**Mathematical Models in Economics and Finance**

**Topic 3 – Fundamental theorem of asset pricing**

3.1 Law of one price and Arrow securities

3.2 No-arbitrage theory and risk neutral probability measure

3.3 Valuation of contingent claims

3.4 Binomial option pricing model

## 3.1 Law of one price and Arrow securities

- The initial prices of $M$ risky securities, denoted by $S_1(0), \cdots, S_M(0)$, are positive scalars that are known at $t = 0$.

- Their values at $t = 1$ are random variables, which are defined with respect to a sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$ of $K$ possible outcomes (or states of the world).

- At $t = 0$, the investors know the list of all possible outcomes, but which outcome does occur is revealed only at the end of the investment period $t = 1$.

- A probability measure $P$ satisfying $P(\omega) > 0$, for all $\omega \in \Omega$, is defined on $\Omega$.

- We use $\boldsymbol{S}$ to denote the price process $\{\boldsymbol{S}(t) : t = 0, 1\}$, where $\boldsymbol{S}(t)$ is the row vector $\boldsymbol{S}(t) = (S_1(t)\ S_2(t) \cdots S_M(t))$.

Consider 3 risky assets with time-0 price vector

$$\boldsymbol{S}(0) = (S_1(0) \quad S_2(0) \quad S_3(0)) = (1 \quad 2 \quad 3).$$

At time 1, there are 2 possible states of the world:

$$\begin{aligned}
\omega_1 &= \text{Hang Seng index is at or above } 22,000 \\
\omega_2 &= \text{Hang Seng index falls below } 22,000.
\end{aligned}$$

If $\omega_1$ occurs, then

$$\boldsymbol{S}(1;\omega_1) = (1.2 \quad 2.1 \quad 3.4);$$

otherwise, $\omega_2$ occurs and

$$\boldsymbol{S}(1;\omega_2) = (0.8 \quad 1.9 \quad 2.9).$$

- The possible values of the asset price process at $t = 1$ are listed in the following $K \times M$ matrix

$$S(1;\Omega) = \begin{pmatrix} S_1(1;\omega_1) & S_2(1;\omega_1) & \cdots & S_M(1;\omega_1) \\ S_1(1;\omega_2) & S_2(1;\omega_2) & \cdots & S_M(1;\omega_2) \\ \cdots & \cdots & \cdots & \cdots \\ S_1(1;\omega_K) & S_2(1;\omega_K) & \cdots & S_M(1;\omega_K) \end{pmatrix}.$$

- Since the assets are limited liability securities, the entries in $S(1;\Omega)$ are non-negative scalars.

- Existence of a strictly positive riskless security or bank account, whose value is denoted by $S_0$. Without loss of generality, we take $S_0(0) = 1$ and the value at time 1 to be $S_0(1) = 1 + r$, where $r \geq 0$ is the deterministic interest rate over one period.

- We define the discounted price process by

$$S^*(t) = S(t)/S_0(t), \quad t = 0, 1,$$

that is, we use the riskless security as the *numeraire* or *accounting unit*.

- The payoff matrix of the discounted price processes of the $M$ risky assets and the riskless security can be expressed in the form

$$\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & S_1^*(1;\omega_1) & \cdots & S_M^*(1;\omega_1) \\ 1 & S_1^*(1;\omega_2) & \cdots & S_M^*(1;\omega_2) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & S_1^*(1;\omega_K) & \cdots & S_M^*(1;\omega_K) \end{pmatrix}.$$

*Trading strategies*

- An investor adopts a *trading strategy* by selecting a portfolio of the $M$ assets at time 0. A trading strategy is characterized by asset holding in the portfolio.

- The number of units of asset $m$ held in the portfolio from $t = 0$ to $t = 1$ is denoted by $h_m, m = 0, 1, \cdots, M$.

- The scalars $h_m$ can be positive (long holding), negative (short selling) or zero (no holding).

- An investor is endowed with an initial endowment $V_0$ at time 0 to set up the trading portfolio. How do we choose the portfolio holding of the assets such that the expected portfolio value at time 1 is maximized?

*Portfolio value process*

- Let $V = \{V_t : t = 0, 1\}$ denote the value process that represents the total value of the portfolio over time. It is seen that

$$V_t = h_0 S_0(t) + \sum_{m=1}^{M} h_m S_m(t), \quad t = 0, 1.$$

- Let $G$ be the random variable that denotes the total gain generated by investing in the portfolio. We then have

$$G = h_0 r + \sum_{m=1}^{M} h_m \Delta S_m, \quad \Delta S_m = S_m(1) - S_m(0).$$

*Account balancing*

- If there is no withdrawal or addition of funds within the investment horizon, then

$$V_1 = V_0 + G.$$

- Suppose we use the bank account as the numeraire, and define the discounted value process by $V_t^* = V_t/S_0(t)$ and discounted gain by $G^* = V_1^* - V_0^*$, we then have

$$\begin{aligned} V_t^* &= h_0 + \sum_{m=1}^{M} h_m S_m^*(t), \quad t = 0, 1; \\ G^* &= V_1^* - V_0^* = \sum_{m=1}^{M} h_m \Delta S_m^*. \end{aligned}$$

## Dominant trading strategies

A trading strategy $\mathcal{H}$ is said to be *dominant* if there exists another trading strategy $\widehat{\mathcal{H}}$ such that

$$V_0 = \widehat{V}_0 \qquad \text{and} \quad V_1(\omega) > \widehat{V}_1(\omega) \quad \text{for all } \omega \in \Omega.$$

- Suppose $\mathcal{H}$ dominates $\widehat{\mathcal{H}}$, we define a new trading strategy $\widetilde{\mathcal{H}} = \mathcal{H} - \widehat{\mathcal{H}}$. Let $\widetilde{V}_0$ and $\widetilde{V}_1$ denote the portfolio value of $\widetilde{\mathcal{H}}$ at $t = 0$ and $t = 1$, respectively. We then have $\widetilde{V}_0 = 0$ and $\widetilde{V}_1(\omega) > 0$ for all $\omega \in \Omega$.

- This trading strategy is dominant since it dominates the strategy which starts with zero value and does no investment at all.

- Equivalent definition: A dominant trading strategy exists if and only if there exists a trading strategy satisfying $V_0 < 0$ and $V_1(\omega) \geq 0$ for all $\omega \in \Omega$.

## Asset span

- Consider two risky securities whose discounted payoff vectors are

$$\boldsymbol{S}_1^*(1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{S}_2^*(1) = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

- The payoff vectors are used to form the discounted terminal payoff matrix

$$S^*(1) = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

- Let the current prices be represented by the row vector $\boldsymbol{S}^*(0) = (1 \quad 2)$.

- We write $\boldsymbol{h}$ as the column vector whose entries are the portfolio holding of the securities in the portfolio. The trading strategy is characterized by specifying $\boldsymbol{h}$. The current portfolio value and the discounted portfolio payoff are given by $\boldsymbol{S}^*(0)\boldsymbol{h}$ and $S^*(1)\boldsymbol{h}$, respectively.

- The set of all portfolio payoffs via different holding of securities is called the *asset span* $\mathcal{S}$. The asset span is seen to be the column space of the payoff matrix $S^*(1)$, which is a subspace in $\mathbb{R}^K$ spanned by the columns of $S^*(1)$.

$$\begin{aligned}\text{asset span} &= \text{column space of } S^*(1)\\ &= \text{span}(\boldsymbol{S}_1^*(1)\cdots\boldsymbol{S}_M^*(1))\end{aligned}$$

Recall that

$$\begin{aligned}\text{column rank} &= \text{dimension of column space}\\ &= \text{number of independent columns.}\end{aligned}$$

It is well known that number of independent columns = number of independent rows, so column rank = row rank = rank $\leq \min(K, M)$.

- In the above numerical example, the asset span consists of all vectors of the form $h_1\begin{pmatrix}1\\2\\3\end{pmatrix}+h_2\begin{pmatrix}3\\1\\2\end{pmatrix}$, where $h_1$ and $h_2$ are scalars.

*Redundant security and complete model*

- If the discounted terminal payoff vector of an added security lies inside $\mathcal{S}$, then its payoff can be expressed as a linear combination of $\boldsymbol{S}_1^*(1)$ and $\boldsymbol{S}_2^*(1)$. In this case, it is said to be a *redundant security*. The added security is said to be replicable by some combination of existing securities.

- A securities model is said to be *complete* if every payoff vector lies inside the asset span. That is, all new securities can be replicated by existing securities. This occurs if and only if the dimension of the asset span equals the number of possible states, that is, the asset span becomes the whole $\mathbb{R}^K$.

Given the securities model with 4 risky securities and 3 possible states of world:

$$S^*(1;\Omega) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 8 & 11 \end{pmatrix}, \quad \boldsymbol{S}^*(0) = (1 \quad 2 \quad 4 \quad 7).$$

asset span = span$(\boldsymbol{S}_1^*(1), \boldsymbol{S}_2^*(1))$, which has dimension $= 2 < 3 =$ number of possible states. Hence, the securities model is not complete! For example, the following security

$$\boldsymbol{S}_\beta^*(1;\Omega) = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

does not lie in the asset span of the securities model. There is no solution to

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 8 & 11 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

## Pricing problem

Given a new security that is replicable by existing securities, its price with reference to a given securities model is given by the cost of setting up the replicating portfolio.

Consider a new security with discounted payoff at $t = 1$ as given by

$$\boldsymbol{S}^*_\alpha(1;\Omega) = \begin{pmatrix} 5 \\ 8 \\ 13 \end{pmatrix},$$

which is seen to be

$$\boldsymbol{S}^*_\alpha(1;\Omega) = \boldsymbol{S}^*_2(1;\Omega) + \boldsymbol{S}^*_3(1;\Omega) = \boldsymbol{S}^*_1(1;\Omega) + 2\boldsymbol{S}^*_2(1;\Omega).$$

This new security is redundant. Unfortunately, the price of this security can be either

$$\boldsymbol{S}^*_2(0) + \boldsymbol{S}^*_3(0) = 6 \quad \text{or} \quad \boldsymbol{S}^*_1(0) + 2\boldsymbol{S}^*_2(0) = 5.$$

There are two possible prices, corresponding to two different choices of replicating portfolios.

*Question*

How to modify $\boldsymbol{S}^*(0)$ so as to avoid the above ambiguity that portfolios with the same terminal payoff have different initial prices (failure of law of one price).

Note that $S_3^*(1;\Omega) = S_1^*(1;\Omega)+S_2^*(1;\Omega)$ and $S_4^*(1;\Omega) = S_1^*(1;\Omega)+S_3^*(1;\Omega)$, both the third and fourth security are redundant securities. To achieve the law of one price, we modify $S_3^*(0)$ and $S_4^*(0)$ such that

$$S_3^*(0) = S_1^*(0) + S_2^*(0) = 3 \quad \text{and} \quad S_4^*(0) = 2S_1^*(0) + S_2^*(0) = 4.$$

*Conjecture*

If there are no redundant securities, then the law of one price holds. Mathematically, non-existence of redundant securities means $S^*(1;\Omega)$ has full column rank. That is, column rank = number of columns. This gives a sufficient condition for "law of one price".

**Law of one price (pricing of securities that lie in the asset span)**

1. The law of one price states that all portfolios with the same terminal payoff have the same initial price.

2. Consider two portfolios with different portfolio weights $\boldsymbol{h}$ and $\boldsymbol{h}'$. Suppose these two portfolios have the same discounted payoff, that is, $S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}'$, then the law of one price infers that $\boldsymbol{S}^*(0)\boldsymbol{h} = \boldsymbol{S}^*(0)\boldsymbol{h}'$.

3. The trading strategy $\boldsymbol{h}$ is obtained by solving

$$S^*(1)\boldsymbol{h} = \boldsymbol{S}^*_\alpha(1).$$

Solution exists if $\boldsymbol{S}^*_\alpha(1)$ lies in the asset span. Uniqueness of solution is equivalent to null space of $S^*(1)$ having zero dimension. There is only one trading strategy that replicates the security with discounted terminal payoff $\boldsymbol{S}^*_\alpha(1)$. In this case, the law of one price always holds.

## Law of one price and dominant trading strategy

If the law of one price fails, then it is possible to have two trading strategies $\boldsymbol{h}$ and $\boldsymbol{h}'$ such that $S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}'$ but $\boldsymbol{S}^*(0)\boldsymbol{h} > \boldsymbol{S}^*(0)\boldsymbol{h}'$.

Let $G^*(\omega)$ and $G^{*\prime}(\omega)$ denote the respective discounted gain corresponding to the trading strategies $\boldsymbol{h}$ and $\boldsymbol{h}'$. We then have $G^{*\prime}(\omega) > G^*(\omega)$ for all $\omega \in \Omega$, so there exists a dominant trading strategy. The corresponding dominant trading strategy is $\boldsymbol{h}' - \boldsymbol{h}$ so that $V_0 < 0$ but $V_1^*(\omega) = 0$ for all $\omega \in \Omega$.

