Committee adopted Huntington's reasoning that the Equal Proportions Method is preferred (the Method occupies mathematically a neutral position with respect to emphasis on larger and smaller states.)

Key result

The divisor method based on $d(a)$ is the Hungtington method based on $r(p,a) = p/d(a)$.

$p_i/a_i =$ average district size;
$a_i/p_i =$ per capita share of a representative

- Dean's Method – absolute difference in average district sizes: $\left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right|$
- Webster's Method – absolute difference in per capita shares of a representative: $\left|\frac{a_i}{p_i} - \frac{a_j}{p_j}\right|$
- Hill's Method – relative differences in both district sizes and shares of a representative: $\left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right| \Big/ \min\left(\frac{p_i}{a_i}, \frac{p_j}{a_j}\right)$
- Adams' Method – absolute representative surplus: $a_i - \frac{p_i}{p_j}a_j$ is the amount by which the allocation for state i exceeds the number of seats it would have if its allocation was directly proportional to the actual allocation for state j
- Jefferson's Method – absolute representation deficiency: $\frac{p_j}{p_i}a_i - a_j$

Let $r(p,a)$ be any real valued function of two real variables called a *rank-index*, satisfying $r(p,a) > r(p,a+1) \geq 0$, and $r(p,a)$ can be plus infinity. Given a rank-index, a Huntington Method M of apportionment is the set of solutions obtained recursively as follows:

(i) $f_i(\boldsymbol{p},0) = 0, \quad 1 \leq i \leq S$;

(ii) If $a_i = f_i(\boldsymbol{p},h)$ is an apportionment for h of M, and k is some state for which

$$r(p_k,a_k) \geq r(p_i,a_i) \quad \text{for} \quad 1 \leq i \leq S,$$

then

$$f_k(\boldsymbol{p},h+1) = a_k + 1 \quad \text{and} \quad f_i(\boldsymbol{p},h+1) = a_i \quad \text{for} \quad i \neq k.$$

The Huntington method based on $r(p,a)$ is

$$M(\boldsymbol{p},h) \;=\; \left\{ \boldsymbol{a} \geq 0 : \sum_{i=1}^{S} a_i = h, \max_i r(p_i,a_i) \leq \min_{a_j>0} r(p_j,a_j-1) \right\}.$$

Debate between Webster's and Hill's methods

- In 1922 apportionment, the two methods produced significantly different outcomes. By this time, the number of seats in the House had been fixed by law. Consequently, the 1912 seat totals were held over without any reapportionment whatsover.

- In 1932 apportionment, Webster's and Hill's methods produced identical apportionment.

- For the 1942 apportionment, Webster's and Hill's method came very close except that Hill's method gave an extra seat to Arkansas at the expense of Michigan. Democrats favored Hill's since Arkansas tended to vote for Democrats. Since the Democrat-s had the majority, it was Hill's method that passed through Congress. President Franklin Roosevelt (Democrat) signed the method into "permanent" law and it has been used ever since.

Court challenges

- In 1991, for the first time in US history, the constitutionality of an apportionment method was challenged in court, by Montana and Massachusetts in separate cases.

  - Montana proposed two methods as alternatives to EP (current method). Both HM and SD give Montana 2 seats instead of the single seat allocated by EP, but would not have increased Massachusetts' EP allocation of 10 seats. [Favoring small states.]

  - Massachusetts proposed MF, which would have allocated 11 seats to Massachusetts, and 1 to Montana. [Favoring medium states.]

"Apportionment Methods for the House of Representatives and the Court Challenges", by Lawrence R. Ernst, *Management Science*, vol. 40(10), p.1207-1227 (1994). Ernst is the author of the declarations on the mathematical and statistical issues used by the defense in these cases.

Supreme court case No. 91–860

US Department of Commerce versus Montana

1990 census	**Montana**	**Washington**
population	$803,655$	$4,887,941$
quota	1.40 seats	8.53 seats
Based on Hill's method	one seat	nine seats
district size	$803,655$	4,887,941/9=543,104.55

$$\begin{aligned} \text{absolute difference} &= 260,550.44 = 803,655 - 543,104.55 \\ \text{relative difference} &= 0.480 = \frac{260,550.44}{543,104.55}. \end{aligned}$$

How about the transfer of one seat from Washington to Montana?

New district size $401,827.5$ $610,992.625$

new absolute difference $= 209,165.125 = 610,992.625 - 401,827.5$

new relative difference $= 0.521 = \dfrac{209,165.125}{401,827.5}$.

A transfer of one seat from Washington to Montana results in a decrease of the absolute difference of the district sizes. According to Dean's method, this transfer should then happen.

The same transfer leads to an increase in the relative difference of the district sizes, and hence violates the stipulation of Hill's method.

The Supreme Court rejected the argument that Hill's method violates the Constitution and Montana did not gain a second seat. However, it ruled that apportionment methods are justiciable（可供裁判）, opening the door to future cases.

Theorem – Quota properties of Huntington family of methods

There exists no Huntington method satisfying quota. Of these five "known workable" method, only the Smallest Divisors Method satisfies upper quota and only the Jefferson Method satisfies lower quota.

			Apportionment for 36				
Party	Votes received	Exact quota	**SD**	**HM**	**EP**	**W**	**J**
A	$27,744$	9.988	10	10	10	10	11
B	$25,178$	9.064	9	9	9	9	9
C	$19,947$	7.181	7	7	7	8	7
D	$14,614$	5.261	5	5	6	5	5
E	$9,225$	3.321	3	4	3	3	3
F	$3,292$	1.185	2	1	1	1	1
	$100,000$	$36,000$	36	36	36	36	36

Quota Method

Uses the same rule as in the Jefferson Method to determine which state receives the next seat, but rules this state ineligible if it will violate the upper quota.

Definition of eligibility

If f is an apportionment solution and $f_i(\boldsymbol{p}, h) = a_i$ and $q_i(\boldsymbol{p}, h)$ denotes the quota of the i^{th} state, then state i is eligible at $h+1$ for its $(a_i + 1)^{\text{st}}$ seat if $a_i < q_i(\boldsymbol{p}, h+1) = (h+1)p_i/P$. Write

$$E(\boldsymbol{a}, h+1) = \{i \in N_s : i \text{ is eligible for } a_i + 1 \text{ at } h+1\}.$$

Algorithm

The quota method consists of all apportionment solutions $f(\boldsymbol{p}, h)$ such that

$$f(\boldsymbol{p}, 0) = 0 \quad \text{for all } i$$

and if $k \in E(\boldsymbol{a}, h+1)$ and

$$\frac{p_k}{a_k + 1} \geq \frac{p_j}{a_j + 1} \text{ for all } j \in E(\boldsymbol{a}, h+1),$$

then

$$\begin{aligned} f_k(\boldsymbol{p}, h+1) &= a_k + 1 \quad \text{for one such } k \text{ and} \\ f_i(\boldsymbol{p}, h+1) &= a_i \quad \text{for all } i \neq k. \end{aligned}$$

- Allocate seats to political parties proportionally to their respective votes.

Party	Votes received	*Exact* proportionality	Possible allocations					
			SD	GR, HM	EP	MF	Q	GD
A	27,744	9.988	10	10	10	10	10	11
B	25,179	9.064	9	9	9	9	10	9
C	19,947	7.181	7	7	7	8	7	7
D	14,614	5.261	5	5	6	5	5	5
E	9,225	3.321	3	4	3	3	3	3
F	3,292	1.185	2	1	1	1	1	1
	100,000	36.000	36	36	36	36	36	36

$\longrightarrow$ favoring larger parties

SD: Smallest Divisor, Adams; GR: Greatest Remainder, Hamilton;

HM: Harmonic Means, Dean; EP: Equal Proportions, Huntington-Hill;

MF: Major Fractions, Webster; GD: Greatest Divisor, Jefferson;

Q: Quota (Balinski-Young)

1.6 Analysis of bias and notion of marginal equity measure

- An apportionment that gives a_1 and a_2 seats to states having populations $p_1 > p_2 > 0$ favors the larger state over the smaller state if $a_1/p_1 > a_2/p_2$ and favors the smaller state over the larger state if $a_1/p_1 < a_2/p_2$.

- Over many pairs $(p_1, p_2), p_1 > p_2$, whether a method tends more often to favor the larger state over the smaller or vice versa.

- There are many ways to measure "bias" and there are different probabilistic models by which a tendency toward bias can be revealed theoretically.

- A casual inspection shows the order: Adams, Dean, Hill, Webster, Jefferson that the apportionment methods tend increasingly to favor the larger states.

Apportionment of 6 states and 36 seats

Votes	*Adams*	*Dean*	*Hill*	*Webster*	*Jefferson*
27,744	10	10	10	10	11
25,178	9	9	9	9	9
19,951	7	7	7	8	7
14,610	5	5	6	5	5
9,225	3	4	3	3	3
3,292	2	1	1	1	1
100,000	36	36	36	36	36

- The apportionment in any column leads to the apportionment in the next column by the transfer of one seat from a smaller state to a larger state.

Majorization ordering

Reference "A majorization comparison of apportionment methods in proportional representation," A Marshall, I. Olkin, and F. Fukelsheim, *Social Choice Welfare* (2002) vol. 19, p.885-900.

Majorization provides an ordering between two vectors

$$\boldsymbol{m} = (m_1 \cdots m_\ell) \quad \text{and} \quad \boldsymbol{m}' = (m'_1 \cdots m'_\ell)$$

with ordered elements

$$m_1 \geq \cdots \geq m_\ell \quad \text{and} \quad m'_1 \geq \cdots \geq m'_\ell,$$

and with an identical component sum

$$m_1 + m_2 + \cdots + m_\ell = m'_1 + m'_2 + \cdots + m'_\ell = M.$$

The ordering states that all partial sums of the k largest components in $\boldsymbol{m}$ are dominated by the sum of the k largest components in $\boldsymbol{m}'$.

$$\begin{array}{rcl} m_1 & \leq & m'_1 \\ m_1 + m_2 & \leq & m'_1 + m'_2 \\ & \vdots & \\ m_1 + \cdots + m_k & \leq & m'_1 + \cdots + m'_k \\ & \vdots & \\ m_1 + \cdots + m_\ell & = & m'_1 + \cdots + m'_\ell \end{array}$$

$\boldsymbol{m} \prec \boldsymbol{m}', \boldsymbol{m}$ is majorized by $\boldsymbol{m}'$ or $\boldsymbol{m}'$ majorizes $\boldsymbol{m}$.

Suppose it never occurs that $m_i > m'_i$ and $m_j < m'_j$, for all $i < j$, (larger state has more seats while smaller state has less seats for apportionment $\boldsymbol{m}$), then apportionment $\boldsymbol{m}$ is majorized by $\boldsymbol{m}'$.

Divisor methods and signpost sequences

A divisor method of apportionment is defined through the number $s(k)$ in the interval $[k, k+1]$, called "signpost" or "dividing point" that splits the interval $[k, k+1]$. A number that falls within $[k, s(k)]$ is rounded down to k and it is rounded up to $k+1$ if it falls within $(s(k), k+1)$. If the number happens to hit $s(k)$, then there is an option to round down to k or to round up to $k+1$.

Power-mean signposts

$$s(k,p) = \left[\frac{k^p}{2} + \frac{(k+1)^p}{2}\right]^{1/p}, \quad -\infty \le p \le \infty.$$

$p = -\infty, s(k, -\infty) = k$ (Adams); $p = \infty, s(k, \infty) = k+1$ (Jefferson); $p = 0$ (Hills); $p = -1$ (Dean); $p = 1$ (Webster).

- For Hill's method, we consider

$$\begin{aligned}
& \ln\left(\lim_{p\to 0^+}\left[\frac{k^p}{2}+\frac{(k+1)^p}{2}\right]^{1/p}\right) \\
= & \lim_{p\to 0^+}\frac{1}{p}\ln\left(\frac{k^p}{2}+\frac{(k+1)^p}{2}\right) \\
= & \lim_{p\to 0^+}\frac{\frac{k^p}{2}\ln k+\frac{(k+1)^p}{2}\ln(k+1)}{\frac{k^p}{2}+\frac{(k+1)^p}{2}} \quad \text{(by Hospital's rule)} \\
= & \ln\frac{k(k+1)}{2}
\end{aligned}$$

so that

$$\lim_{p\to 0^+}\left[\frac{k^p}{2}+\frac{(k+1)^p}{2}\right]^{1/p}=\sqrt{k(k+1)},\ k=0,1,2,\ldots.$$

- For Jefferson's method, we consider

$$\begin{aligned}
\lim_{p\to\infty}\left[\frac{k^p}{2}+\frac{(k+1)^p}{2}\right]^{1/p} &= \lim_{p\to\infty}[(k+1)^p]^{1/p}\lim_{p\to\infty}\left[\frac{1}{2}\left(\frac{k}{k+1}\right)^p+\frac{1}{2}\right]^{1/p} \\
&= k+1.
\end{aligned}$$

Proposition 1

Let A be a divisor method with signpost sequence: $s(0), s(1), \cdots$, and a similar definition for another divisor method A'. Method A is majorized by Method A' if and only if the signpost ratio $s(k)/s'(k)$ is strictly increasing in k.

For example, suppose we take A to be Adams and A' to be Jefferson, then $\frac{s(k)}{s'(k)} = \frac{k}{k+1} = 1 - \frac{1}{k+1}$ which is strictly increasing in k.

Proposition 2

The divisor method with power-mean rounding of order p is majorized by the divisor method with power-mean rounding of order p, if and only if $p \leq p'$.

This puts the 5 traditional divisor methods into the following majorization ordering

$$\text{Adams} \prec \text{Dean} \prec \text{Hill} \prec \text{Webster} \prec \text{Jefferson}.$$

Definition

A method M' favors small states relative to M if for every M-apportionment $\boldsymbol{a}$ and M'-apportionment $\boldsymbol{a}'$ for $\boldsymbol{p}$ and h,

$$p_i < p_j \Rightarrow a'_i \geq a_i \quad \text{or} \quad a'_j \leq a_j.$$

That is, it cannot happen that simultaneously a smaller district loses seats and a larger district gains seats.

Theorem

If M and M' are divisor methods with divisor criteria $d(a)$ and $d'(a)$ satisfying

$$\frac{d'(a)}{d'(b)} > \frac{d(a)}{d(b)} \text{ for all integers } a > b \geq 0,$$

then M' favors small states relative to M.

Proof

By way of contradiction, for some $\boldsymbol{a} \in M(\boldsymbol{p}, h)$ and $\boldsymbol{a}' \in M'(\boldsymbol{p}, h), p_i < p_j, a'_i < a_i$ and $a'_j > a_j$. By population monotonicity of divisor methods,

$$a'_i < a_i \leq a_j < a'_j$$

so $a'_j - 1 > a'_i \geq 0$ and $d'(a'_j - 1) \geq 1$ since $a \leq d'(a) \leq a + 1$ for all a.

Using the min-max property for $\boldsymbol{a}'$, we deduce that

$$\frac{p_j}{d'(a'_j - 1)} \geq \frac{p_i}{d'(a'_i)}$$

and so $d'(a'_i) > 0$. Lastly

$$\frac{p_j}{p_i} \geq \frac{d'(a'_j - 1)}{d'(a'_i)} > \frac{d(a'_j - 1)}{d(a'_i)} \geq \frac{d(a_j)}{d(a_i - 1)}.$$

We then have $\dfrac{p_j}{d(a_j)} > \dfrac{p_i}{d(a_i - 1)}$, a contradiction to the min-max property.

1. One can see that “is majorized by” is less demanding than “favoring small districts relative to”.

2. Since Hamilton’s apportionment is not a divisor method, how about the positioning of the Hamilton method in those ranking?

Proposition

Adams’ method favors small districts relative to Hamilton’s method while Hamilton’s method favors small districts relative to Jefferson’s method. However, Hamilton’s method is incomparable to other divisor methods such as Dean, Hill, and Webster.

Reference “The Hamilton apportionment method is between the Adams method and the Jefferson method,” *Mathematics of Operations Research*, vol. 31(2) (2006) p.390-397.

$A >$ Hamilton $> J$, but not Hamilton $> D, H, W$.

Population	Proportions	*A, D, H, W*	Hamilton	*J*
603	6.70	6	7	8
149	1.66	2	2	1
148	1.64	2	1	1
total = 900	10.00	10	10	10

$A >$ Hamilton $> J$, but not $> D, H, W >$ Hamilton.

Population	Proportions	*A*	Hamilton	*D, H, W, J*
1,600	5.36	5	5	6
1,005	3.37	3	4	3
380	1.27	2	1	1
total = 2,985	10.00	10	10	10

Hamilton happens to be the same as Webster

Population	Proportions	Adams	Webster	Hamilton	Jefferson
603	6.03	5	6	6	7
249	2.49	3	3	3	2
148	1.48	2	1	1	1
total = 1,000	10.00	10	10	10	10

Probabilistic approach

Consider a pair of integer apportionments $a_1 > a_2 > 0$ and ask

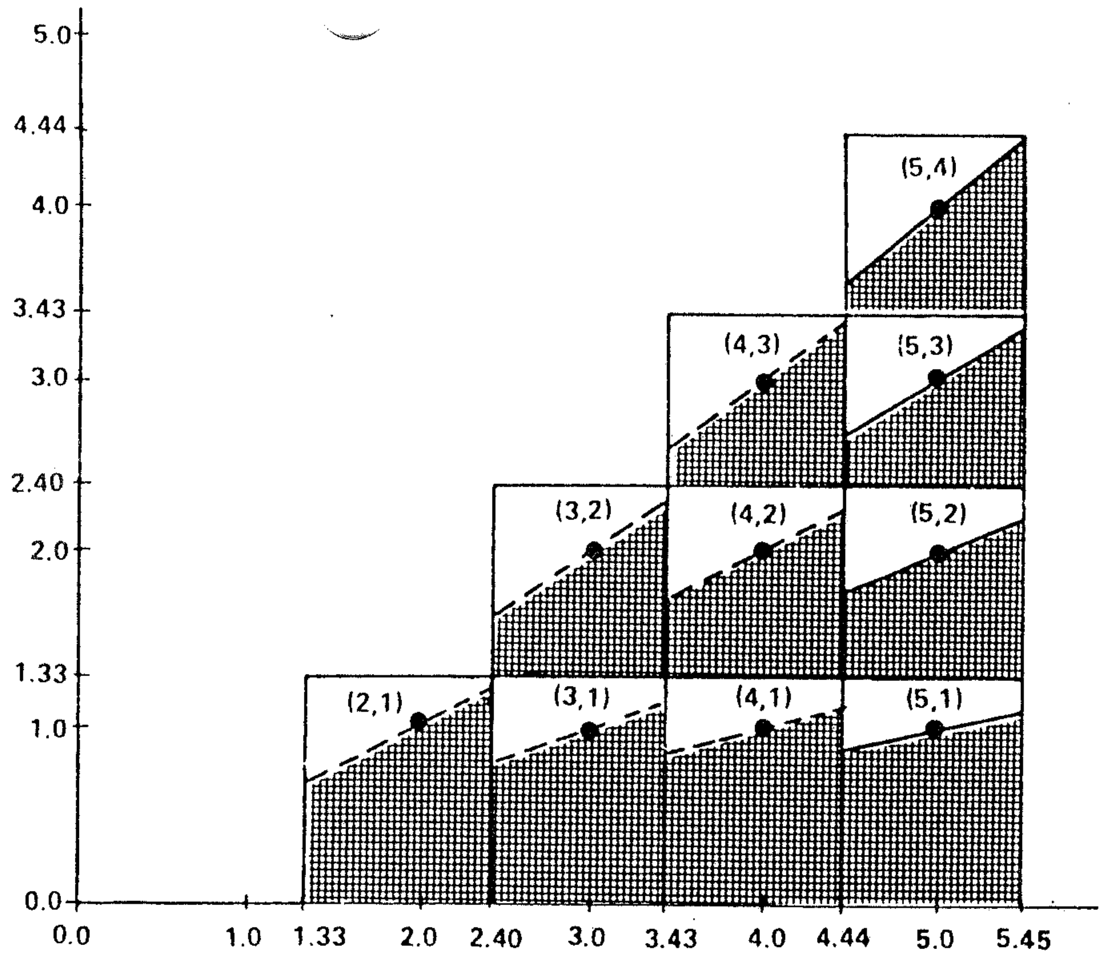
> "If the populations (p_1, p_2) has the M-apportionment (a_1, a_2), how likely is it that the small state (State 2) is favored?"

By population monotonicity, implicitly $p_1 \geq p_2$ since $a_1 > a_2$.