Hence, the non-existence of dominant trading strategy implies the law of one price. However, the converse statement does not hold.

[See later numerical example.]

## Pricing functional

- Given a discounted portfolio payoff $\boldsymbol{x}$ that lies inside the asset span, the payoff can be generated by some linear combination of the securities in the securities model. We have $\boldsymbol{x} = S^*(1)\boldsymbol{h}$ for some $\boldsymbol{h} \in \mathbb{R}^M$. Existence of the solution $\boldsymbol{h}$ is guaranteed since $\boldsymbol{x}$ lies in the asset span, or equivalently, $\boldsymbol{x}$ lies in the column space of $S^*(1)$.

- The current value of the portfolio is $\boldsymbol{S}^*(0)\boldsymbol{h}$, where $\boldsymbol{S}^*(0)$ is the initial price vector.

- We may consider $\boldsymbol{S}^*(0)\boldsymbol{h}$ as a pricing functional $F(\boldsymbol{x})$ on the payoff $\boldsymbol{x}$. If the law of one price holds, then the pricing functional is single-valued. Furthermore, it is a linear functional, that is,

$$F(\alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2) = \alpha_1 F(\boldsymbol{x}_1) + \alpha_2 F(\boldsymbol{x}_2)$$

  for any scalars $\alpha_1$ and $\alpha_2$ and payoffs $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

## Arrow security and state price

- Let $\boldsymbol{e}_k$ denote the $k^{\text{th}}$ coordinate vector in the vector space $\mathbb{R}^K$, where $\boldsymbol{e}_k$ assumes the value 1 in the $k^{\text{th}}$ entry and zero in all other entries. The vector $\boldsymbol{e}_k$ can be considered as the discounted payoff vector of a security, and it is called the Arrow security of state $k$. This Arrow security has unit payoff when state $k$ occurs and zero payoff otherwise.

- Suppose the securities model is complete (all Arrow securities lie in the asset span) and the law of one price holds, then the pricing functional $F$ assigns unique value to each Arrow security. We write $s_k = F(\boldsymbol{e}_k)$, which is called the state price of state $k$. Note that state price must be non-negative. Take

$$\boldsymbol{S}_\alpha^*(1) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{pmatrix} = \sum_{k=1}^K \alpha_k \boldsymbol{e}_k, \text{ then}$$

$$S_\alpha^*(0) = F(\boldsymbol{S}_\alpha^*(1)) = F\left(\sum_{k=1}^K \alpha_k \boldsymbol{e}_k\right) = \sum_{k=1}^K \alpha_k F(\boldsymbol{e}_k) = \sum_{k=1}^K \alpha_k s_k.$$

**Example – State prices**

Given $F\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = 7$ and $F\left(\begin{pmatrix} 4 \\ 2 \end{pmatrix}\right) = 9$, find $F\left(\begin{pmatrix} 5 \\ 3 \end{pmatrix}\right)$.

By the linear property of pricing functional, we deduce that

$$F\left(\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 9 - 7 = 2 \text{ so that } s_1 = 2;$$

$$\begin{aligned} F\left(\frac{1}{2}\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\right) = F\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= \frac{1}{2}\left[F\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) - 3F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right] \\ &= \frac{1}{2}(7 - 3 \times 2) = \frac{1}{2} \end{aligned}$$

so that $s_2 = \frac{1}{2}$.

By the linear property of pricing functional, the fair price of $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ is given by

$$F\left(\begin{pmatrix} 5 \\ 3 \end{pmatrix}\right) = 5F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + 3F\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 5s_1 + 3s_2 = \frac{23}{2}.$$

The actual probabilities of occurrence of the two states are irrelevant in the pricing of the new contingent claim $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

Lastly, we observe the following relation between the state price vector $\begin{pmatrix} 2 & \frac{1}{2} \end{pmatrix}$, payoff matrix and initial price vector $(7 \quad 9)$.

$$\begin{pmatrix} 2 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 9 \end{pmatrix}.$$

## Summary

Given a securities model endowed with $S^*(1;\Omega)$ and $\boldsymbol{S}^*(0)$, can we find a trading strategy to form a portfolio that replicates a new security $\boldsymbol{S}^*_\alpha(1;\Omega)$ (also called a contingent claim) that is outside the universe of the $M$ available risky securities in the securities model?

*Replication* means the terminal payoff of the replicating portfolio matches with that of the contingent claim under all scenarios of occurrence of the state of the world at $t = 1$.

1. Formation of the replicating portfolio is possible if we have *existence of solution* $\boldsymbol{h}$ to the following system

$$S^*(1;\Omega)\boldsymbol{h} = \boldsymbol{S}^*_\alpha(1;\Omega).$$

This is equivalent to the fact that "$\boldsymbol{S}^*_\alpha(1;\Omega)$ lies in the asset span (column space) of $S^*(1;\Omega)$". The solution $\boldsymbol{h}$ is the corresponding trading strategy. Note that $\boldsymbol{h}$ may not be unique.

*Completeness of securities model*

If all contingent claims are replicable, then the securities model is said to be *complete*. This is equivalent to

$$\text{dim(asset span)} = K = \text{number of possible states},$$

that is, asset span $= \mathbb{R}^K$. In this case, solution $\boldsymbol{h}$ always exists.

2. Uniqueness of trading strategy

If $\boldsymbol{h}$ is unique, then there is only one trading strategy that generates the replicating portfolio. This occurs when the columns of $S^*(1;\Omega)$ are independent. Equivalently, column rank $= M$ and all securities are non-redundant. Mathematically, this is equivalent to observe that the homogeneous system

$$S^*(1;\Omega)\boldsymbol{h} = \boldsymbol{0}$$

admits only the trivial zero solution. In other words, the dimension of the null space of $S^*(1;\Omega)$ is zero.

When we have unique solution $\boldsymbol{h}$, the initial cost of setting up the replicating portfolio (price at time 0) as given by $\boldsymbol{S}^*(0)\boldsymbol{h}$ is unique. In this case, law of one price holds.

*Matrix properties of $S^*(1)$ that are related to financial economics concepts*

The securities model is endowed with

(i) discounted terminal payoff matrix $= \left( \boldsymbol{S}_1^*(1) \quad \cdots \quad \boldsymbol{S}_M^*(1) \right)$, and
(ii) initial price vector; $\boldsymbol{S}^*(0) = (S_1^*(0) \cdots S_M^*(0))$.

Recall that

$$\text{column rank } \leq \min(K, M)$$

where $K$ = number of possible states, $M$ = number of risky securities.

List of terms: redundant securities, complete model, replicating portfolio, asset holding, asset span, law of one price, dominant trading strategy, Arrow securities, state prices

Given a risky security with the discounted terminal payoff $S_\alpha^*(1)$, we are interested to explore the existence and uniqueness of solution to

$$S^*(1)\boldsymbol{h} = \boldsymbol{S}_\alpha^*(1).$$

Here, $\boldsymbol{h}$ is the asset holding of the portfolio that replicates $\boldsymbol{S}_\alpha^*(1)$.

(i) column rank $= K$

asset span $= \mathbb{R}^K$, so the securities model is complete. Any risky securities is replicable. In this case, solution $\boldsymbol{h}$ always exists.

(ii) column rank $= M$ (all columns of $S^*(1)$ are independent)

All securities are non-redundant. In this case, $\boldsymbol{h}$ may or may not exist. However, if $\boldsymbol{h}$ exists, then it must be unique. The price of any replicable security is unique.

(iii) column rank $< K$

Solution $\boldsymbol{h}$ exists if and only if $\boldsymbol{S}_\alpha^*(1)$ lies in the asset span. However, there is no guarantee for the uniqueness of solution.

(iv) column rank $< M$

Existence of redundant securities, so the law of one price may fail.

## Law of one price revisited

Law of one price holds if and only if solution to

$$\boldsymbol{\pi} S^*(1) = \boldsymbol{S}(0) \tag{A}$$

exists.

1. Suppose solution to (A) exists, let $\boldsymbol{h}$ and $\boldsymbol{h'}$ be two trading strategies such that their respective discounted terminal payoff $\boldsymbol{V}$ and $\boldsymbol{V}'$ are the same. That is,

   $$S^*(1)\boldsymbol{h} = \boldsymbol{V} = \boldsymbol{V}' = S^*(1)\boldsymbol{h'}.$$

   Since $\boldsymbol{\pi}$ exists, we then have

   $$\boldsymbol{\pi} S^*(1)(\boldsymbol{h} - \boldsymbol{h'}) = 0.$$

   Noting that $\boldsymbol{\pi} S^*(1) = \boldsymbol{S}(0)$, we obtain

   $$\boldsymbol{S}(0)(\boldsymbol{h} - \boldsymbol{h'}) = 0 \quad \text{so that } V_0 = V_0'.$$

2. Suppose solution to $(A)$ does not exist for the given $\boldsymbol{S}(0)$, this implies that $\boldsymbol{S}(0)$ that does not lie in the row space of $S^*(1)$. The row space of $S^*(1)$ does not span the whole $\mathbb{R}^M$. Therefore, dim(row space of $S^*(1)) < M$, where $M$ is the number of securities = number of columns in $S^*(1)$.

   Recall that

$$\text{dim(null space of } S^*(1)) + \text{rank}(S^*(1)) = M$$

   so that dim(null space of $S^*(1)) > 0$.

   Hence, there exists non-zero solution $\boldsymbol{h}$ to

$$S^*(1)\boldsymbol{h} = \boldsymbol{0}.$$

   Note that $\boldsymbol{h}$ is orthogonal to all rows of $S^*(1)$. This is consistent with the property that

$$\text{row space} \;=\; \text{orthogonal complement of null space.}$$

We claim that one can always find non-zero solution $\boldsymbol{h}$ that is not orthogonal to $\boldsymbol{S}(0)$. If otherwise, $\boldsymbol{S}(0)$ lies in the orthogonal complement of the null space (that is, row space). This leads to a contradiction.

Consider the above choice of non-zero $\boldsymbol{h}$, where $S^*(1)\boldsymbol{h} = 0$. We split $\boldsymbol{h} = \boldsymbol{h}_1 - \boldsymbol{h}_2$, where $\boldsymbol{h}_1 \neq \boldsymbol{h}_2$. Then there exist two distinct trading strategies such that

$$S^*(1)\boldsymbol{h}_1 = S^*(1)\boldsymbol{h}_2.$$

The two strategies have the same discounted terminal payoff under all states of the world. However, their initial prices are unequal since

$$\boldsymbol{S}(0)\boldsymbol{h}_1 \neq \boldsymbol{S}(0)\boldsymbol{h}_2,$$

by virtue of the property: $\boldsymbol{S}(0)\boldsymbol{h} \neq 0$. Hence, the law of one price does not hold.

## Linear pricing measure

We consider securities models with the inclusion of the riskfree security. A non-negative row vector $\boldsymbol{q} = (q(\omega_1) \cdots q(\omega_K))$ is said to be a linear pricing measure if for every trading strategy the portfolio values at $t = 0$ and $t = 1$ are related by

$$V_0^* = \sum_{k=1}^{K} q(\omega_k) V_1^*(\omega_k).$$

*Remark*

Here, the same initial price $V_0^*$ is always resulted as there is no dependence of $V_0^*$ on the asset holding of the portfolio. Two portfolios with the same terminal payoff for all states of the world would have the same price. Implicitly, this implies that the law of one price holds. The rigorous justification of the above statement will be presented later. Note that $\boldsymbol{q}$ is not required to be unique.

1. Suppose we take the holding amount of every risky security to be zero, thereby $h_1 = h_2 = \cdots = h_M = 0$, then

$$V_0^* = h_0 = \sum_{k=1}^{K} q(\omega_k) h_0$$

so that

$$\sum_{k=1}^{K} q(\omega_k) = 1.$$

2. Suppose that the securities model is complete. By taking the portfolio to have the same terminal payoff as that of the $k^{\text{th}}$ Arrow security, we obtain

$$s_k = q(\omega_k), \quad k = 1, 2, \cdots, K.$$

That is, the state price of the $k^{\text{th}}$ state is simply $q(\omega_k)$. This is not surprising when we compare

$$V_0^* = \sum_{k=1}^{K} q(\omega_k) V_1^*(\omega_k) \quad \text{and} \quad S_\alpha^*(0) = \sum_{k=1}^{K} \alpha_k s_k.$$

- Since we have taken $q(\omega_k) \geq 0, k = 1, \cdots, K$, and their sum is one, we may interpret $q(\omega_k)$ as a probability measure on the sample space $\Omega$.