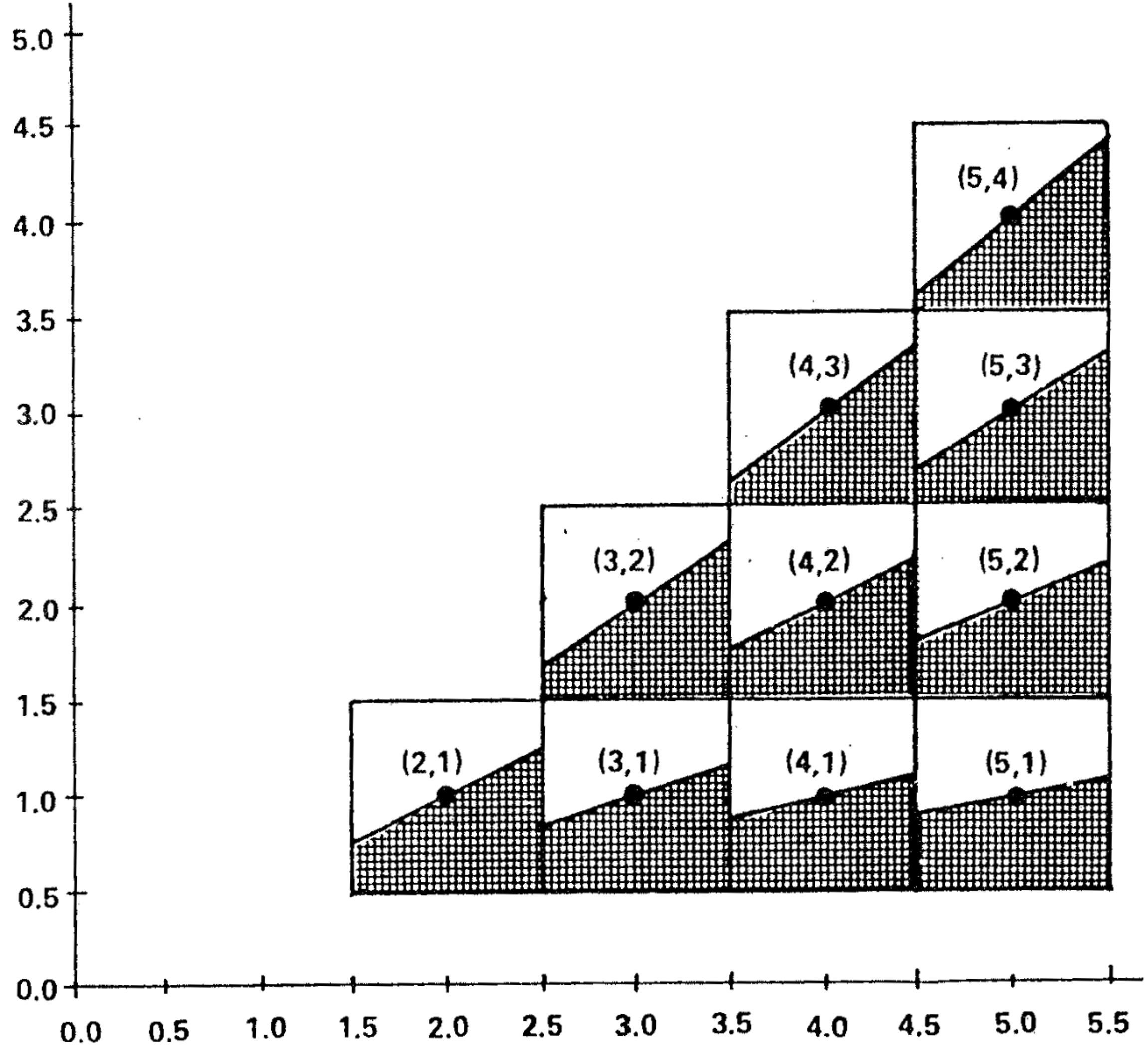
- Take as a probabilistic model that the populations $(p_1, p_2) = \boldsymbol{p} > 0$ are uniformly distributed in the positive quadrant.

  $$R_X(\boldsymbol{a}) = \left\{\boldsymbol{p} > 0 : d(a_i) \geq \frac{p_i}{\lambda} \geq d(a_i - 1)\right\}, \text{ with } d(-1) = 0.$$

  Each region $R_X(\boldsymbol{a})$ is a rectangle containing the point $\boldsymbol{a}$ and having sides of length $d(a_1) - d(a_1 - 1)$ and $d(a_2) - d(a_2 - 1)$.



Populations Favoring Small and Large States — Dean's Methods. Points that are inside the shaded area satisfies $p_1/a_1 > p_2/a_2$, that is, the larger state has smaller value in district size. The shaded area shows those populations that favor the smaller state.



Populations Favoring Small and Large States — Webster's Method. The shaded area shows those populations that favor the smaller state.

"Near the quota" and "Near the ideal"

"Near the quota" property

Instead of requiring "stay within the quota", a weaker version can be stated as: It should not be possible to take a seat from one state and give it to another and simultaneously bring both of them nearer to their quotas. That is, there should be no states i and j such that

$$q_i - (a_i - 1) < a_i - q_i \quad \text{and} \quad a_j + 1 - q_j < q_j - a_j. \tag{1}$$

Alternatively, no state can be brought closer to its quota without moving another state further from its quota. The above definition is in absolute terms.

In relative terms, no state can be brought closer to its quota on a percentage basis without moving another state further from its quota on a percentage basis. For no states i and j do we have

$$1 - \frac{a_i - 1}{q_i} < \frac{a_i}{q_i} - 1 \quad \text{and} \quad \frac{a_j + 1}{q_j} - 1 < 1 - \frac{a_j}{q_j}. \tag{2}$$

It can be checked easily that (1) $\Leftrightarrow$ (2).

Theorem

Webster's method is the unique population monotone method that is near quota.

Proof

(i) Webster method $\Rightarrow$ "near quota" property

If $\boldsymbol{a}$ is not near quota, that is if Eq. (1) holds for some i and j then rearranging, we have

$$1 < 2(a_i - q_i) \quad \text{and} \quad 1 < 2(q_j - a_j)$$

or

$$a_j + \frac{1}{2} < q_j \quad \text{and} \quad a_i - \frac{1}{2} < q_i$$

while implies

$$q_j/(a_j + \frac{1}{2}) > q_i/(a_i - \frac{1}{2}).$$

Hence the min-max inequality for Webster's method is violated, so $\boldsymbol{a}$ could not be a Webster apportionment. Therefore Webster's method is near quota.

(ii) non-Webster method $\Rightarrow$ "non-near quota" property

Conversely, let M be a population monotone method (i.e. a divisor method) different from Webster's. Then *there exists* a 2-state problem (p_1, p_2) in which the M-apportionment is uniquely (a_1+1, a_2), whereas the W-apportionment is uniquely (a_1, a_2+1). By the latter, we deduce the property:

$$p_2/(a_2+1/2) > p_1/(a_1+1/2).$$

At $h = a_1 + a_2 + 1$, the quota of state 1 is

$$\begin{aligned} q_1 &= \frac{p_1 h}{p_1 + p_2} \\ &= \frac{p_1(a_1 + 1/2 + a_2 + 1/2)}{p_1 + p_2} < \frac{p_1(a_1+1/2) + p_2(a_1+1/2)}{p_1+p_2} \\ &= a_1 + 1/2. \end{aligned}$$

State 2's quota is $q_2 = (a_1 + a_2 + 1) - q_1 > a_2 + 1/2$.
Therefore the M-apportionment (a_1+1, a_2) is not near quota.

US Presidential elections and Electoral College

- 538–member Electoral College (EC)

  435 (same apportionment as the House Representatives)

  $+$ 3 from the District of Columbia (same number as the smallest state)

  $+ 2 \times 50$ states

- Presidential elections
  - The winner of the plurality vote in a state is entitled to all the electors from that state (except Maine and Nebraska).
  - Actually the US Constitution gives the states broad powers as to the method of choosing their electors.

- Maine and Nebraska give an elector to the winner of the plurality of votes in each congressional district and give additional two electors corresponding to Senate seats to the winner of the plurality of the statewide vote.

- Most states are small and benefit from having their proportional share in representation augmented by those two electoral votes corresponding to Senate seats (favoring small states over large states).

- In the 2000 election, the 22 smallest states had a total of 98 votes in the EC while their combined population was roughly equal to that of the state of California, which had only 54 votes in the EC. Of those 98 EC votes, 37 went for Gore while 61 went for Bush.

- Gore would win for large House sizes and Bush would win for small House sizes as he did with the House size at 435. This is because Bush won many of the smaller states, where these small states have higher proportional share due to the additional two electoral votes. For House size > 655, Gore is sure to win. Unfortunately, the House size has been fixed in 1941, at that time there was approximately one representative for every $301,000$ citizens. Based on the same ratio of representatives to people today as existed in 1941 then the House based on the 1990 census should have about 830 members.

- A direct election of the president does offer the advantage that it is independent of the House size. One drawback is that a third party candidate that draws votes disproportionately away from one candidate over the other thereby influencing the election.

Electoral college representation is sensitive to the apportionment method

	Hamilton	Jefferson	*Adams*	*Webster*	*Dean*	*Hill**
2000 E.C.	tie	Gore	Bush	Bush	Bush	Bush
Winner	269 – 269	271 – 267	274 – 264	270 – 268	272 – 266	271 – 267

- Since the E.C. has built-in biases favoring small states, an apportionment method that partially offsets this bias might be justifiable.

- The infrequency of apportionment (once every 10 years)

  States that grow most quickly in population end up under-represented later in the life of a given apportionment.

Notion of marginal inequity measure

We formulate all existing apportionment methods (Hamilton and divisor methods) into an unified framework of integer programming with constraint.

The disparity (inequity measure) for state i is represented by the individual inequity function $f_i(a_i, p_i; P, H)$, with dependence on a_i and p_i, while P and H are shown explicitly. Some examples are

(i) Hamilton's method:

$$f_i(a_i, p_i; P, H) = \left(a_i - \frac{p_i H}{P}\right)^2,$$

where $\frac{p_i H}{P} = q_i$ is the quota of state i;

(ii) Webster's method:

$$f_i(a_i, p_i; P, H) = \frac{P}{p_i H}\left(a_i - \frac{p_i H}{P}\right)^2 = \frac{1}{q_i}(a_i - q_i)^2;$$

(iii) Hill's method:

$$f_i(a_i, p_i; P, H) = \frac{1}{a_i}(a_i - q_i)^2;$$

(iv) Parametric divisor method:

$$f_i(a_i, p_i; P, H) = p_i \left(\frac{a_i + \delta - \frac{1}{2}}{p_i} - \frac{H}{P} \right)^2.$$

When $\delta = 0$, it reduces to Webster's method.

The explicit dependence of f_i on p_i, P and H is more general than the dependence on p_i and q_i.

The aggregate inequity for the whole apportionment problem is $\sum_{i=1}^{S} f_i(a_i, p_i; P, H)$. This representation implicitly implies that inequity measure is *counted individually and additively*. As a result, the effect of seat transfers on the aggregate inequity between a subset of states is limited to the states involved in the transfer.

The integer programming with constraint can be formulated as

$$\min \sum_{i=1}^{S} f_i(a_i, p_i; P, H) \text{ subject to } \sum_{i=1}^{S} a_i = H,\ a_i \in \mathbb{Z}_+,$$

where $\mathbb{Z}_+$ is the set of non-negative integer. In other words, the apportionment vector $\boldsymbol{a} = (a_1\ a_2\ \cdots\ a_S)^T$ is given by

$$\boldsymbol{a} = \arg\min_{\boldsymbol{a}} \sum_{i=1}^{S} f_i(a_i, p_i; P, H).$$

Property on the individual inequity function f_i

It is desirable to have f_i to observe convexity property with respect to a_i so that the disparity is minimized with some appropriate choice of a_i (including the possibility of the unlikely scenario of ties between two apportionment methods).

Marginal inequity function

The earlier research works on apportionment methods have been directed to search for the corresponding inequity function f_i for every apportionment method. Unfortunately, the inequity function may not exist for all apportionment methods.

- For example, the Dean method does not possess any functional form of f_i (or yet to be found); the Adams and Jefferson methods lead to f_i with some parameter being assigned $-\infty$ or ∞ (see the power-mean formulas).

- Even when f_i exists, it may not be unique (like Webster's method).

We propose that a more appropriate choice is the *marginal inequity function* ϕ_i that is related to f_i

$$\phi_i(a_i, p_i; P, H) = f_i(a_i + 1, p_i; P, H) - f_i(a_i, p_i; P, H)$$

if $f_i(a_i, p_i; P, H)$ exists.

As deduced from the convexity property of f_i in a_i, we require $\phi_i(a_i, p_i; P, H)$ to be non-decreasing in a_i.

Hamilton's method

$$f_i(a_i, p_i; P, H) = \left(a_i - \frac{p_i H}{P}\right)^2$$

so that

$$\phi_i(a_i, p_i; P, H) = 2a_i + 1 - \frac{2p_i H}{P}.$$

Parametric divisor method

$$f_i(a_i, p_i; P, H) = p_i \left(\frac{a_i + \delta - 0.5}{p_i} - \frac{H}{P}\right)^2$$

so that

$$\phi_i(a_i, p_i; P, H) = \frac{2a_i + \delta + 0.5}{p_i} - \frac{2H}{P}.$$

Hill's method

$$f_i(a_i, p_i; P, H) = \frac{1}{a_i}(a_i - q_i)^2 = a_i - 2q_i + \frac{q_i^2}{a_i}$$

so that

$$\phi_i(a_i, p_i; P, H) = 1 - \frac{p_i^2 H^2}{P^2} \frac{1}{a_i(a_i + 1)}.$$

Webster's method

$$f_i(a_i, p_i; P, H) = \left(a_i - \frac{p_i H}{P}\right)^2 \frac{P}{p_i H}$$

so that

$$\phi_i(a_i, p_i; P, H) = \frac{P}{p_i H}\left(2a_i + 1 - \frac{2p_i H}{P}\right).$$

In all of the above cases, $\phi_i(a_i, p_i; P, H)$ is increasing in a_i.

Remark Given f_i, we can always compute ϕ_i; but not vice versa. For the known apportionment methods, like Hamilton's method and divisor methods, we can always find the corresponding ϕ_i.

A *necessary condition* for $\boldsymbol{a}$ to be the solution to the apportionment problem is that no transfer between any two states can lower the aggregate inequity measure. Observing that inequity is counted separately and additively, for any pair of states i and j, we can deduce the following necessary condition for $\boldsymbol{a}$:

$$
\begin{aligned}
& f_i(a_i, p_i; P, H) + f_j(a_j, p_j; P, H) \\
& \leq f_i(a_i + 1, p_i; P, H) + f_j(a_j - 1, p_i; P, H) \\
\Leftrightarrow \quad & \phi_j(a_j - 1, p_j; P, H) \leq \phi_i(a_i, p_i; P, H).
\end{aligned}
$$

Interpretation

The above inequality dictates a useful condition on the ordering of ϕ_i and ϕ_j among any pair of states i and j. Suppose $a_j - 1$ seats have been apportioned to state j and a_i seats have been apportioned to state i. Assume that the above inequality holds, then the next seat will be apportioned to state j in favor of state i.

Algorithm

Let the starting value of $\boldsymbol{a}$ be $(0, 0, \ldots, 0)$. Choose state i whose $\phi_i(a_i, p_i; P, H)$ is the smallest among all states, and increase a_i by 1. Repeat the procedure until $\sum_{i=1}^{S} a_i = H$ is satisfied.

The above iterative scheme implicitly implies

$$\max_i \phi_i(a_i - 1, p_i; P, H) \leq \min_i \phi_i(a_i, p_i; P, H).$$

This is in a similar spirit to the rank index method, where

$$\max_i \frac{d(a_i - 1)}{p_i} \leq \min_i \frac{d(a_i)}{p_i}.$$

Here, $d(a_i)$ is the signpost function of the divisor method whose common divisor λ satisfies

$$d(a_i - 1) \leq \frac{p_i}{\lambda} \leq d(a_i) \Longleftrightarrow \frac{p_i}{d(a_i)} \leq \lambda \leq \frac{p_i}{d(a_i - 1)} \text{ for all } i.$$

For the divisor method with signpost function $d(a)$, we may set the corresponding $\phi_i(a_i, p_i)$ to be $d(a_i)/p_i$ (which is independent of P and H, and satisfies non-decreasing property in a_i).

Alabama paradox

For a given apportionment method, if the ordering of ϕ_i is not affected by the house size H, then the method will not produce Alabama paradox.

- The marginal inequity measure of Hamilton's method is

$$\phi_i(a_i, p_i; P, H) = 2a_i + 1 - \frac{2p_i H}{P},$$

  where an increase of H by one will cause ϕ_i to decrease by $2p_i/P$ (with dependence on state population p_i as well). The seat apportionment order has to be modified accordingly.

- For all divisor methods, the ordering of ϕ_i only depends on $\dfrac{d(a_i)}{p_i}$, which is independent of H. Note that there are various possible forms of f_i, hence ϕ_i, for Hill's method and Webster's method (both are divisor methods). Some of these forms may lead to ϕ_i that is dependent on H.

Uniformity

An apportionment method is said to be consistent if the restriction of the apportionment problem to a subset of the universe of states still produces the same result for the states involved.

Lemma

For a given apportionment method, if the ordering of ϕ_i is not affected by the value of P and H, then the method is consistent.

In the subproblem with k states, the total population is lowered to $P' = \sum_{i=1}^{k} p_i$ and the total number of seats is changed to $H' = \sum_{i=1}^{k} a_i$. Now, the marginal inequity function $\phi_i(a_i, p_i; P, H)$ is changed to $\phi_i(a_i, p_i; P', H')$. If the ordering of ϕ_i is unchanged by the changes in P and H, then the same apportionment solution will be resulted in the subproblem.

Corollary All divisor methods are uniform.

Bias analysis

Let ϕ_i and ϕ_i' denote the marginal inequity measure of $M(\boldsymbol{p}, H)$ and $M'(\boldsymbol{p}, H)$, respectively. Suppose for any $i > j$, $\phi_j(a_j - 1, p_j; P, H) < \phi_i(a_i, p_i; P, H)$ always implies $\phi_j'(a_j - 1, p_j; P, H) < \phi_i'(a_i, p_i; P, H)$. The last inequality is equivalent to

$$f_i'(a_i, p_i; P, H) + f_j'(a_j, p_j; P, H) < f_i'(a_i + 1, p_i; P, H) + f_j'(a_j - 1, p_j; P, H).$$

The above inequality implies that when the apportionment method is changed from $M(\boldsymbol{p}, H)$ to $M'(\boldsymbol{p}, H)$, the more populous state will never lose seats to the less populous state.

Example

Given two divisor methods with the signpost function $d(k)$ and $d'(k)$. For $d'(k)$ to be majorized by $d(k)$, where $\phi_i(a_i, p_i) = \dfrac{d(a_i)}{p_i}$, we require

$$\frac{d(a_j - 1)}{p_j} < \frac{d(a_i)}{p_i} \Rightarrow \frac{d'(a_j - 1)}{p_j} < \frac{d'(a_i)}{p_i}, \qquad p_i > p_j$$

Suppose $\dfrac{d(k)}{d'(k)}$ is decreasing in k, we always have

$$\frac{d(a_i)}{d'(a_i)} < \frac{d(a_j - 1)}{d'(a_j - 1)} \quad \text{(since } a_i > a_j - 1 \text{ when } p_i > p_j\text{)}.$$

We then deduce that

$$\frac{d'(a_j - 1)}{d'(a_i)} < \frac{d(a_j - 1)}{d(a_i)} < \frac{p_j}{p_i}.$$

Therefore, $d(k)/d'(k)$ decreasing in k is sufficient for $d'(k)$ to be majorized by $d(k)$.

Hamilton's method favors less populous states compared to Jefferson's method

Recall that

$$\phi_i(a_i, p_i) = \frac{a_i + 1}{p_i} \quad \text{for Jefferson's method}$$

and

$$\phi_i'(a_i, p_i; P, H) = 2a_i + 1 - \frac{2p_i H}{P} \quad \text{for Hamilton's method.}$$

If $(a_1, a_2, \ldots, a_S)$ is the Jefferson apportionment, then for $p_i > p_j$,

$$\phi_j(a_j - 1, p_j) < \phi_i(a_i, p_i) \Leftrightarrow \frac{a_j}{p_j} < \frac{a_i + 1}{p_i}.$$

Given $a_j < \frac{p_j}{p_i}(a_i + 1)$, we need to establish that

$$\begin{aligned}\phi_j'(a_j - 1, p_j; P, H) < \phi_i'(a_i, p_i; P, H) &\Leftrightarrow 2a_j - 1 - \frac{2p_j H}{P} < 2a_i + 1 - \frac{2p_i H}{P} \\ &\Leftrightarrow a_j < a_i + 1 + \frac{H}{P}(p_j - p_i).\end{aligned}$$

For $a_j < \frac{p_j}{p_i}(a_i + 1)$, we can establish

$$\begin{aligned} a_i + 1 - a_j + \frac{H}{P}(p_j - p_i) &> (a_i + 1)\left(1 - \frac{p_j}{p_i}\right) - \frac{H}{P}p_i\left(1 - \frac{p_j}{p_i}\right) \\ &= (a_i + 1 - q_i)\left(1 - \frac{p_j}{p_i}\right) > 0 \end{aligned}$$

since $p_i > p_j$ and $a_i + 1 > q_i$ (Jefferson's method observes the lower quota property). We conclude that Hamilton's method favors less populous states compared to Jefferson's method.

1.7 Cumulative voting and proportional representation

Plurality voting

- In *single-winner* plurality voting, each voter is allowed to vote for only one candidate; and the winner of the election is whichever candidate represents a plurality of voters.

- In *multi-member* constituencies, referred to as an *exhaustic counting system*, one member is elected at a time and the process repeated until the number of vacancies is filled.

  Example

  With 8,000 voters and 5 to be elected, under plurality voting, a coalition C of 4001 members can elect 5 candidates of its choice by giving each of the 5 candidates 4,001 votes.

Cumulative voting

Cumulative voting is a multiple-winner voting system intended to promote proportional representation while also being simple to understand.

You may offer up to 3 votes			
1	2	3	
○	○	○	Chan
●	●	○	Lee
○	○	○	Cheung
○	○	●	Wong
○	○	○	Ho

2 votes for Lee and 1 vote for Wong

Voters can 'plump' their votes, conferring all n votes on a single candidate or distributing their n votes as they please.

In cumulative voting, each voter is allotted the same number of votes, while allowing for expression of intensity of candidate preference.