- Note that $q(\omega_k)$ is not related to the actual probability of occurrence of the state $k$, though the current security price is given by the discounted expectation of the security payoff one period later under the linear pricing measure.

- By taking the portfolio weights to be zero except for the $m^{\text{th}}$ security, we have

$$S_m^*(0) = \sum_{k=1}^{K} q(\omega_k) S_m^*(1;\omega_k), \quad m = 0, 1, \cdots, M.$$

In matrix form, we have

$$\widehat{\boldsymbol{S}}^*(0) = \boldsymbol{q}\widehat{S}^*(1;\Omega), \quad \boldsymbol{q} \geq \mathbf{0}.$$

*Numerical example*

Take $S^*(1) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\boldsymbol{S}^*(0) = (1 \quad 1\frac{1}{3})$, then

$$\boldsymbol{q} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

is a linear pricing measure. The linear pricing measure is not unique! Actually, we have $q(\omega_1) = \frac{1}{3}$ and $q_2(\omega_2) + q(\omega_3) = \frac{2}{3}$.

- The securities model is not complete. Though $\boldsymbol{e}_1$ is replicable and its initial price is $\frac{1}{3}$, but $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are not replicable so the state price of $\omega_2$ and $\omega_3$ do not exist.

Suppose we add the new risky security with discounted terminal payoff $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and initial price $\frac{2}{3}$ into the securities model, then the securities model becomes complete. We have the following state prices

$$s_1 = \frac{1}{3}, \quad s_2 = -\frac{1}{3} \quad s_3 = 1.$$

In this case, law of one price holds but dominant trading strategy exists. For example, we may take

$$V_1^*(\omega) = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} > \mathbf{0}, \quad V_0^* = 3s_1 + 6s_2 + s_3 = 0.$$

*Remark* To explore "law of one price", one has to consider the existence of solution to the linear system of equations: $\boldsymbol{S}^*(0) = \boldsymbol{\pi} S^*(1)$.

**Example – Law of one price**

Take $\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 2 & 6 & 9 \\ 1 & 3 & 3 & 7 \\ 1 & 6 & 12 & 19 \end{pmatrix}$, the sum of the first 3 columns gives the fourth column. The first column corresponds to the discounted terminal payoff of the riskfree security under the 3 possible states of the world. The third risky security is a redundant security.

Let $\widehat{\boldsymbol{S}}^*(0) = (1 \quad 2 \quad 3 \quad k)$. We observe that solution to

$$(1 \quad 2 \quad 3 \quad k) = (\pi_1 \quad \pi_2 \quad \pi_3) \begin{pmatrix} 1 & 2 & 6 & 9 \\ 1 & 3 & 3 & 7 \\ 1 & 6 & 12 & 19 \end{pmatrix} \qquad (A)$$

exists if and only if $k = 6$. That is, $S_3^*(0) = S_0^*(0) + S_1^*(0) + S_2^*(0)$.

When $k \neq 6$, the law of one price does not hold. The last equation: $9\pi_1 + 7\pi_2 + 19\pi_3 = k \neq 6$ is inconsistent with the first 3 equations. One may check that $(1 \quad 2 \quad 3 \quad 6)$ can be expressed as a linear combination of the rows of $\widehat{S}^*(1;\Omega)$.

We consider the linear system

$$\widehat{\boldsymbol{S}}^*(0) = \boldsymbol{\pi}\widehat{S}^*(1;\Omega),$$

solution exists if and only if $\widehat{\boldsymbol{S}}^*(0)$ lies in the row space of $\widehat{S}^*(1;\Omega)$. Uniqueness follows if the rows of $\widehat{S}^*(1;\Omega)$ are independent.

Since

$$\boldsymbol{S}_3^*(1;\Omega) = \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_1^*(1;\Omega) + \boldsymbol{S}_2^*(1;\Omega),$$

the third risky security is replicable by holding one unit of each of the riskfree security and the first two risky securities. The initial price must observe the same relation in order that the law of one price holds.

Here, we have redundant securities. Actually, one may show that the law of one price holds if and only if we have existence of solution to the linear system. In this example, when $k = 6$, we obtain

$$\boldsymbol{\pi} = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & -\frac{1}{6} \end{pmatrix}.$$

This is *not* a linear pricing measure.

**Example – Law of one price holds while dominant trading strategies exist**

Consider a securities model with 2 risky securities and the riskfree security, and there are 3 possible states. The current discounted price vector $\widehat{\boldsymbol{S}}^*(0)$ is $(1 \quad 4 \quad 2)$ and the discounted payoff matrix at $t = 1$ is $\widehat{S}^*(1) = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 4 \end{pmatrix}$. Here, the law of one price holds since the only solution to $\widehat{S}^*(1)\boldsymbol{h} = \boldsymbol{0}$ is $\boldsymbol{h} = \boldsymbol{0}$. This is because the columns of $\widehat{S}^*(1)$ are independent so that the dimension of the nullspace of $\widehat{S}^*(1)$ is zero.

The linear pricing probabilities $q(\omega_1), q(\omega_2)$ and $q(\omega_3)$, if exist, should satisfy the following equations:

$$\begin{aligned} 1 &= q(\omega_1) + q(\omega_2) + q(\omega_3) \\ 4 &= 4q(\omega_1) + 3q(\omega_2) + 2q(\omega_3) \\ 2 &= 3q(\omega_1) + 2q(\omega_2) + 4q(\omega_3). \end{aligned}$$

Solving the above equations, we obtain $q(\omega_1) = q(\omega_2) = 2/3$ and $q(\omega_3) = -1/3$.

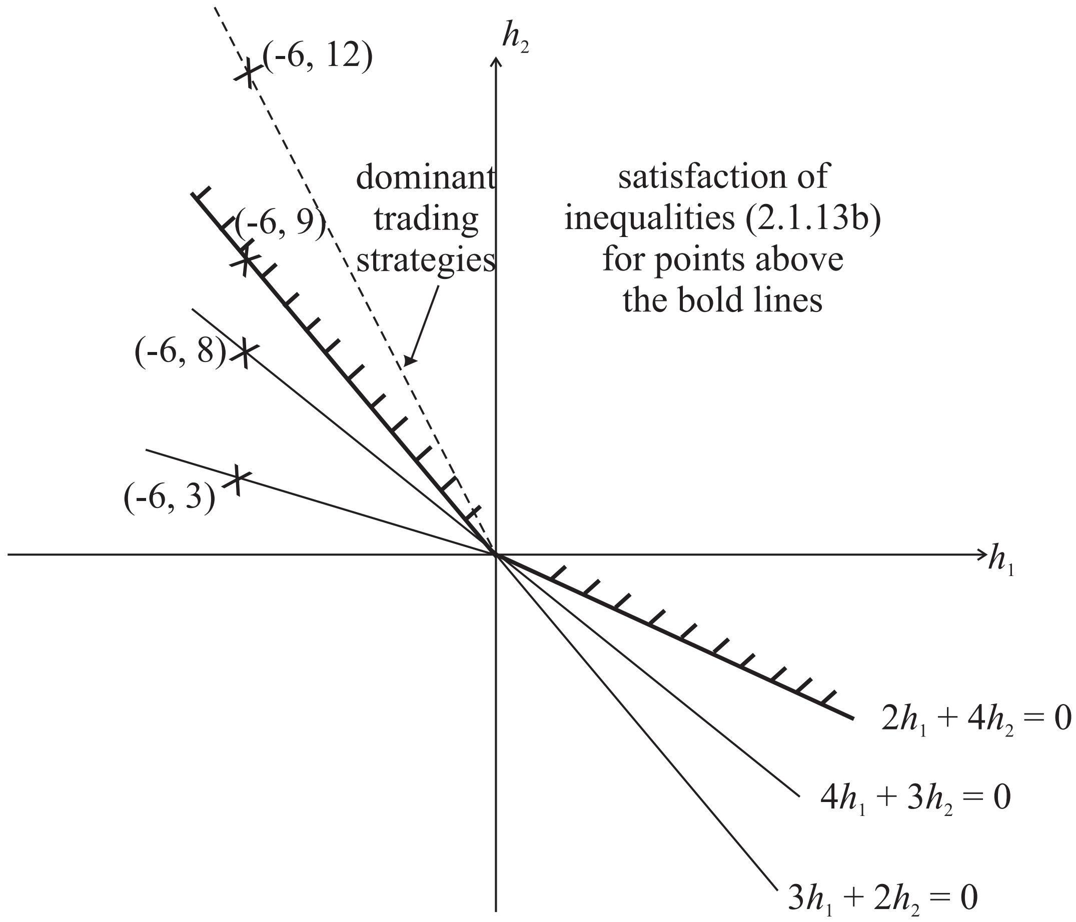
- Since not all the pricing probabilities are non-negative, the linear pricing measure does not exist for this securities model.

*Existence of dominant trading strategies*

- For convenience of graphical interpretation, we consider trading strategies that take $h_0 = 0$. Can we find a trading strategy $(h_1\ h_2)$ such that $V_0^* = 4h_1 + 2h_2 = 0$ but $V_1^*(\omega_k) > 0, k = 1, 2, 3$? This is equivalent to ask whether there exist $h_1$ and $h_2$ such that $4h_1 + 2h_2 = 0$ and

$$\begin{aligned} 4h_1 + 3h_2 &> 0 \\ 3h_1 + 2h_2 &> 0 \\ 2h_1 + 4h_2 &> 0. \end{aligned} \qquad (A)$$

- The region is found to be lying on the top right sides above the two bold lines: (i) $3h_1 + 2h_2 = 0, h_1 < 0$ and (ii) $2h_1 + 4h_2 = 0, h_1 > 0$. It is seen that all the points on the dotted half line: $4h_1 + 2h_2 = 0, h_1 < 0$ represent dominant trading strategies that start with zero wealth but end with positive wealth with certainty.




The region above the two bold lines represents trading strategies that satisfy inequalities (A). The trading strategies that lie on the dotted line: $4h_1 + 2h_2 = 0, h_1 < 0$ are dominant trading strategies.

- Suppose the initial discounted price vector is changed from $(4 \quad 2)$ to $(3 \quad 3)$, the new set of linear pricing probabilities will be determined by

$$\begin{aligned} 1 &= q(\omega_1) + q(\omega_2) + q(\omega_3) \\ 3 &= 4q(\omega_1) + 3q(\omega_2) + 2q(\omega_3) \\ 3 &= 3q(\omega_1) + 2q(\omega_2) + 4q(\omega_3), \end{aligned}$$

which is seen to have the solution: $q(\omega_1) = q(\omega_2) = q(\omega_3) = 1/3$. Now, all the pricing probabilities have non-negative values, the row vector $\boldsymbol{q} = (1/3 \quad 1/3 \quad 1/3)$ represents a linear pricing measure.

- The line $3h_1 + 3h_2 = 0$ always lies outside the region above the two bold lines.

- We cannot find $(h_1 \quad h_2)$ such that $3h_1 + 3h_2 = 0$ together with $h_1$ and $h_2$ satisfying all these inequalities.

**Theorem**

There exists a linear pricing measure if and only if there are no dominant trading strategies.

The above linear pricing measure theorem can be seen to be a direct consequence of the Farkas Lemma.