Use of cumulative voting system in the US electoral systems

- Under the usual one-member district system (winner-take-all), voters can elect just one representative from that district, even if another candidate won a substantial percentage of votes.

- Between 1870 and 1980, voters of a state congressional district were able to elect 3 candidates for the Illinois House of Representatives. This allowed for the election of "political minorities". Voters did not understand the cumulative voting system. In 1960s, nearly 45% of Illinois House elections involved only 3 candidates for 3 seats.

- New York City ended cumulative voting in the 1950s because of the election of a communist from Harlem.

"Pros" of cumulative voting systems

- Since 1980, Illinois tried "redrawing political districts" in order to guarantee election of political minorities. This takes power away from the people and gives it to politicians and to the courts.
  - There is nothing in the Illinois Constitution or the US Constitution that requires single-member districts.
  - Proportional voting is the system in most European countries. If 7% of the voters support the Green Party, the Green Party gets 7% of the seats.

- Minority group voters do not have to be made into majorities of voters in order to elect a candidate. The need to manipulate *district lines* is largely, if not completely, eliminated.

Assuring a certain representation

- Voting literature frequently mentions "thresholds", which designate a fraction of population for which a cohesive group whose population fraction is above the threshold can assure itself a certain level of representation under a method of voting.

- For example, a like-minded grouping of voters that is 20% of a city would be well positioned to elect one out of five seats.

- Let P be the total number of voters (population) and n the number of seats to be elected, $P > n$.

- We want the fraction of population x/P over which the group can elect k of n, if the group desires to do so and if they vote strategically. Everybody has n votes.

Negative remarks

It does usually provide proportional representation. However, it may promote factional strife and thus seriously affect the efficiency of the company. It also paves the way for "extremists".

Fair apportionment of seats

- Cumulative voting can guarantee a minority the opportunity to elect representatives in the same number that they would receive by one of the apportionment methods.

- A minority can never guarantee itself greater representation by cumulative voting than that would be allotted and deemed fair by Webster or Jefferson apportionment.

Theorem

Assume that there are P voters and n seats. Under cumulative voting, a coalition C of x voters can guarantee the election of $\left\lfloor \frac{x}{P} n \right\rfloor$ candidates.

Example

Suppose $x = 46, P = 81, n = 8$, a coalition of 46 voters can elect $\left\lfloor \frac{46}{81} \times 8 \right\rfloor = 4$ candidates by giving each of its four candidates $\frac{46 \times 8}{4} = 92$ votes.

Actually, the coalition can elect 5 candidates by giving each of them $\frac{368}{5}$ votes.

Proof

Let $k = \left\lfloor \frac{x}{P}n \right\rfloor$. Coalition C may cast $\left\lfloor \frac{x}{k}n \right\rfloor$ votes for each of these k candidates. It suffices to show that it is impossible to have $n-k+1$ candidates to receive at least $\frac{x}{k}n$ votes.

Since $k \leq \frac{x}{P}n$, so

$$\frac{n-k+1}{k} \geq \frac{n - \frac{x}{P}n + 1}{\frac{x}{P}n}.$$

Rearranging, we obtain

$$(n-k+1)\frac{x}{k}n \geq \left(n - \frac{x}{P}n + 1\right)\frac{xn}{\frac{x}{P}n} = Pn - xn + P > (P-x)n.$$

where $(P-x)n$ is the maximum number of votes that can be casted by voters outside the coalition. The number of votes required to win $n-k+1$ candidates is beyond the maximum number of votes held.

Recall that $\frac{x}{k}$ is the number of voters represented by each candidate for the minority if k candidates are chosen, and similarly, that for the majority is $\dfrac{P-x}{n-k+1}$ if $n-k+1$ candidates are chosen. There is a threshold head counts x required in order to guarantee the election of k candidates.

Lemma

Under cumulative voting, a coalition C of x voters can guarantee the election of k candidates if and only if

$$\frac{x}{k} > \frac{P-x}{n-k+1} \quad \Leftrightarrow \quad \frac{x}{P} > \frac{k}{n+1}.$$

Example

Let $P = 81$ and $n = 8$. A coalition of size $x = 46$ can guarantee the election of 5 candidates since $46 \times 9 > 5 \times 81$.

Proof

(i) $\dfrac{x}{k} > \dfrac{P-x}{n-k+1} \Rightarrow$ election of k candidates.

A coalition of x voters can give each of k candidates $\dfrac{xn}{k}$ votes. The least popular of $n-k+1$ other candidates could receive no more than $\dfrac{(P-x)n}{n-k+1}$ votes. Thus the coalition of x voters can guarantee the election of k candidates if

$$\frac{xn}{k} > \frac{(P-x)n}{n-k+1} \Leftrightarrow \frac{x}{k} > \frac{P-x}{n-k+1} \Leftrightarrow \frac{x}{P} > \frac{k}{n+1}.$$

(ii) election of k candidates $\Rightarrow \dfrac{x}{k} > \dfrac{P-x}{n-k+1}$

By contradiction, suppose $\dfrac{x}{k} \leq \dfrac{P-x}{n-k+1}$, then the other $P-x$ voters can block the election of the k^{th} candidate of coalition C. This is because $\dfrac{(P-x)n}{n-k+1}$ votes is more than $\dfrac{xn}{k}$ votes.

- The commonly cited "threshold of exclusion" for cumulative voting $\dfrac{1}{n+1}$ above which a minority can assure itself representation is just a special case with $k = 1$.

- How do we compare with the generalized plurality multimember voting, where every voter has n votes but no plumping is allowed? The most votes that each of a coalition's k candidates receives is x. However, the $(n-k+1)^{\text{st}}$ candidate can receive $P - x$ votes. To elect k candidates, the coalition needs

$$x > P - x \quad \text{or} \quad \frac{x}{P} > \frac{1}{2}.$$

  This result is independent of k, so to assure any representation under generalized plurality voting, a coalition must be a population majority.

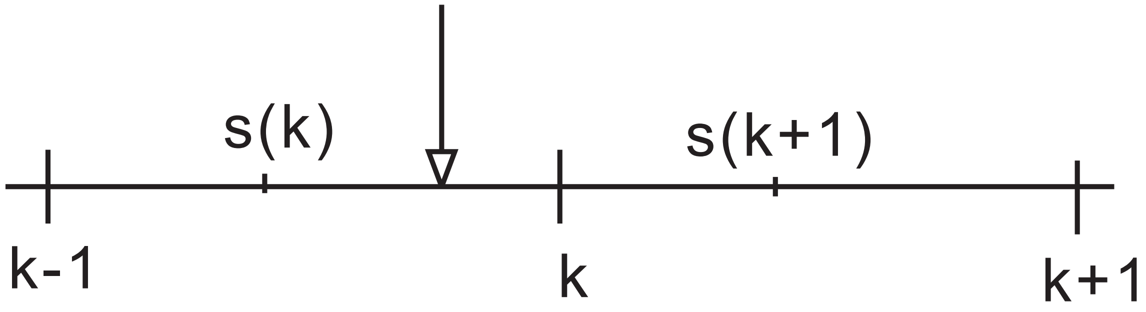
Fair representation

- Webster's method minimizes the absolute difference between all pairs of states, in the numbers of representatives per person, known as "per capita representation". That is, $\left|\frac{a_i}{p_i} - \frac{a_j}{p_j}\right|$ is minimized between any pair of states.

- Consider representation that is apportioned to reflect minority and majority subsets of a population, Dean's method would be more favorable to the minority than Hill's method, which would be more favorable than Webster's method. Recall biases toward larger states: Dean (harmonic mean) $<$ Hill (geometric mean) $<$ Webster (arithmetic mean).

- Suppose that there are 2 groups: minority with population x and majority with population $P - x$. The eligible quota for the minority is $\frac{x}{P}n$.

If the quota falls within $[s(k), s(k+1)]$, then the minority wins k seats.



Recall that $s(k)$ is some chosen form of mean of $k-1$ and k.

For example, the population threshold x for the Webster-fair representation is given by

$$\frac{x}{P} > \frac{s_{\text{Web}}(k)}{n} = \frac{k - \frac{1}{2}}{n}.$$

Reference

"The potential of cumulative voting to yield fair representation", by Duane A. Cooper, *Journal of Theoretical Politics*, vol.19, (2007) p.277-295.

In summary, to deserve k of n seats, the group's quota (as derived from the population threshold x) must be greater than the mean of $k-1$ and k.

Hill-fair representation

$$\frac{x}{P} > \frac{\sqrt{(k-1)k}}{n}$$

Dean-fair representation

$$\frac{x}{P} > \frac{\frac{2}{\frac{1}{k-1}+\frac{1}{k}}}{n} = \frac{k(k-1)}{\left(k-\frac{1}{2}\right)n}.$$

The above means observe the following order: HM $<$ GM $<$ AM

$$\frac{k(k-1)}{\left(k-\frac{1}{2}\right)n} < \frac{\sqrt{k(k-1)}}{n} < \frac{k-\frac{1}{2}}{n}$$

On one hand, minority coalition of population fraction $\frac{x}{P}$ can win k of n seats under cumulative voting method if and only if

$$\frac{1}{2} > \frac{x}{P} > \frac{k}{n+1}.$$

On the other hand, Webster-fair representation requires $\frac{x}{P} > \frac{k-\frac{1}{2}}{n}$.

Comparing $\frac{k-\frac{1}{2}}{n}$ and $\frac{k}{n+1}$, we deduce the algebraic property:

$$\frac{k-\frac{1}{2}}{n} < \frac{k}{n+1} \Leftrightarrow \frac{k}{n+1} < \frac{1}{2}.$$

$$\frac{k-\frac{1}{2}}{n} < \frac{k}{n+1} < \frac{x}{P} < \frac{1}{2}$$

For any minority, cumulative voting can be deemed more favorable to the majority than Webster's method in that a greater threshold is required for the cumulative voting electoral possibilities than is necessary in the measure of Webster-fairness. This counter claims that cumulative voting would be unfairly advantageous to minority populations.

Fairness of cumulative voting

- How often does cumulative voting yield the opportunity for a minority to elect its fair share against a majority?

- When cumulative voting does not make it possible for minority voting strength to elect a fair share, it is possible to elect only one less representative than the Webster-fair amount.

Theorem

In an election for n representatives of the population under cumulative voting, the probability that the minority is unable to elect its Webster-fair share of the n seats is

$$\begin{cases} \frac{1}{4}\frac{n}{n+1}, \text{if } n \text{ is even} \\ \\ \frac{1}{4}\frac{n-1}{n}, \text{if } n \text{ is odd.} \end{cases}$$

Moreover, if the minority's Webster-fair share is $k_w \geq 1$, then it has the voting strength to elect either k_w or $k_w - 1$ representatives.

Proof

Under the scenario of winning k out of n seats for minority $\left(\frac{x}{P} < \frac{1}{2}\right)$, the Webster threshold $\frac{k-\frac{1}{2}}{n}$ is less than the cumulative voting threshold $\frac{k}{n+1}$.

1. The minority cannot elect any more than the Webster-fair number of representation, say, k_w+1. If otherwise, the Webster-fair representation would be at least k_w+1.

2. Also, a minority is able to elect at least k_w-1 representatives. If otherwise, we could have

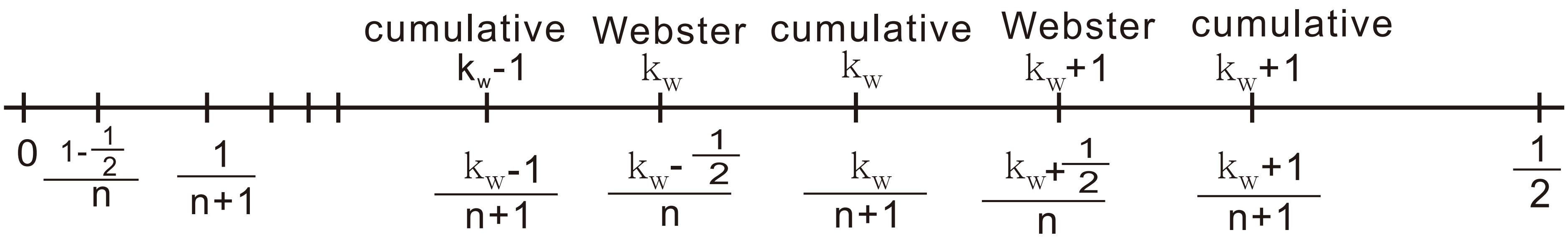
$$\frac{k_w-\frac{1}{2}}{n}<\frac{x}{P}<\frac{k_w-1}{n+1}.$$

   (a) The left inequality arises since the Webster-fair representation is k_w;

   (b) The right inequality arises when cumulative voting is assumed to elect less than k_w-1 representatives.

This is impossible since

$$\frac{k_w - 1}{n+1} < \frac{k_w - 1}{n} < \frac{k_w - \frac{1}{2}}{n}.$$



By virtue of the above inequality and $\frac{k-\frac{1}{2}}{n} < \frac{k}{n+1}$, the interval $\left(0, \frac{1}{2}\right)$ can be partitioned by an alternating sequence of Webster- and cumulative voting thresholds as follows:

$$0, \frac{1-\frac{1}{2}}{n}, \frac{1}{n+1}, \frac{2-\frac{1}{2}}{n}, \frac{2}{n+1}, \dots, \frac{\lfloor\frac{n}{2}\rfloor - \frac{1}{2}}{n}, \frac{\lfloor\frac{n}{2}\rfloor}{n+1}, \frac{1}{2},$$

where

$$\lfloor\frac{n}{2}\rfloor = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Consider a population of size P. Consider a minority fraction of the population $\frac{x}{P}$ chosen from the uniform distribution on $\left(0, \frac{1}{2}\right) \cap Q$, where Q is the set of rational numbers. The remaining $\frac{P-x}{P}$ constitutes the population's majority.

The probability that cumulative voting does not make it possible for the minority to attain its Webster-fair representation is the probability that the minority has the voting strength to elect $k_w - 1$ representatives but not k_w, which is just the probability that $\frac{x}{P}$ belongs to one of the subintervals

$$\left(\frac{k-\frac{1}{2}}{n}, \frac{k}{n+1}\right)$$

of $\left(0, \frac{1}{2}\right)$, where $1 \leq k \leq \frac{n}{2}$. This probability is just

$$\begin{aligned} & \left|\bigcup_k \left(\frac{k-\frac{1}{2}}{n}, \frac{k}{n+1}\right)\right| \Big/ \left|\left(0, \frac{1}{2}\right)\right| \\ = & \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} \left(\frac{k}{n+1} - \frac{k-\frac{1}{2}}{n}\right) \Big/ \frac{1}{2}. \end{aligned}$$

Case 1: n is even.

$$\begin{aligned}
\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{k}{n+1} - \frac{k - \frac{1}{2}}{n} \right) &= \sum_{k=1}^{\frac{n}{2}} \left(\frac{k}{n+1} - \frac{k}{n} + \frac{1}{2n} \right) \\
&= \frac{\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}}{n+1} - \frac{\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}}{n} + \frac{1}{2n} \cdot \frac{n}{2} \\
&= \frac{(n^2 + 2n) - (n^2 + n)}{8(n+1)} \\
&= \frac{n}{8(n+1)}.
\end{aligned}$$

Case 2: n is odd.

$$\begin{aligned}
\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{k}{n+1} - \frac{k - \frac{1}{2}}{n} \right) &= \sum_{k=1}^{\frac{n-1}{2}} \left(\frac{k}{n+1} - \frac{k}{n} + \frac{1}{2n} \right) \\
&= \frac{\frac{\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)}{2}}{n+1} - \frac{\frac{\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)}{2}}{n} + \frac{1}{2n} \cdot \frac{n-1}{2} \\
&= \frac{n-1}{8} - \frac{n^2-1}{8n} + \frac{2n-2}{8n} \\
&= \frac{n-1}{8n}.
\end{aligned}$$

Therefore, the probability that cumulative voting does not make it possible for the minority to attain its Webster-fair representation is

$$\begin{cases} \frac{n}{8(n+1)} \Big/ \frac{1}{2} = \frac{n}{4(n+1)} & \text{if } n \text{ is even} \\ \frac{n-1}{8n} \Big/ \frac{1}{2} = \frac{n-1}{4n} & \text{if } n \text{ is odd.} \end{cases}$$

Conclusion

Under cumulative voting, a minority of arbitrary size is able, if it chooses, to elect its Webster-fair share of n seats against the majority more than 75% of the time. In the remaining instances, the minority can do no worse than one less than its Webster-fair share.

Example

Consider a population of 500, divided into a polarized majority and minority of 340 and 160 people, respectively, and suppose a five-member representative body is to be elected. The minority – at 32 per cent – has more that $\frac{1}{6}$, but less than $\frac{2}{6}$, of the population; thus under cumulative voting the minority has the electoral strength to elect one, but not two, representatives.

Recall the population threshold for the cumulative voting method to elect k out of n is $\frac{k}{n+1}$. With $n = 5$, the threshold values are $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$.

If the actual population fraction falls within $\frac{k}{n+1}$ and $\frac{k+1}{n+1}$, $k < n$, then k is elected out of n.

- *Webster's appointment*

  Were the five-member body apportioned by Webster's method, the minority's quota of $\frac{160}{500} \cdot 5 = 1.6$ would be rounded up to deserve 2 seats, and the majority's quota of $\frac{340}{500} \cdot 5 = 3.4$ would be rounded down to deserve three seats. It can be readily verified that the absolute difference in per capita representation, $\frac{2}{160} - \frac{3}{340} \approx 0.00368$, is the minimum value for all possible apportionments.

- The population fraction $\frac{160}{500} = 0.32$ exceeds the threshold $\frac{2 - 1/2}{5} = 0.3$ for deserving two of five seats by Webster's method but fails to attain the threshold $\frac{2}{5+1} = \frac{1}{3} \approx 0.333$ to assure two of five seats under cumulative voting.

- Continuing with the consideration of a total population of 500, a minority in the range of 151 to 166 people in a polarized electorate would have to settle for one less than its Webster-fair share of two representatives.

  (i) 151 people can attain 2 seats under Webster apportionment.

  (ii) 167 people are required to attain 2 seats under cumulative voting method.

- Likewise, a minority of size from 51 to 83 would deserve one of five seats by Webster but would not reach the threshold of exclusion necessary for representation by cumulative voting.

- Minorities of sizes 1–49, 84–149, or 167–249 could earn their Webster-fair share of representatives under cumulative voting, comprising about 80 per cent of the possible minority sizes for total population $P = 500$. This is consistent with the theorem's predicted result, where $\left.\frac{n-1}{4n}\right|_{n=5} = 20\%$.

Theorem – Cumulative voting and Jefferson's method

A population of size P is partitioned into 2 subgroups of x and $P-x$, with n seats. The number of seats each group can be assured under cumulative voting is equivalent to the number of seats each group would be assigned by Jefferson's method of apportionment.

Numerical example - Jefferson's apportionment

- To apportion the seats under Jefferson's method, again with a majority of 340 and a minority of 160, we would start with divisor $d = \dfrac{500}{5} = 100$, divide that into the populations, and round down, repeating until an appropriate divisor is determined to allocate five seats.

- At first, we have

$$\left\lfloor \frac{340}{100} \right\rfloor = \lfloor 3.4 \rfloor = 3 \quad \text{and} \quad \left\lfloor \frac{160}{100} \right\rfloor = \lfloor 1.6 \rfloor = 1,$$

but $3 + 1 = 4 \neq 5$.

We see that $d = 85$ works (as will any d satisfying $80 < d \leq 85$), yielding

$$\left\lfloor \frac{340}{85} \right\rfloor = \lfloor 4.0 \rfloor = 4 \quad \text{and} \quad \left\lfloor \frac{160}{85} \right\rfloor = \lfloor 1.88 \rfloor = 1,$$

with $4 + 1 = 5$, so the majority is allotted four seats and the minority gets one, the same result achieved by cumulative voting for these subpopulations.

Comparison between Jefferson's and Webster's apportionment

The total over-representation of this Jefferson (4–1) apportionment, $\frac{4}{340} - \frac{5}{500} \approx 0.00176$, is the minimum for all possible apportionments; in particular, it is less than the over-representation $\frac{2}{160} - \frac{5}{500} = 0.00250$ of the Webster (3–2) apportionment.

Concurrently, the total under-representation of the Jefferson apportionment, $\frac{5}{500} - \frac{1}{160} = 0.00375$, is greater than the under-representation $\frac{5}{500} - \frac{3}{340} \approx 0.00118$ of the Webster apportionment.

Proof

By Jefferson's method, we apportion the n seats by finding a divisor d such that $\left\lfloor \frac{x}{d} \right\rfloor + \left\lfloor \frac{P-x}{d} \right\rfloor = n$. We begin by considering $d = \frac{P}{n}$. If $\left\lfloor \frac{x}{P/n} \right\rfloor + \left\lfloor \frac{P-x}{P/n} \right\rfloor = n$, then the population subgroups occur in a ratio that can precisely be represented proportionally among the n seats. Cumulative voting would give the same proportional representation to the subpopulations, if they choose, with appropriate strategy in this case.

For example, suppose we take $x = 100$, $P = 400$, so $P - x = 300$; also, we take $n = 12$. Minority and majority receive 3 and 9 seats, respectively. Minority (majority) puts all 1,200 (3,600) votes into 3 (9) candidates.

Otherwise, and more commonly, we have $\left\lfloor \frac{x}{P/n} \right\rfloor + \left\lfloor \frac{P-x}{P/n} \right\rfloor < n$. Thus, some $d < \frac{P}{n}$ must be determined to get $\left\lfloor \frac{x}{d} \right\rfloor + \left\lfloor \frac{P-x}{d} \right\rfloor = n$.