*Farkas Lemma*

There does not exist $\boldsymbol{h} \in \mathbb{R}^M$ such that

$$\widehat{S}^*(1;\Omega)\boldsymbol{h} > \boldsymbol{0} \quad \text{and} \quad \widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = 0$$

if and only if there exists $\boldsymbol{q} \in \mathbb{R}^K$ such that

$$\widehat{\boldsymbol{S}}^*(0) = \boldsymbol{q}\widehat{S}^*(1;\Omega) \quad \text{and} \quad \boldsymbol{q} \geq \boldsymbol{0}.$$

Given that the security lies in the asset span, we can deduce that law of one price holds by observing either

(i) null space of $S^*(1)$ has zero dimension, or

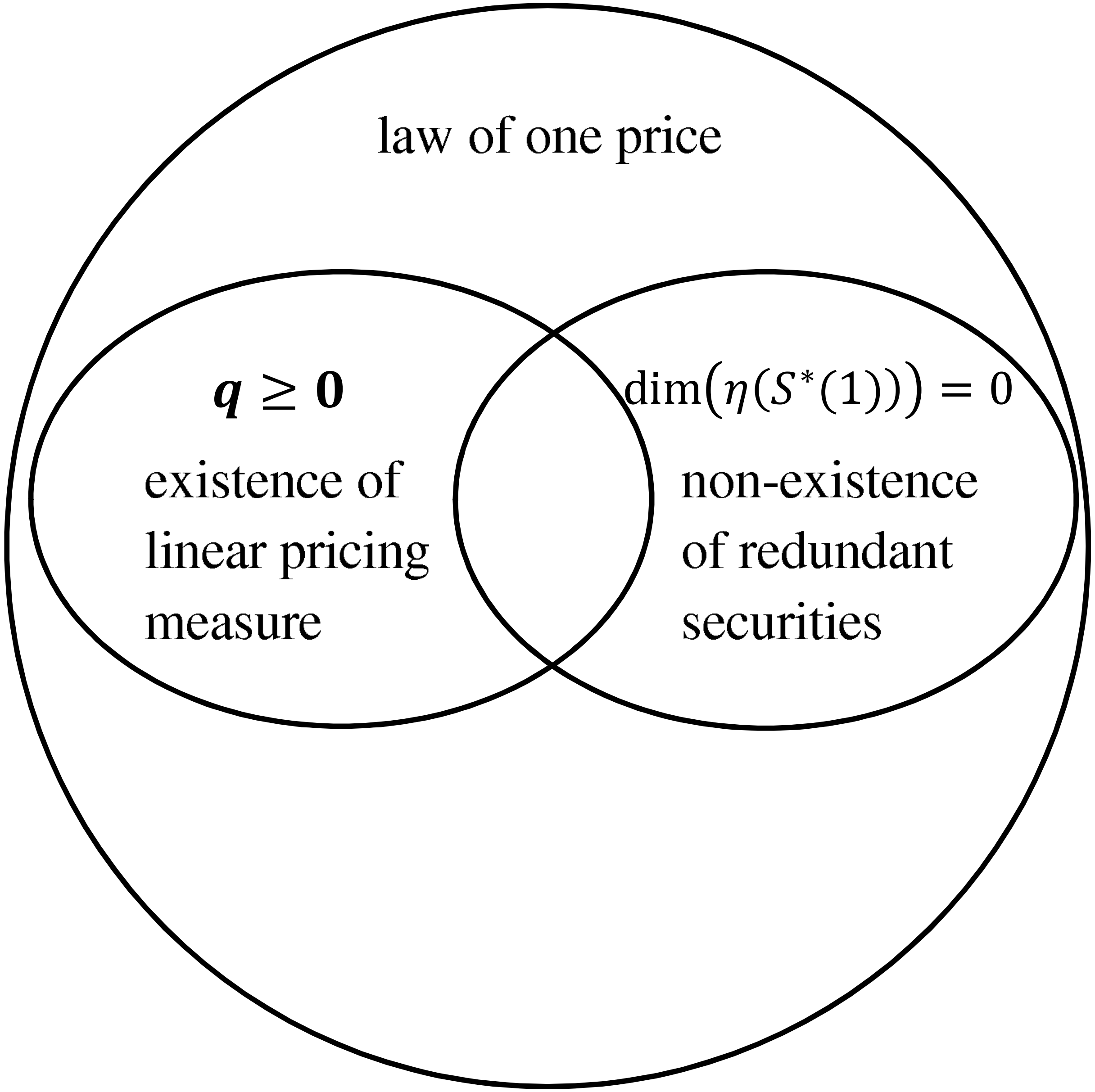
(ii) existence of a linear pricing measure.

Both (i) and (ii) represent the various forms of sufficient condition for the law of one price.

*Remarks*

1. Condition (i) is equivalent to non-existence of redundant securities.

2. Condition (i) and condition (ii) are not equivalent.




Various forms of sufficient condition for the law of one price

## 3.2 No-arbitrage theory and risk neutral probability measure

- An *arbitrage opportunity* is some trading strategy that has the following properties: (i) $V_0^* = 0$, (ii) $V_1^*(\omega) \geq 0$ with strict inequality at least for one state.

- The existence of a dominant strategy requires a portfolio with initial zero wealth to end up with a *strictly* positive wealth in all states.

- The existence of a dominant trading strategy implies the existence of an arbitrage opportunity, but the converse is not necessarily true.

**Risk neutral probability measure**

A probability measure $Q$ on $\Omega$ is a risk neutral probability measure if it satisfies

(i) $Q(\omega) > 0$ for all $\omega \in \Omega$, and

(ii) $E_Q[\Delta S_m^*] = 0, m = 0, 1, \cdots, M$, where $E_Q$ denotes the expectation under $Q$. The expectation of the discounted gain of any security in the securities model under $Q$ is zero.

Note that $E_Q[\Delta S_m^*] = 0$ is equivalent to $S_m^*(0) = \sum_{k=1}^{K} Q(\omega_k) S_m^*(1; \omega_k)$.

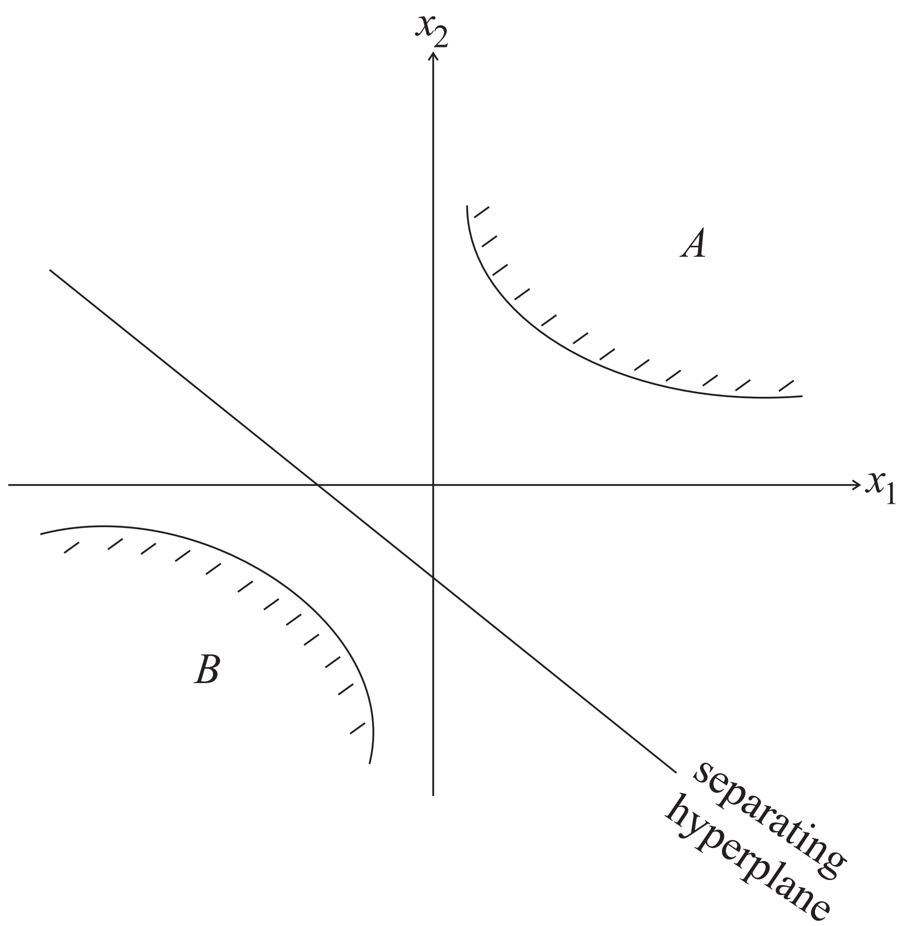
- In financial markets with no arbitrage opportunities, every investor should use such risk neutral probability measure (though not necessarily unique) to find the fair value of a contingent claim, independent of the subjective assessment of the probabilities of occurrence of different states.

## Fundamental theorem of asset pricing

No arbitrage opportunities exist if and only if there exists a risk neutral probability measure $Q$.

- The proof of the Theorem requires the Separating Hyperplane Theorem.
- The Separating Hyperplane Theorem states that if $A$ and $B$ are two non-empty disjoint convex sets in a vector space $V$, then they can be separated by a hyperplane.

*Geometric illustration of separation*




The hyperplane (represented by a line in $\mathbb{R}^2$) separates the two convex sets $A$ and $B$ in $\mathbb{R}^2$. A set $C$ is convex if any convex combination $\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}, 0 \leq \lambda \leq 1$, of a pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $C$ also lies in $C$.

The hyperplane $[\boldsymbol{f}, \alpha]$ separates the sets $A$ and $B$ in $\mathbb{R}^n$ if there exists $\alpha$ such that $\boldsymbol{f} \cdot \boldsymbol{x} \geq \alpha$ for all $\boldsymbol{x} \in A$ and $\boldsymbol{f} \cdot \boldsymbol{y} < \alpha$ for all $\boldsymbol{y} \in B$. In $\mathbb{R}^2$ and $\mathbb{R}^3$, the vector $\boldsymbol{f}$ has the geometric interpretation that it is the normal vector to the hyperplane.

For example, the hyperplane $\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 0\right]$ separates the two disjoint convex sets $A = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\right\}$

and $B = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 < 0, x_2 < 0, x_3 < 0\right\}$ in $\mathbb{R}^3$.

Note that the hyperplane is not necessarily unique. In the above example, $\left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, 0\right]$ is another possible choice of the separating hyperplane.

*Proof*

"$\Leftarrow$ part".

Assume that a risk neutral probability measure $Q$ exists, that is, $\widehat{\boldsymbol{S}}^*(0) = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)$, where $\boldsymbol{\pi} = (Q(\omega_1) \cdots Q(\omega_K))$ and $\boldsymbol{\pi} > \mathbf{0}$. Under such assumption, we would like to show that it is never possible to construct a trading strategy that represents an arbitrage opportunity.

Consider a trading strategy $\boldsymbol{h} = (h_0 \ h_1 \quad \cdots \quad h_M)^T \in \mathbb{R}^{M+1}$ such that $\widehat{S}^*(1;\Omega)\boldsymbol{h} \geq 0$ in all $\omega \in \Omega$ and with strict inequality in at least one state. Now consider $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)\boldsymbol{h}$, it is seen that $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} > \mathbf{0}$ since all entries in $\boldsymbol{\pi}$ are strictly positive and entries in $\widehat{S}^*(1;\Omega)\boldsymbol{h}$ are either zero or strictly positive. It is then impossible to have $\widehat{\boldsymbol{S}}(0)\boldsymbol{h} = 0$ and $S^*(1;\Omega)\boldsymbol{h} \geq \mathbf{0}$ in all $\omega \in \Omega$, with strict inequality in at least one state. Hence, no arbitrage opportunities exist.

"$\Rightarrow$ part".

First, we define the subset $U$ in $\mathbb{R}^{K+1}$ which consists of vectors of the form $\begin{pmatrix} -\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} \\ \widehat{\boldsymbol{S}}^*(1;\omega_1)\boldsymbol{h} \\ \vdots \\ \widehat{\boldsymbol{S}}^*(1;\omega_K)\boldsymbol{h} \end{pmatrix}$, where $\widehat{\boldsymbol{S}}^*(1;\omega_k)$ is the $k^{\text{th}}$ row in $\widehat{S}^*(1;\Omega)$ and $\boldsymbol{h} \in \mathbb{R}^{M+1}$ represents a trading strategy. This subset is seen to be a subspace since $U$ contains the zero vector and $\alpha_1\boldsymbol{h}_1 + \alpha_2\boldsymbol{h}_2$ remains to be a trading strategy for any scalar multiples $\alpha_1$ and $\alpha_2$. The convexity property of $U$ is obvious.

Consider another subset $\mathbb{R}_+^{K+1}$ defined by

$$\mathbb{R}_+^{K+1} = \{\boldsymbol{x} = (x_0\ x_1 \cdots x_K)^T \in \mathbb{R}^{K+1} : x_i \geq 0 \quad \text{for all} \quad 0 \leq i \leq K\},$$

which is a convex set in $\mathbb{R}^{K+1}$.

We claim that the non-existence of arbitrage opportunities implies that $U$ and $\mathbb{R}_+^{K+1}$ can only have the zero vector in common.

Assume the contrary, suppose there exists a non-zero vector $\boldsymbol{x} \in U \cap \mathbb{R}_+^{K+1}$. Since there is a trading strategy vector $\boldsymbol{h}$ associated with every vector in $U$, it suffices to show that the trading strategy $\boldsymbol{h}$ associated with $\boldsymbol{x}$ always represents an arbitrage opportunity.

We consider the following two cases: $-\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = 0$ or $-\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} > 0$.