In order for the subpopulation of x people to be allotted exactly k of the n seats under Jefferson's apportionment, the following two inequalities must be satisfied:

$$k \leq \frac{x}{d} < k+1 \quad \text{and} \quad n-k \leq \frac{P-x}{d} < (n-k)+1.$$

Rearranging the inequalities to solve for d, we obtain

$$\frac{x}{k+1} < d \leq \frac{x}{k} \quad \text{and} \quad \frac{P-x}{(n-k)+1} < d \leq \frac{P-x}{n-k}.$$

Now combining these results, we must have $\frac{P-x}{(n-k)+1} < \frac{x}{k}$; solving for $\frac{x}{P}$, we find the equivalent inequality, $\frac{x}{P} > \frac{k}{n+1}$.

Similarly, the statements imply that

$$\frac{x}{k+1} < \frac{P-x}{n-k} \Leftrightarrow \frac{x}{P} < \frac{k+1}{n+1}.$$

Putting the two results together, we obtain

$$\frac{k}{n+1} < \frac{x}{P} < \frac{k+1}{n+1}.$$

Interpretation: When there are minority and majority groups only (two states), the Jefferson apportionment gives k seats out of n seats if the fraction of population satisfies the above pair of inequalities.

The subpopulation of size x has the electoral strength to win k of n seats under cumulative voting, but not $k+1$ seats. The k seats are the same as the allotment from Jefferson's method.

- The only remaining consideration is what happens when the population fraction $\frac{x}{P}$ equals a threshold value $\frac{k}{n+1}$. In this instance, both the electoral result of cumulative voting and the apportionment of Jefferson's method are indeterminate.

- When $\frac{x}{P} = \frac{k}{n+1}$, if the two polarized subpopulations of size x and $P-x$ vote perfectly strategically, a tie breaker would be necessary to determine whether the x voters get k or $k-1$ seats and, correspondingly, whether the $P-x$ voters receive $n-k$ or $(n-k)+1$ seats.

Can the result be extended to more than 2 subgroups?

1. Jefferson apportionment results cannot always be guaranteed by cumulative voting. As a counterexample, consider the subpopulations X_1, X_2, X_3 of size $x_1 = 350, x_2 = 350, x_3 = 200$, respectively. Using a divisor of 180, we realize that X_1, X_2, X_3 are awarded one seat each, as $\left\lfloor \frac{350}{180} \right\rfloor + \left\lfloor \frac{350}{180} \right\rfloor + \left\lfloor \frac{200}{180} \right\rfloor = 1 + 1 + 1 = 3.$ However, X_3 does *not* have the electoral strength to elect one of three representatives by cumulative voting, as its population does not exceed the threshold of exclusion, that is, $\frac{200}{900} \leq \frac{1}{3+1}$.

2. We can prove for more than two population subgroups that a subpopulation can never use cumulative voting to guarantee more seats than would be assigned to it by Jefferson apportionment.

Theorem

Consider a population of size P partitioned into subsets $X_1, X_2, \cdots, X_m$ of size $x_1, x_2, \cdots, x_m$, respectively, with a representative body of n seats to be determined. For $i = 1, \cdots, m$, if X_i has the electoral strength to guarantee at least k seats under cumulative voting, then X_i would receive at least k seats by the Jefferson apportionment.

Proof

Suppose population subgroup X_i has the electoral strength to guarantee at least k seats under cumulative voting. Recall that this means their fraction of the population must exceed the necessary threshold, that is,

$$\frac{x_i}{P} > \frac{k}{n+1}.$$

By contradiction, let us suppose that X_i receives fewer than k seats by Jefferson's apportionment. This means that for the divisor d that achieves the Jefferson apportionment, we have

$$\left\lfloor \frac{x_i}{d} \right\rfloor \leq k - 1.$$

Therefore, $\frac{x_i}{d} < k$ and so $d > \frac{x_i}{k}$.

The remaining seats are alloted to the remaining $m-1$ population subgroups, so $\sum_{j \neq i} \left\lfloor \frac{x_j}{d} \right\rfloor \geq n - (k-1)$. Therefore,

$$\begin{aligned} n-k+1 &\le \sum_{j\neq i}\left\lfloor\frac{x_j}{d}\right\rfloor \le \left\lfloor\sum_{j\neq i}\frac{x_j}{d}\right\rfloor \\ &= \left\lfloor\frac{\sum_{j\neq i} x_j}{d}\right\rfloor = \left\lfloor\frac{P-x_i}{d}\right\rfloor \le \frac{P-x_i}{d}. \end{aligned}$$

Thus, $d \le \dfrac{P-x_i}{n-k+1}$ which, in conjunction with the already established $d > \dfrac{x_i}{k}$, implies that

$$\frac{x_i}{k} < \frac{P-x_i}{n-k+1}.$$

It follows that

$$\frac{n-k+1}{k} < \frac{P}{x_i} - 1 \quad \Leftrightarrow \quad \frac{x_i}{P} < \frac{k}{n+1}.$$

But this contradicts the hypothesis that X_i has the electoral strength to guarantee at least k seats under cumulative voting! Hence, X_i must receive at least k seats by Jefferson's apportionment.

Conclusion

- Cumulative voting's electoral potential is never more advantageous than apportionment by Jefferson's method and would favor a majority over a minority in some situations.

- Cumulative voting gives the Webster-fair representation more often than not.

- Since cumulative voting's potential is "bounded above" in a sense by the Jefferson apportionment, we know that cumulative voting would provide no incentives for groups to splinter into smaller factions.

- Groups may find it advantageous to join forces in coalition. Jefferson's method is the one method of its type that invariably encourages coalitions: subgroups who join forces could gain but could never lose seats; Dean's, Hill's, and Webster's methods do not share this property.

- Cumulative voting might prove more palatable and practicable for use in the United States, with its two-party domination, where rigorous proportional representation methods would be generally unpopular as a means of assuring or bolstering representation by race.

- The nature of cumulative voting, with each voter having n votes, allows individual freedom to express multiple preferences that transcend a single party, race, or political issue. For example, a voter might not strategically vote to maximize the race's chances of electability, choosing instead to distribute votes for all competing interests, such as race, environmental policy, and candidate locality.

1.8 Fair majority voting - eliminate Gerrymandering

- "Districting determines elections, not votes."
- District boundaries are likely to be drawn to maximize the political advantage of the party temporarily dominant in public affairs (谁人掌权).

On one hand, every member of the House of Representatives represents a district.

On the other hand, representatives should represent their districts, their states, and their parties.

Rationale behind fair majority voting (FMV)

Voters cast ballots in single-member districts. In voting for a candidate, each gives a vote to the *candidate's* party.

1. The requisite number of representatives each party receives is calculated by Jefferson's method of apportionment on the basis of the total party votes.

2. The candidates elected, exactly one in each district, and the requisite number from each party are determined by a biproportional procedure.

2004 Connecticut congressional elections: votes.

District	1st	2nd	3d	4th	5th	Total
Republican	73,273	165,558	68,810	149,891	165,440	622,972
Democratic	197,964	139,987	199,652	136,481	105,505	779,589

- The Democratic candidates as a group out-polled the Republican candidates by over $156,000$ votes. However, only 2 were elected to the Republican's 3.

- By the method of Jefferson, the Republicans should have elected only 2 representatives while the Democratic 3.
- In the FMV approach, the 5 Republicans compete for their 2 seats while the 5 Democrats compete for their 3 seats.

Difficulty

- Among the Republicans, the 2 with the most votes have the strongest claims to seats; and similarly for the 3 Democrats with the most votes.
- However, some of these "party-winners" may be in the same district (see the 2nd district). Who, then, should be elected? Consider the 4th district where the race is very competitive

Method One

- All the Democratic votes should be scaled up until one more of the Democrats' justified-votes exceeds that of his/her Republican opponent.

- This happens when the scaling factor f or the Democratic Party is

$$\frac{149,892}{136,481} \approx 1.0983.$$

2004 Connecticut congressional elections: justified-votes (Democratic candidates' votes all scaled up, district-winners in bold).

District	multiplier	1st	2nd	3d	4th	5th
Republican	1	73,273	**165,558**	68,810	149,891	**165,440**
Democratic	1.0983	**217,416**	153,743	**219,270**	**149,892**	115,872

- Now, the Democratic Party wins the seat in the 4th district.

Method Two

- If every column (district) has exactly one party-winner, they are elected. In Connecticut, the second district has 2 party-winners, the fourth district none.

- Those in districts with more than one winner should be decreased, while the relative votes between the candidates in each district must remain the same.

2004 Connecticut congressional elections: justified-votes (2nd district's candidates' votes both scaled down, party-winners in bold). The scale down makes the Democratic candidate in the 4th district to emerge as the party-winner.

District	1st	2nd	3d	4th	5th
Republican	73,273	**161,410**	68,810	149,891	**165,440**
Democratic	**197,964**	136,480	**199,652**	**136,481**	105,505
multiplier	1	0.9749	1	1	1

Multiply the votes of the 2nd district by $136,480/139,987 \approx 0.9749$.

When there are *exactly 2 parties*, a very simple rule yields the FMV result.

(a) Compute the *percentage* of the votes for each of the 2 candidates in each district.

(b) Elect for each party the number of candidates it deserves, taking those with the highest percentages.

2004 Connecticut congressional elections: percentage of votes in districts (FMV winners in bold). Look at the percentages, rather than the actual vote count.

District	1st	2nd	3d	4th	5th
Republican	27.0%	**54.2%**	25.6%	52.3%	**61.1%**
Democratic	**73.0%**	45.8%	**74.4%**	**47.7%**	38.9%

- It eliminates the possibility of defining electoral districts for partisan political advantage. The great loss in district 1 for the Republicans leads to the loss of the seat in the 4th district.

Pros of FMV

- Since parties are allocated seats on the basis of their total votes in all districts, the necessity of strict equality in the number of inhabitants per district is attenuated (less important). This permits districting boundaries to be drawn that respect traditional political, administrative, natural frontiers, and communities of common interest.

- *FMV makes every vote count*. A state like Massachusetts has no Republican representatives at all seems ridiculous. Certainly at least 10% of the potential voters in Massachusetts have preferences for the Republican party, and should be represented by at least one of the state's 10 representatives.

- FMV would prevent a minority of voters from electing a majority in the House.

- If FMV becomes the electoral system, it is inconceivable that a major party would not present a candidate in every district. Even as little as 10% or 20% of the votes against a very strong candidate would help the opposition party to elect one of its candidates in another district. The anomaly of large numbers of unopposed candidates would disappear.

Cons of FMV

It is possible that a district's representative could have received fewer votes than her opponent in the district.

Results of 2002, 2004 and 2006 congressional elections.