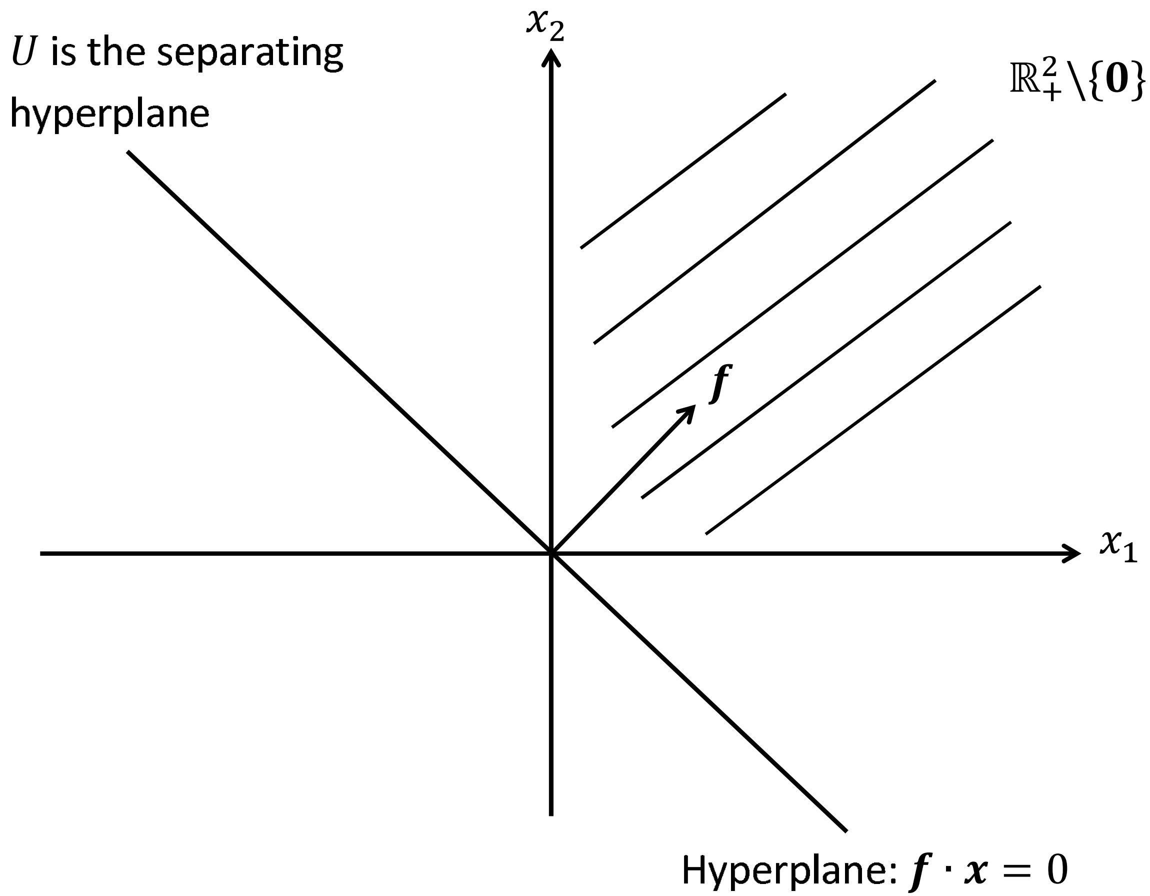
(i) When $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = 0$, since $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{x} \in R_+^{K+1}$, then the entries $\widehat{\boldsymbol{S}}(1;\omega_k)\boldsymbol{h}, k = 1, 2, \cdots K$, must be all greater than or equal to zero, with at least one strict inequality. In this case, $\boldsymbol{h}$ is seen to represent an arbitrage opportunity.

(ii) When $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} < 0$, all the entries $\widehat{\boldsymbol{S}}(1;\omega_k)\boldsymbol{h}, k = 1, 2, \cdots, K$ must be all non-negative. Correspondingly, $\boldsymbol{h}$ represents a dominant trading strategy and in turns $\boldsymbol{h}$ is an arbitrage opportunity.

Since $U \cap R_+^{K+1} = \{\mathbf{0}\}$, by the Separating Hyperplane Theorem, there exists a hyperplane that separates the pair of disjoint convex sets: $\mathbb{R}_+^{K+1} \backslash \{\mathbf{0}\}$ and $U$. One can show easily that this hyperplane must go through the origin, so its equation is of the form $[\boldsymbol{f}, 0]$. Let $\boldsymbol{f} \in \mathbb{R}^{K+1}$ be the normal to this hyperplane, then we have $\boldsymbol{f} \cdot \boldsymbol{x} > \boldsymbol{f} \cdot \boldsymbol{y}$, for all $\boldsymbol{x} \in \mathbb{R}_+^{K+1} \backslash \{\mathbf{0}\}$ and $\boldsymbol{y} \in U$.

[*Remark*: We may have $\boldsymbol{f} \cdot \boldsymbol{x} < \boldsymbol{f} \cdot \boldsymbol{y}$, depending on the orientation of the normal vector $\boldsymbol{f}$. However, the final conclusion remains unchanged.]

*Two-dimensional case*




(i) $\boldsymbol{f} \cdot \boldsymbol{y} = 0$ for all $\boldsymbol{y} \in U$;

(ii) $\boldsymbol{f} \cdot \boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathbb{R}^2_+ \backslash \{\mathbf{0}\}$.

Since $U$ is a linear subspace so that a negative multiple of $\boldsymbol{y} \in U$ also belongs to $U$. Note that $\boldsymbol{f} \cdot \boldsymbol{x} > \boldsymbol{f} \cdot \boldsymbol{y}$ and $\boldsymbol{f} \cdot \boldsymbol{x} > \boldsymbol{f} \cdot (-\boldsymbol{y})$ both holds only if $\boldsymbol{f} \cdot \boldsymbol{y} = 0$ for all $\boldsymbol{y} \in U$.

*Remark*

Interestingly, all vectors in $U$ lie in the hyperplane $\boldsymbol{f} \cdot \boldsymbol{y} = 0$ through the origin. This hyperplane separates $U$ (hyperplane itself) and $R_+^{K+1} \backslash \{0\}$.

We have $\boldsymbol{f} \cdot \boldsymbol{x} > 0$ for all $\boldsymbol{x}$ in $\mathbb{R}_+^{K+1} \backslash \{\mathbf{0}\}$. This requires all entries in $\boldsymbol{f}$ to be strictly positive. Note that if at least one of the components (say, the $i^{\text{th}}$ component) of $\boldsymbol{f}$ is zero or negative, then we choose $\boldsymbol{x}$ to be the $i^{\text{th}}$ coordinate vector. This gives $\boldsymbol{f} \cdot \boldsymbol{x} \leq 0$, a violation of $\boldsymbol{f} \cdot \boldsymbol{x} > 0$.

From $\boldsymbol{f} \cdot \boldsymbol{y} = 0$, we have

$$-f_0 \widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} + \sum_{k=1}^{K} f_k \widehat{\boldsymbol{S}}^*(1;\omega_k)\boldsymbol{h} = 0$$

for all $\boldsymbol{h} \in \mathbb{R}^{M+1}$, where $f_j, j = 0, 1, \cdots, K$ are the entries of $\boldsymbol{f}$. We then deduce that

$$\widehat{\boldsymbol{S}}^*(0) = \sum_{k=1}^{K} Q(\omega_k)\widehat{\boldsymbol{S}}^*(1;\omega_k), \text{ where } Q(\omega_k) = f_k/f_0.$$

Consider the first component in the vectors on both sides of the above equation. They both correspond to the current price and discounted payoff of the riskless security, and all are equal to one. We then obtain

$$1 = \sum_{k=1}^{K} Q(\omega_k).$$

We obtain the risk neutral probabilities $Q(\omega_k), k = 1, \cdots, K$, whose sum is equal to one and they are all strictly positive since $f_j > 0, j = 0, 1, \cdots, K$.

Corresponding to each risky asset, we have

$$S_m^*(0) = \sum_{k=1}^{K} Q(\omega_k) S_m^*(1; \omega_k), \quad m = 1, 2, \cdots, M.$$

Hence, the current price of any one of risky securities in the securities model is given by the expectation of the discounted payoff under the risk neutral measure $Q$.

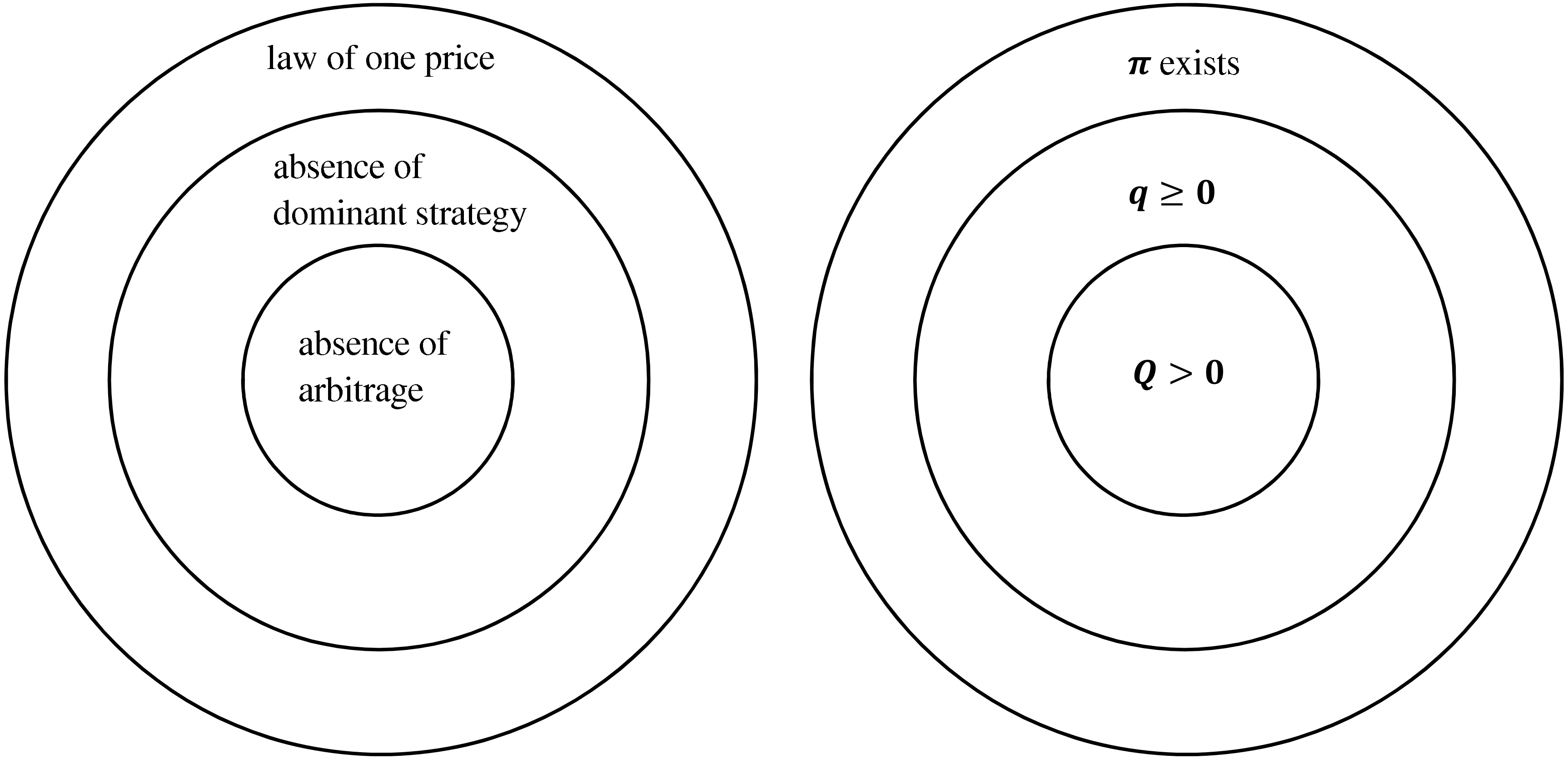
*Remark*

The existence of the separating hyperplane leads to the existence of $Q(\omega_k)$, $k = 1, \cdots, K$, as determined by the ratio of some appropriate entries in the normal vector $\boldsymbol{f}$ to the hyperplane. The non-uniqueness of the separating hyperplane leads to non-uniqueness of the risk neutral measure.

*Financial Economics* *Linear Algebra*: $\boldsymbol{\pi} S^*(1) = \boldsymbol{S}(0)$

$$\boldsymbol{q}\widehat{S}^*(1) = \widehat{\boldsymbol{S}}(0)$$

$$\boldsymbol{Q}\widehat{S}^*(1) = \widehat{\boldsymbol{S}}(0)$$




*Remark* The securities model contains the riskfree asset when we consider the linear pricing measure $\boldsymbol{q}$ and martingale pricing measure $\boldsymbol{Q}$.