	2002	2004	2006
Incumbent candidates reelected	380	389	371
Elected candidates ahead by $\geq 20\%$ of votes	356	361	318
Elected candidates ahead by $\geq 16\%$ of votes	375	384	348
Elected candidates ahead by $\leq 10\%$ of votes	36	22	56
Elected candidates ahead by $\leq 6\%$ of votes	24	10	39
Candidates elected without opposition	81	66	59
Republicans elected	228	232	202
Democrats elected	207	203	233

"Without opposition" means without the opposition of a Democrat or a Republican.

- California's last redistributing is particularly comfortable: every one of its districts has returned a candidate of the same party since 2002. Fifty were elected by a margin of at least 20% in 2002.

Mathematical formulation

Let $x = (x_{ij})$, with $x_{ij} = 1$ if the candidate of party i is elected in district j and $x_{ij} = 0$ otherwise.

FMV selects a $(0,1)$-valued matrix **x** that satisfies

$$\sum_i x_{ij} = 1,\ j = 1,2,\ldots,n,\ \ \sum_j x_{ij} = a_i,\ i = 1,2,\ldots,m.$$

Does a feasible delegation always exist?

	1st	2nd	3d	4th	5th	6th	7th	seats
party 1	+	+	+	+	+	+	+	2
party 2	+	+	+	+	+	+	+	1
party 3	+	+	+	O	O	O	O	4

- 4 districts (4th to 7th) cast all their votes for parties 1 and 2 that together deserve only 3 seats.

- Party 3 deserves 4 seats but receives all its votes from only 3 districts.

Feasible apportionment $\boldsymbol{a}$ for a given vote matrix V

A *problem* $(V, \boldsymbol{a})$ defined by an $m \times n$ matrix of votes V and an apportionment $\boldsymbol{a}$ satisfying $\sum a_i = n$ is said to be *feasible* if it has at least one feasible delegation x.

Justified-votes

Given row-multipliers $\lambda = (\lambda_i) > 0$ and column-multipliers $\rho = (\rho_j) > 0$, the matrices $\lambda \circ v = (\lambda_i v_{ij})$, $v \circ \rho = (v_{ij} \rho_j)$, and $\lambda \circ v \circ \rho = (\lambda_i v_{ij} \rho_j)$ are the *justified-votes* of the candidates of the different parties in the various districts.

1.9 Proportionality in matrix apportionment

Statement of the problem

The Zurich Canton Parliament is composed of seats that represent electoral districts as well as political parties.

- Each district, $j = 1, 2, \ldots, n$, is represented by a number of seats r_j that is proportional to its population (preset before the election).
- Each political party, $i = 1, 2, \ldots, m$, gets c_i seats proportional to its total number of votes (constitutional requirement).
- The vote count in district j of party i is denoted by v_{ij}. The vote counts are assembled into a vote matrix $V \in \mathbb{N}^{m \times n}$.

Vote Numbers for the Zurich City Council Election on February 12, 2006

		District									
		1 + 2	3	4 + 5	6	7 + 8	9	10	11	12	
Party	125	12	16	13	10	17	16	12	19	10	Total
SP	44	28,518	45,541	26,673	24,092	61,738	42,044	35,259	56,547	13,215	333,627
SVP	24	15,305	22,060	8,174	9,676	27,906	31,559	19,557	40,144	10,248	184,629
FDP	19	21,833	10,450	4,536	10,919	51,252	12,060	15,267	19,744	3,066	149,127
Greens	14	12,401	17,319	10,221	8,420	25,486	9,154	9,689	12,559	2,187	107,436
CVP	10	7,318	8,661	4,099	4,399	14,223	11,333	8,347	14,762	4,941	78,083
EVP	6	2,829	2,816	1,029	3,422	10,508	9,841	4,690	11,998	0	47,133
AL	5	2,413	7,418	9,086	2,304	5,483	2,465	2,539	3,623	429	35,760
SD	3	1,651	3,173	1,406	1,106	2,454	5,333	1,490	6,226	2,078	24,917
Total		92,268	117,438	65,224	64,338	199,050	123,789	96,838	165,603	36,164	960,712
Total no. of voters		7,891	7,587	5,269	6,706	12,180	7,962	8,344	9,106	3,793	68,838

- The district magnitudes are based on population counts and are known prior to the election. For example, district 9 has 16 seats.

- Each voter has as many votes as there are seats in the corresponding district. Voters in district 9 has 16 votes.

- The table does not include parties that do not pass the threshold of 5% of the votes in at least one district. So, total number of votes in Table $<$ number of actual votes.

- District 12 has the least percentage of population coming to vote (politically less engaged).

District marginals

District 12 has 5.5% of the voters ($3,793$ out of $68,838$), but is set to receive 8.0% of the seats (10 out of 125). This is because *population counts* from the basis for the allocation of seats to districts.

District quota

This is the proportion of seats that a party should receive within each district.

Example: The Greens received $9,154$ votes out of $123,789$ votes in district 9; so

$$\begin{aligned} & \text{district quota for the Greens in district 9} \\ = \; & 16 \times \frac{9,154}{123,789} = 1.18. \end{aligned}$$

District Quotas for the Zurich City Council Election on February 12, 2006

		District									
		1 + 2	3	4 + 5	6	7 + 8	9	10	11	12	
Party	125	12	16	13	10	17	16	12	19	10	Total
SP	44	3.71	6.20	5.32	3.74	5.27	5.43	4.37	6.49	3.65	44.19
SVP	24	1.99	3.01	1.63	1.50	2.38	4.08	2.42	4.61	2.83	24.45
FDP	19	2.84	1.42	0.90	1.70	4.38	1.56	1.89	2.27	0.85	17.81
Greens	14	1.61	2.36	2.04	1.31	2.18	1.18	1.20	1.44	0.60	13.92
CVP	10	0.95	1.18	0.82	0.68	1.21	1.46	1.03	1.69	1.37	10.41
EVP	6	0.37	0.38	0.21	0.53	0.90	1.27	0.58	1.38	0.00	5.62
AL	5	0.31	1.01	1.81	0.36	0.47	0.32	0.31	0.42	0.12	5.13
SD	3	0.21	0.43	0.28	0.17	0.21	0.69	0.18	0.71	0.57	3.47
Total		12.00	16.00	13.00	10.00	17.00	16.00	12.00	19.00	10.00	125.00

- Summing all district quota for the Greens across all 12 districts gives the sum 13.92.

- The percentage of population count of each district is *not* the same as the district's percentage of voters count, reflecting the varying levels of engagement in politics in the districts.

- Suppose we use the total aggregate votes across all districts as the basis for computing the quota for the Greens, we obtain

$$\text{eligible quota for the Greens (out of 125 seats)} = \frac{107,436}{960,712} \times 125 = 13.97 \text{ (slightly different from 13.92).}$$

$$\text{Also, eligible quota for the Greens in district 9} = \frac{9,154}{960,712} \times 125 = 1.19.$$

Super apportionment

- Party seats are allocated on the basis of the total party ballots in the whole electoral region.
- Respond to the constitutional demand that all voters contribute to the electoral outcome equally, no matters whether voters cast their ballots in districts that are large or small.
- For a given party, we divide the vote counts in each district by its corresponding district magnitude (rounding to the nearest integer), and sum over all districts. This gives the *support size* for each party – *number of people supporting a party*.

Zurich City Parliament election of 12 February 2006, Superapportionment:

	SP	SVP	FDP	Greens	CVP	EVP	AL	SD	City divisor
Support size	23180	12633	10300	7501	5418	3088	2517	1692	530
Seats 125	44	24	19	14	10	6	5	3	

For example, consider Party SP:

$$\frac{28,518}{12} + \underset{\substack{\uparrow \\ \text{each voter} \\ \text{in district 2} \\ \text{has 16 votes}}}{\frac{45,541}{16}} + \cdots + \frac{56,547}{19} + \frac{13,215}{10} \approx 23,180$$

Apply the divisor 530 so that

$$\begin{aligned} & \left[\frac{23,180}{530}\right] + \left[\frac{12,633}{530}\right] + \cdots + \left[\frac{2,517}{530}\right] + \left[\frac{1,692}{530}\right] \\ = \; & [43.7] + [23.8] + \cdots + [4.7] + [3.19] \\ = \; & 44 + 24 + \cdots + 5 + 3 = 125. \end{aligned}$$

Subapportionment

Concerned with the allocation of the seats to the parties within the districts.

- Each vote count of a party in a district is divided by its corresponding district divisor and party divisor. The quotient is rounded using the standard apportionment schemes to obtain the seat number.

Mathematical formulation

$\boldsymbol{r} = (r_1 \ldots r_m) > \mathbf{0}$ and $\boldsymbol{c} = (c_1 \ldots c_n) > \mathbf{0}$ are integer-valued vectors whose sums are equal. That is,

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j = h = \text{total number of seats.}$$

We need to find row multipliers λ_i and column multipliers μ_j such that

$$x_{ij} = [\lambda_i v_{ij} \mu_j], \text{ for all } i \text{ and } j,$$

such that the row-sum and column-sum requirements are fulfilled. Here, [ ] denotes some form of rounding.

An apportionment solution is a matrix $X = (x_{ij})$, where $x_{ij} > 0$ and integer-valued such that

$$\sum_{j=1}^{n} x_{ij} = r_i \text{ for all } i \text{ and } \sum_{i=1}^{m} x_{ij} = c_j \text{ for all } j.$$

- Assign integer values to the elements of a matrix that are proportional to a given input matrix, such that a set of row-sum and column-sum requirements are fulfilled.

- In a divisor-based method for biproportional apportionment, the problem is solved by computing appropriate row-divisors and column-divisors, and by rounding the quotients.

Result of Zurich City Council Election on February 12, 2006

		District									
		1 + 2	3	4 + 5	6	7 + 8	9	10	11	12	
Party	125	12	16	13	10	17	16	12	19	10	Divisor $1/\lambda_i$
SP	44	4	7	5	4	5	6	4	6	3	1.006
SVP	24	2	3	2	1	2	4	3	4	3	1.002
FDP	19	3	1	1	2	5	2	2	2	1	1.010
Greens	14	2	3	2	1	2	1	1	1	1	0.970
CVP	10	1	1	1	1	1	1	1	2	1	1.000
EVP	6	0	0	0	1	1	1	1	2	0	0.880
AL	5	0	1	2	0	1	0	0	1	0	0.800
SD	3	0	0	0	0	0	1	0	1	1	1.000
Divisor $1/\mu_j$		7,000	6,900	5,000	6,600	11,200	7,580	7,800	9,000	4,000	

The divisors are those that were published by the Zurich City administration. In district $1+2$, the Greens had $12,401$ ballots and were awarded by two seats. This is because $12,401/(7,000\times0.97) \approx 1.83$, which is rounded up to 2.

- For the politically less active districts, like district 12, the divisor (number of voters represented by each seat) is smaller ($1/\mu_j = 4,000$).

- The matrix apportionment problem can be formulated as an integer programming problem with constraints, which are given by the row sums and column sums. We solve for the multipliers λ_i and μ_j through an iterative algorithm.