**Example (arbitrage opportunities but no dominant trading strategies)**

Consider the securities model

$$\begin{pmatrix}1 & 2 & 3 & 6\end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2 & \pi_3\end{pmatrix} \begin{pmatrix}1 & 2 & 3 & 6 \\ 1 & 3 & 4 & 8 \\ 1 & 6 & 7 & 14\end{pmatrix},$$

where the number of non-redundant securities is only 2. Note that

$$\boldsymbol{S}_2^*(1;\Omega) = \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_1^*(1;\Omega) \quad \text{and}$$

$$\boldsymbol{S}_3^*(1;\Omega) = \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_1^*(1;\Omega) + \boldsymbol{S}_2^*(1;\Omega),$$

and the initial prices have been set such that

$$S_2^*(0) = S_0^*(0) + S_1^*(0) \quad \text{and} \quad S_3^*(0) = S_0^*(0) + S_1^*(0) + S_2^*(0),$$

so we expect to have the existence of solution. However, since $2 =$ number of non-redundant securities $<$ number of states $= 3$, we do not have uniqueness of solution. Indeed, we obtain

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\end{pmatrix} + t\begin{pmatrix}3 & -4 & 1\end{pmatrix}, \quad t \text{ any value}.$$

For example, when we take $t = 1$, then

$$(\pi_1 \quad \pi_2 \quad \pi_3) = (4 \quad -4 \quad 1).$$

In terms of linear algebra, we have existence of solution if the equations are consistent. Consider the present example, we have

$$\begin{aligned} \pi_1 + \pi_2 + \pi_3 &= 1 \\ 2\pi_1 + 3\pi_2 + 6\pi_3 &= 2 \\ 3\pi_1 + 4\pi_2 + 7\pi_3 &= 3 \\ 6\pi_1 + 8\pi_2 + 14\pi_3 &= 6 \end{aligned}$$

Note that the last two redundant equations are consistent. Alternatively, we can interpret that the row vector $\boldsymbol{S}^*(0) = (1 \quad 2 \quad 3 \quad 6)$ lies in the row space of $\widehat{S}^*(1;\Omega)$, which is spanned by $\{(1 \quad 2 \quad 3 \quad 6)$, $(0 \quad 1 \quad 1 \quad 2)\}$.

In this securities model, we cannot find a risk neutral measure where $(Q_1 \quad Q_2 \quad Q_3) > \mathbf{0}$. This is easily seen since $\pi_2 = -4t$ and $\pi_3 = t$, and they always have opposite sign. However, a linear prcing measure exists. One such example is $(q_1 \quad q_2 \quad q_3) = (1 \quad 0 \quad 0) > \mathbf{0}$.

Since $Q$ does not exist, the securities model admits arbitrage opportunities. One such example is $\boldsymbol{h} = (-11 \quad 1 \quad 1 \quad 1)^T$, where

$$\boldsymbol{S}^*(0)\boldsymbol{h} = 0 \quad \text{and} \quad S^*(1;\Omega)\boldsymbol{h} = \begin{pmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 & 8 \\ 1 & 6 & 7 & 14 \end{pmatrix} \begin{pmatrix} -11 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 14 \\ 5 \end{pmatrix}.$$

The discounted portfolio value at $t = 1$ is guaranteed to be non-negative, with strict positivity for at least one state.

However, the securities model does not admit dominant trading s-trategies since a linear pricing measure exists. This is evidenced by showing that one cannot find a trading strategy $\boldsymbol{h} = (h_0 \quad h_1 \quad h_2 \quad h_3)^T$ such that

$$h_0 + 2h_1 + 3h_2 + 6h_3 = 0$$

while

$$h_0 + 2h_1 + 3h_2 + 6h_3 > 0, \quad h_0 + 3h_1 + 4h_2 + 8h_3 > 0,$$

$$h_0 + 6h_1 + 6h_2 + 14h_3 > 0.$$

The first inequality can never be satisfied when we impose $h_0 + 2h_1 + 3h_2 + 6h_3 = 0$. Indeed, when $\boldsymbol{S}^*(0) = \boldsymbol{S}^*(1;\omega_k)$ for some $\omega_k$, then a linear pricing meaure exists where $\boldsymbol{q} = \boldsymbol{e}_k^T$.

- Martingale property is defined for adapted stochastic processes*. In the context of one-period model, given the information on the initial prices and terminal payoff values of the security prices at $t = 0$,

$$S_m^*(0) = E_Q[S_m^*(1;\Omega)] = \sum_{k=1}^{K} S_m^*(1;\omega_k)Q(\omega_k), \quad m = 1, 2, \cdots, M. \tag{1}$$

The discounted security price process $S_m^*(t)$ is said to be a martingale† under $Q$.

Martingale is associated with the wealth process of a gambler in a fair game. In a fair game, the expected value of the gambler's wealth after any number of plays is always equal to her initial wealth.

*A stochastic process is adapted to a filtration with respect to a measure. Suppose $S_m^*$ is adapted to $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$, we say $S_m^*(t)$ is $\mathcal{F}_t$-measurable.

†Martingale property with respect to $Q$ and $\mathbb{F}$:

$$S_m^*(t) = E_Q[S_m^*(s+t)|\mathcal{F}_t] \text{ for all } t \geq 0, s \geq 0.$$

**Equivalent martingale measure**

- The risk neutral probability measure $Q$ is commonly called the equivalent martingale measure. "Equivalent" refers to the equivalence between the physical measure $P$ and martingale measure $Q$ [observing $P(\omega) > 0 \Leftrightarrow Q(\omega) > 0$ for all $\omega \in \Omega$]*. The linear pricing measure falls short of this equivalence property since $q(\omega)$ can be zero.

*$P$ and $Q$ may not agree on the assignment of probability values to individual events, but they always agree as to which events are possible or impossible.

*Martingale property of discounted portfolio value (assuming the existence of $Q$ or equivalently, the absence of arbitrage in the securities model)*

- Let $V_1^*(\Omega)$ denote the discounted payoff of a portfolio. Since $V_1^*(\Omega) = \widehat{S}^*(1;\Omega)\boldsymbol{h}$ for some trading strategy $\boldsymbol{h} = (h_0 \ \cdots \ h_M)^T$, by Eq. (1),

$$
\begin{aligned}
V_0^* &= (S_0^*(0) \cdots S_M^*(0))\boldsymbol{h} \\
&= (E_Q[S_0^*(1;\Omega)] \cdots E_Q[S_M^*(1;\Omega)])\boldsymbol{h} \\
&= \sum_{m=0}^{M} \left[ \sum_{k=1}^{K} S_m^*(1;\omega_k) Q(\omega_k) \right] h_m \\
&= \sum_{k=1}^{K} Q(\omega_k) \left[ \sum_{m=0}^{M} S_m^*(1;\omega_k) h_m \right] = E_Q[V_1^*(\Omega)].
\end{aligned}
$$

- The equivalent martingale measure $Q$ is not necessarily unique. Since "absence of arbitrage opportunities " implies "law of one price", the expectation value $E_Q[V_1^*(\Omega)]$ is single-valued under all equivalent martingale measures.

## Finding the set of risk neutral measures

Consider the earlier securities model with the riskfree security and only one risky security, where $\widehat{S}(1;\Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}$ and $\widehat{\boldsymbol{S}}(0) = (1 \quad 3)$. The risk neutral probability measure

$$\boldsymbol{Q} = (Q(\omega_1) \quad Q(\omega_2) \quad Q(\omega_3)),$$

if exists, will be determined by the following system of equations

$$(Q(\omega_1) \quad Q(\omega_2) \quad Q(\omega_3)) \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} = (1 \quad 3).$$

Since there are more unknowns than the number of equations, the solution is not unique. The solution is found to be $\boldsymbol{Q} = (\lambda \quad 1-2\lambda \quad \lambda)$, where $\lambda$ is a free parameter. Since all risk neutral probabilities are all strictly positive, we must have $0 < \lambda < 1/2$.

*Under market completeness, if the set of risk neutral measures is non-empty, then it must be a singleton.*

Under market completeness, column rank of $\widehat{S}^*(1;\Omega)$ equals the number of states. Since column rank = row rank, then all rows of $\widehat{S}^*(1;\Omega)$ are independent. If solution $\boldsymbol{Q}$ exists for

$$\boldsymbol{Q}\widehat{S}^*(1;\Omega) = \widehat{\boldsymbol{S}}^*(0),$$

then it must be unique. Note that $\boldsymbol{Q} > \boldsymbol{0}$.

Conversely, suppose the set of risk neutral measures is a singleton, one can show that the securities model is complete (see later discussion).

*Numerical example*

Suppose we add the second risky security with discounted payoff $\boldsymbol{S}_2^*(1) = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ and current discounted value $S_2^*(0) = 3$. With this new addition, the securities model becomes complete.

With the new equation $3Q(\omega_1)+2Q(\omega_2)+4Q(\omega_3) = 3$ added to the system, this new securities model is seen to have the unique risk neutral measure $(1/3 \quad 1/3 \quad 1/3)$.

Indeed, when the securities model is complete, all Arrow securities are replicable. Their prices (called state prices) are simply equal to the risk neutral measures. In this example, we have

$$s_1 = Q(\omega_1) = \frac{1}{3}, \quad s_2 = Q(\omega_2) = \frac{1}{3}, \quad s_3 = Q(\omega_3) = \frac{1}{3}.$$

## Subspace of discounted gains

Let $W$ be a subspace in $\mathbb{R}^K$ which consists of discounted gains corresponding to some trading strategy $\boldsymbol{h}$. Note that $W$ is spanned by the set of vectors representing discounted gains of the risky securities.

In the above securities model, the discounted gains of the first and second risky securities are $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, respectively.

The discounted gain subspace is given by

$$W = \left\{ h_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + h_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \text{ where } h_1 \text{ and } h_2 \text{ are scalars} \right\}.$$

*Orthogonality of the discounted gain vector and $\boldsymbol{Q}$*

Let $G^*$ denote the discounted gain of a portfolio. For any risk neutral probability measure $Q$, we have

$$
\begin{aligned}
E_Q G^* &= \sum_{k=1}^{K} Q(\omega_k) \left[ \sum_{m=1}^{M} h_m \Delta S_m^*(\omega_k) \right] \\
&= \sum_{m=1}^{M} h_m E_Q[\Delta S_m^*] = 0.
\end{aligned}
$$

Under the absence of arbitrage opportunities, the expected discounted gain from any risky portfolio is simply zero. Apparently, there is no risk premium derived from the risky investment. Therefore, the financial economics term "risk neutrality" is adopted under this framework of asset pricing.

For any $\boldsymbol{G}^* = (G(\omega_1) \cdots G(\omega_K))^T \in W$, we have

$$\boldsymbol{Q}\boldsymbol{G}^* = 0, \text{ where } \boldsymbol{Q} = (Q(\omega_1) \cdots Q(\omega_K)).$$

## Characterization of the set of neutral measures

Since the sum of risk neutral probabilities must be one and all probability values must be positive, the risk neutral probability vector $\boldsymbol{Q}$ must lie in the following subset

$$P^+ = \{\boldsymbol{y} \in \mathbb{R}^K : y_1 + y_2 + \cdots + y_K = 1 \quad \text{and} \quad y_k > 0, k = 1, \cdots K\}.$$

Also, the risk neutral probability vector $\boldsymbol{Q}$ must lie in the orthogonal complement $W^\perp$. Let $R$ denote the set of all risk neutral measures, then $R = P^+ \cap W^\perp$.

In the above numerical example, $W^\perp$ is the line through the origin in $\mathbb{R}^3$ which is perpendicular to $(1 \quad 0 \quad -1)^T$ and $(0 \quad -1 \quad 1)^T$. The line should assume the form $\lambda(1 \quad 1 \quad 1)$ for some scalar $\lambda$. We obtain the risk neutral probability vector $\boldsymbol{Q} = (1/3 \quad 1/3 \quad 1/3)$.

## 3.3 Valuation of contingent claims

- A contingent claim can be considered as a random variable $Y$ that represents a terminal payoff whose value depends on the occurrence of a particular state $\omega_k$, where $\omega_k \in \Omega$.

- Suppose the holder of the contingent claim is promised to receive the preset contingent payoff, how much should the writer of such contingent claim charge at $t = 0$ so that the price is *fair* to both parties.

- Consider the securities model with the riskfree security whose values at $t = 0$ and $t = 1$ are $S_0(0) = 1$ and $S_0(1) = 1.1$, respectively, and a risky security with $S_1(0) = 3$ and $\boldsymbol{S}_1(1) = \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$.

The set of $t = 1$ payoffs that can be generated by certain trading strategy is given by $h_0 \begin{pmatrix} 1.1 \\ 1.1 \\ 1.1 \end{pmatrix} + h_1 \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$ for some scalars $h_0$ and $h_1$.

For example, the contingent claim $\begin{pmatrix} 5.5 \\ 4.4 \\ 3.3 \end{pmatrix}$ can be generated by the trading strategy: $h_0 = 1$ and $h_1 = 1$, while the other contingent claim $\begin{pmatrix} 5.5 \\ 4.0 \\ 3.3 \end{pmatrix}$ cannot be generated by any trading strategy associated with the given securities model.

A contingent claim $Y$ is said to be *attainable* if there exists some trading strategy $\boldsymbol{h}$, called the *replicating portfolio*, such that $V_1 = Y$ for all possible states occurring at $t = 1$.

The price at $t = 0$ of the replicating portfolio is given by

$$V_0 = h_0 S_0(0) + h_1 S_1(0) = 1 \times 1 + 1 \times 3 = 4.$$

Suppose there are no arbitrage opportunities (equivalent to the existence of a risk neutral probability measure), then the law of one price holds and so $V_0$ is unique.

## Pricing of attainable contingent claims

Let $V_1^*(1;\Omega)$ denote the value of the replicating portfolio that matches with the payoff of the attainable contingent claim at every state of the world. Suppose the associated trading strategy to generate the replicating portfolio is $\boldsymbol{h}$, then

$$V_1^* = \widehat{S}^*(1;\Omega)\boldsymbol{h}.$$

The initial cost of setting up the replicating portfolio is

$$V_0^* = \widehat{\boldsymbol{S}}^*(0)\boldsymbol{h}.$$

Assuming $\boldsymbol{\pi}$ exists, where $\widehat{\boldsymbol{S}}^*(0) = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)$ so that

$$\begin{aligned} V_0^* &= \boldsymbol{\pi}\widehat{S}^*(1;\Omega)\boldsymbol{h} = \boldsymbol{\pi}V_1^*(1;\Omega) \\ &= \sum_{k=1}^{K} \pi_k V_1^*(1;\omega_k), \text{ independent of } \boldsymbol{h}. \end{aligned}$$

Even when $\boldsymbol{\pi}$ is not a risk neutral measure or linear pricing measure, the above pricing relation remains valid. Though $\boldsymbol{\pi}$ may not be unique, by virtue of law of one price, we have the same value for $V_0^*$.

- Consider a given attainable contingent claim $Y$ which is generated by certain trading strategy. The associated discounted gain $G^*$ of the trading strategy is given by $G^* = \sum_{m=1}^{M} h_m \Delta S_m^*$. Now, suppose a risk neutral probability measure $Q$ associated with the securities model exists, we have

$$V_0 = E_Q V_0^* = E_Q[V_1^* - G^*].$$

Since $E_Q[G^*] = 0$ and $V_1^* = Y/S_0(1)$, we obtain

$$V_0 = E_Q[Y/S_0(1)].$$

**Risk neutral valuation principle**

The price at $t = 0$ of an attainable claim $Y$ is given by the expectation under any risk neutral measure $Q$ of the discounted value of the contingent claim.

**Attainability of a contingent claim and uniqueness of $E_Q[Y^*]$**

- Recall that the existence of the risk neutral probability measure implies the law of one price. Does $E_Q[Y/S_0(1)]$ assume the same value for every risk neutral probability measure $Q$?

Provided that $Y$ is attainable, this must be true by virtue of the law of one price since we cannot have two different values for $V_0$ corresponding to the same attainable contingent claim $Y$.

**Theorem**

Suppose the securities model admits no arbitrage opportunities. The contingent claim $Y$ is attainable if and only if $E_Q[Y^*]$ takes the same value for every $Q \in M$, where $M$ is the set of risk neutral measures.

*Proof*

$\Longrightarrow$ part

existence of $Q \Longleftrightarrow$ absence of arbitrage $\Longrightarrow$ law of one price. For an attainable $Y$, $E_Q[Y^*]$ is constant with respect to all $Q \in M$, otherwise this leads to violation of the law of one price.

$\Longleftarrow$ part

It suffices to show that if the contingent claim $Y$ is not attainable then $E_Q[Y^*]$ does not take the same value for all $Q \in M$.

Let $\boldsymbol{y}^* \in \mathbb{R}^K$ be the discounted payoff vector corresponding to $Y^*$. Since $Y$ is not attainable, then there is no solution to

$$\widehat{S}^*(1)\boldsymbol{h} = \boldsymbol{y}^*$$

(non-existence of trading strategy $\boldsymbol{h}$). It then follows that there must exist a non-zero row vector $\boldsymbol{\pi} \in \mathbb{R}^K$ such that

$$\boldsymbol{\pi}\widehat{S}^*(1) = \mathbf{0} \quad \text{and} \quad \boldsymbol{\pi}\boldsymbol{y}^* \neq 0.$$

*Remark*

Recall that the orthogonal complement of the column space is the left null space. The dimension of the left null space equals $K-$ column rank, and it is non-zero when the column space does not span the whole $\mathbb{R}^K$. The above result indicates that when $\boldsymbol{y}^*$ is not in the column space of $S^*(1)$, then there exists a non-zero vector $\boldsymbol{\pi}$ in the left null space of $S^*(1)$ such that $\boldsymbol{y}^*$ and $\boldsymbol{\pi}$ are not orthogonal.

Write $\boldsymbol{\pi} = (\pi_1 \cdots \pi_K)$. Let $\widehat{Q} \in M$ be arbitrary, and let $\lambda > 0$ be small enough such that

$$Q(\omega_k) = \widehat{Q}(\omega_k) + \lambda \pi_k > 0, \quad k = 1, 2, \cdots, K.$$

We would like to show that $Q(\omega_k)$ is also a risk neutral measure by virtue of the relation: $\boldsymbol{\pi} \widehat{S}^*(1) = \mathbf{0}$.

1. Note that $\boldsymbol{\pi}\mathbf{1} = \sum_{k=1}^{K} \pi_k = 0$, so $\sum_{k=1}^{K} Q(\omega_k) = 1$.

2. For the discounted price process $S_n^*$ of the $n^{\text{th}}$ risky securities in the securities model, we have

$$\begin{aligned} E_Q[S_n^*(1)] &= \sum_{k=1}^{K} Q(\omega_k) S_n^*(1;\omega_k) \\ &= \sum_{k=1}^{K} \widehat{Q}(\omega_k) S_n^*(1;\omega_k) + \lambda \sum_{k=1}^{K} \pi_k S_n^*(1;\omega_k) \\ &= \sum_{k=1}^{K} \widehat{Q}(\omega_k) S_n^*(1;\omega_k) = S_n(0). \end{aligned}$$

$Q$ satisfies the martingale property, together with $Q(\omega_k) > 0$ and $\sum_{k=1}^{K} Q(\omega_k) = 1$ so it is also a risk neutral measure.

Lastly, we consider

$$\begin{aligned} E_Q[Y^*] &= \sum_{k=1}^{K} Q(\omega_k)Y^*(\omega_k) \\ &= \sum_{k=1}^{K} \widehat{Q}(\omega_k)Y^*(\omega_k) + \lambda \sum_{k=1}^{K} \pi_k Y^*(\omega_k). \end{aligned}$$

The last term is non-zero since $\boldsymbol{\pi y}^* \neq 0$ and $\lambda > 0$. Therefore, we have

$$E_Q[Y^*] \neq E_{\widehat{Q}}[Y^*].$$

Thus, when $Y$ is not attainable, $E_Q[Y^*]$ does not take the same value for all risk neutral measures.

*Corollary* Given that the set of risk neutral measures $R$ is non-empty. The securities model is complete if and only if $R$ consists of exactly one risk neutral measure.

An earlier proof of "$\Longrightarrow$ part" has been shown on p.69. Alternatively, we may prove by contradiction: non-uniqueness of $Q \Rightarrow$ non-completeness.

Suppose there exist two distinct $Q$ and $\widehat{Q}$, that is, $Q(\omega_k) \neq \widehat{Q}(\omega_k)$ for some state $\omega_k$. Let $Y^* = \begin{cases} 1 & \text{if } \omega = \omega_k \\ 0 & \text{otherwise} \end{cases}$, which is the $k^{th}$ Arrow security. Obviously,

$$E_Q[Y^*] = Q(\omega_k) \neq \widehat{Q}(\omega_k) = E_{\widehat{Q}}[Y^*],$$

so $E_Q[Y^*]$ is not unique. By the theorem, $Y^*$is not attainable so the securities model is not complete.

$\Longleftarrow$ part: If the risk neutral measure is unique, then for any contingent claim $Y, E_Q[Y^*]$ takes the same value for any $Q$ (actually single $Q$). Hence, any contingent claim is attainable so the market is complete.

*Remarks*

- When the securities model is complete and admits no arbitrage opportunities, all Arrow securities lie in the asset span and risk neutral measures exist. The state price of state $\omega_k$ exists for any state and it is equal to the unique risk neutral probability $Q(\omega_k)$. This represents the best scenario of applying the risk neutral valuation procedure for pricing any contingent claim (which is always attainable due to completeness).

- On the other hand, suppose there are two risk neutral probability values for the same state $\omega_k$, the state price of that state cannot be defined in proper sense without contradicting the law of one price. Actually, by the theorem, the Arrow security of that state would not be attainable, so the state price of that state is not defined. Furthermore, we deduce that the securities model cannot be complete.

**Example**

Suppose

$$Y^* = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix},$$

$Y^*$ is seen to be attainable. We have seen that the risk neutral probability is given by

$$\boldsymbol{Q} = (\lambda \quad 1-2\lambda \quad \lambda), \text{ where } 0 < \lambda < 1/2.$$

The price at $t = 0$ of the contingent claim is given by

$$V_0 = 5\lambda + 4(1-2\lambda) + 3\lambda = 4,$$

which is independent of $\lambda$. This verifies the earlier claim that $E_Q[Y/S_0(1)]$ assumes the same value for any risk neutral measure $Q$.

Suppose $Y^*$ is changed to $(5 \quad 4 \quad 4)^T$, then $V_0 = E_Q[Y^*] = 4 + \lambda$, which is not unique. This is expected since the new $Y^*$ is non-attainable.

## Complete markets - summary of results

Recall that a securities model is complete if every contingent claim $Y$ lies in the asset span, that is, $Y$ can be generated by some trading strategy.

Consider the augmented terminal payoff matrix

$$\widehat{S}(1;\Omega) = \begin{pmatrix} S_0(1;\omega_1) & S_1(1;\omega_1) & \cdots & S_M(1;\omega_1) \\ \vdots & \vdots & & \vdots \\ S_0(1;\omega_K) & S_1(1;\omega_K) & \cdots & S_M(1;\omega_K) \end{pmatrix},$$

$Y$ always lies in the asset span if and only if the column space of $\widehat{S}(1;\Omega)$ is equal to $\mathbb{R}^K$.

- Since the dimension of the column space of $\widehat{S}(1;\Omega)$ cannot be greater than $M+1$, a necessary condition for market completeness is that $M+1 \geq K$.

- When $\widehat{S}(1;\Omega)$ has independent columns and the asset span is the whole $\mathbb{R}^K$, then $M+1=K$. Now, the trading strategy that generates $Y$ must be unique since there are no redundant securities. In this case, any contingent claim is replicable and its price is unique. Though law of one price holds, there is no guarantee that arbitrage opportunities do not exist.

- When the asset span is the whole $\mathbb{R}^K$ but some securities are redundant, the trading strategy that generates $Y$ would not be unique. Suppose absence of arbitrage is observed, the price at $t=0$ of the contingent claim is unique under risk neutral pricing, independent of the chosen trading strategy. This is a consequence of the law of one price, which holds since a risk neutral measure exists.

- Non-existence of redundant securities is a sufficient but not necessary condition for law of one price.

## Non-attainable contingent claim

Suppose a risk neutral measure $Q$ exists, risk neutral valuation fails when we price a *non-attainable contingent claim*. However, we may specify an interval $(V_-(Y), V_+(Y))$ where a reasonable price at $t = 0$ of the contingent claim should lie. The lower and upper bounds are given by

$$\begin{aligned} V_+(Y) &= \inf\{E_Q[\widetilde{Y}/S_0(1)] : \widetilde{Y} \geq Y \text{ and } \widetilde{Y} \text{ is attainable}\} \\ V_-(Y) &= \sup\{E_Q[\widetilde{Y}/S_0(1)] : \widetilde{Y} \leq Y \text{ and } \widetilde{Y} \text{ is attainable}\}. \end{aligned}$$

Here, $V_+(Y)$ is the minimum value among all prices of attainable contingent claims that dominate the non-attainable claim $Y$, while $V_-(Y)$ is the maximum value among all prices of attainable contingent claims that are dominated by $Y$.

Note that there exists a sufficiently large scalar $\lambda$ such that $\lambda S_0(1) > Y$, so $V_+(Y)$ is finite and well defined. Since $E_Q[\widetilde{Y}/S_0(1)]$ is constant with respect to all $Q \in R$ and $\widetilde{Y} \geq Y$, so $V^+(Y)$ is bounded below by $\sup\{E_Q[Y^*] : Q \in R\}$.

*Proof of the upper bound*

Suppose $V(Y) > V_+(Y)$, then an arbitrageur can lock in riskless profit by selling the contingent claim to receive $V(Y)$ and use $V_+(Y)$ to construct the replicating portfolio that generates the attainable $\widetilde{Y}$. The upfront positive gain is $V(Y) - V_+(Y)$ and the terminal gain is $\widetilde{Y} - Y$.

Alternatively, based on the linear programming duality theory, we have the following results:

If $R \neq \phi$, then for any contingent claim $Y$, we have

$$V_+(Y) = \text{sup}\{E_Q[Y^*] : Q \in R\},$$

$$V_-(Y) = \text{inf}\{E_Q[Y^*] : Q \in R\}.$$

If $Y$ is attainable, then $V_+(Y) = V_-(Y)$.

**Example**

Consider the securities model: $\widehat{\boldsymbol{S}}(0) = (1 \quad 3)$ and $\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}$, and the non-attainable contingent claim $Y^* = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}$. The risk neutral measure is

$$\boldsymbol{Q} = (\lambda \quad 1 - 2\lambda \quad \lambda), \quad \text{where } 0 < \lambda < 1/2.$$

Note that $E_Q[Y^*] = 4 + \lambda$ so that

$$V_+ = \sup\{E_Q[Y^*] : Q \in R\} = 9/2 \quad \text{and} \quad V_- = \inf\{E_Q[Y^*] : Q \in R\} = 4.$$

The attainable contingent claim corresponding to $V_+$ is

$$\tilde{Y}^* = \begin{pmatrix} 5 \\ 4.5 \\ 4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0.5\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad \text{where } E_Q[\widetilde{Y_+^*}] = 4.5.$$

On the other hand, the attainable contingent claim corresponding to $V_-$ is

$$\tilde{Y}^* = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad \text{where } E_Q[\widetilde{Y_-^*}] = 4.$$

Any reasonable initial price of the non-attainable contingent claim $Y^* = (5 \quad 4 \quad 4)^T$ should lie between the interval $(4, 4.5)$.

## Linear programming formulation

Recall that the set of all risk neutral measures $R$ is given by

$$R = W^{\perp} \cap P^{+};$$

and $W = \{\boldsymbol{x} \in \mathbb{R}^K : \boldsymbol{x} = \boldsymbol{G}^* \text{ for some trading strategy } h\}$, where

$$\text{discount gain } = \boldsymbol{G}^* = \sum_{m=1}^{M} h_m \Delta S_m^*.$$

$$\begin{aligned} W^{\perp} &= \{\boldsymbol{y} \in \mathbb{R}^K : \boldsymbol{x}^T \boldsymbol{y} = 0 \quad \text{for all } \boldsymbol{x} \in W\} \\ P^{+} &= \{\boldsymbol{x} \in \mathbb{R}^K : x_1 + \cdots + x_K = 1, x_1 > 0, \cdots, x_K > 0\}. \end{aligned}$$

Let $J$ be the dimension of $W^{\perp}$, $\boldsymbol{Q}_j \in R = W^{\perp} \cap P^{+}, j = 1, \cdots J$; and they are chosen to be independent vectors, thus forming a basis of $W^{\perp}$. Then

$$W = \{\boldsymbol{x} \in \mathbb{R}^K : \boldsymbol{x}^T \boldsymbol{Q}_j = 0, \quad j = 1, 2, \cdots, J\}.$$

For an attainable contingent claim $X$, whose terminal payoff vector is $\boldsymbol{x}$, how to find the upper bound $V_+(X)$?

Solve the following linear program

$$\text{minimize } \lambda$$

subject to

$$\begin{aligned} \boldsymbol{y} &\geq \boldsymbol{x} \\ \boldsymbol{y}^* &= \boldsymbol{y}/S_0(1) \\ \lambda &= \boldsymbol{y}^{*T}\boldsymbol{Q}_1 \\ &\vdots \\ \lambda &= \boldsymbol{y}^{*T}\boldsymbol{Q}_J \end{aligned}$$

$\lambda \in \mathbb{R}, \boldsymbol{y} \in \mathbb{R}^K$.

We enforce the condition that $E_Q[Y/S_0(1)]$ takes the same value for every risk neutral measure $Q$.

*Justification of the linear programming formulation*

Let $\boldsymbol{e}$ denote the vector whose components are all one.

Suppose $Y$ is an attainable contingent claim with initial price $\lambda$. Since $V_1^* = V_0 + G^*$, this is equivalent to say

$$\boldsymbol{y}^* - \lambda \boldsymbol{e} \in W.$$

Since $\boldsymbol{e}^T \boldsymbol{Q}_j = 1$, so we have $\boldsymbol{y}^{*T} \boldsymbol{Q}_j = \lambda$ for $j = 1, 2, \ldots, J$.

The feasible region is the set of all attainable contingent claims $Y$ with $\boldsymbol{y} \geq \boldsymbol{x}$.

If $\lambda$ and $Y$ are part of an optimal solution of the linear programming problem, then $V_+(X) = \lambda$ and $Y$ is an attainable contingent claim with $\boldsymbol{y} \geq \boldsymbol{x}$ and initial price is $V_+(X)$.

An optimal solution always exists since the feasible region is nonempty and the objective function is bounded below.

**Summary** *Arbitrage opportunity* 無風險套利機會

An arbitrage strategy is requiring no initial investment, having no chance of occurrence of negative value at expiration, and yet having some possibility of a positive terminal portfolio value.

- It is commonly assumed that there are no arbitrage opportunities in well functioning and competitive financial markets.

1. absence of arbitrage opportunities
   $\Rightarrow$ absence of dominant trading strategies
   $\Rightarrow$ law of one price

2. absence of arbitrage opportunities $\Leftrightarrow$ existence of risk neutral measure

   absence of dominant trading strategies $\Leftrightarrow$ existence of linear pricing measure.

3. Under market completeness, the state prices are non-negative when a linear pricing measure exists and they become strictly positive when a risk neutral measure exists.

4. Under the absence of arbitrage opportunities, the risk neutral valuation principle can be applied to find the fair price of an attainable contingent claim.

## 3.4 Binomial option pricing model: continuous limit to the Black-Scholes equation

By buying the asset and borrowing cash (in the form of riskless money market account) in appropriate proportions, one can replicate the position of a call.

Under the binomial random walk model, the asset prices after one period $\Delta t$ will be either $uS$ or $dS$ with probability $q$ and $1 - q$, respectively.

We assume $u > 1 > d$ so that $uS$ and $dS$ represent the up-move and down-move of the asset price, respectively. The jump parameters $u$ and $d$ will be related to the asset price dynamics.

Let $R$ denote the growth factor of riskless investment over one period so that \$1 invested in a riskless money market account will grow to \$$R$ after one period. In order to avoid riskless arbitrage opportunities, we must have $u > R > d$.

For example, suppose $u > d > R$, then we borrow as much as possible for the riskfree asset and use the loan to buy the risky asset. Even the downward move of the risky asset generates a return better than the riskfree rate. This represents an arbitrage.

Suppose we form a portfolio which consists of $\alpha$ units of asset and cash amount $M$ in the form of riskless money market account. After one period $\triangle t$, the value of the portfolio becomes

$$\begin{cases} \alpha uS + RM & \text{with probability } q \\ \alpha dS + RM & \text{with probability } 1 - q. \end{cases}$$

*Valuation of a call option using the approach of replication*

The portfolio is used to replicate the long position of a call option on a non-dividend paying asset.

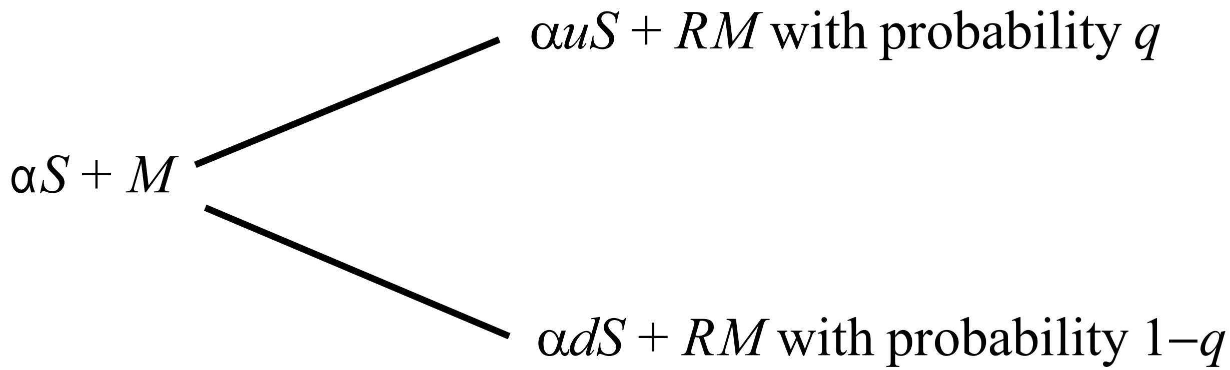
As there are two possible states of the world: asset price goes up or down, the call is thus a contingent claim.

Suppose the current time is only one period $\triangle t$ prior to expiration. Let $c$ denote the current call price, and $c_u$ and $c_d$ denote the call price after one period (which is the expiration time in the present context) corresponding to the up-move and down-move of the asset price, respectively.

Let $X$ denote the strike price of the call. The payoff of the call at expiry is given by

$$\begin{cases} c_u = \max(uS - X, 0) & \text{with probability } q \\ c_d = \max(dS - X, 0) & \text{with probability } 1 - q. \end{cases}$$

One can establish easily that $uc_d - dc_u \leq 0$.




*Evolution of the asset price $S$ and money market account $M$ after one time period under the binomial model.*

*Concept of replication revisited*

The above portfolio containing the risky asset and money market account is said to replicate the long position of the call if and only if the values of the portfolio and the call option match for each possible outcome, that is,

$$\alpha u S + RM = c_u \quad \text{and} \quad \alpha d S + RM = c_d.$$

Solving the equations, we obtain

$$\alpha = \frac{c_u - c_d}{(u-d)S} > 0, \qquad M = \frac{u c_d - d c_u}{(u-d)R} < 0.$$

- Apparently, we are fortunate to have two instruments in the replicating portfolio and two states of the world so that the number of equations equals the number of unknowns. The securities model is complete.

1. The parameters $\alpha$ and $M$ are seen to have opposite sign since cash is paid to acquire stock when the call is exercised.

2. $u/d < c_u/c_d$ due to the leverage effect inherited in the call option. That is, when a given upside growth/downside drop is experienced in the stock, the corresponding ratio is higher in the call.

- The number of units of asset held is seen to be the ratio of the difference of call values $c_u - c_d$ to the difference of asset values $uS - dS$.

- The call option can be replicated by a portfolio of the two basic securities: risky asset and riskfree money market account.

## Binomial option pricing formula

By no-arbitrage argument, the current value of the call is given by the current value of the portfolio, that is,

$$\begin{aligned} c &= \alpha S + M = \frac{\frac{R-d}{u-d}c_u + \frac{u-R}{u-d}c_d}{R} \\ &= \frac{pc_u + (1-p)c_d}{R} \qquad \text{where} \quad p = \frac{R-d}{u-d}. \end{aligned}$$

- The probability $q$, which is the subjective probability about upward or downward movement of the asset price, does not appear in the call value. The parameter $p$ can be shown to be $0 < p < 1$ since $u > R > d$ and so $p$ can be interpreted as a probability.

*Query* Why not perform the simple discounted expectation procedure using the subjective probabilities $q$ and $1 - q$, where

$$c = \frac{qc_u + (1-q)c_d}{R}?$$

*Answer* This price depends on the subjective probabilities taken by individual investors and cannot enforce the price. The replication procedure enforces the price.

The relation

$$puS + (1-p)dS = \frac{R-d}{u-d}\,uS + \frac{u-R}{u-d}\,dS = RS$$

shows that the expected rate of returns on the asset with $p$ as the probability of upside move is just equal to the riskless interest rate:

$$S = \frac{1}{R}E^*[S^{\Delta t}|S],$$

where $E^*$ is expectation under this probability measure. We may view $p$ as the *risk neutral probability*.

*Treating the binomial model as a one-period securities model*

The securities model consists of the riskfree asset and one risky asset with initial price vector: $\boldsymbol{S}^*(0) = (1 \quad S)$ and discounted terminal payoff matrix: $S^*(1) = \begin{pmatrix} 1 & \frac{uS}{R} \\ 1 & \frac{dS}{R} \end{pmatrix}$.

The risk neutral probability measure $\boldsymbol{Q}(\omega) = (Q(\omega_u) \quad Q(\omega_d))$ is obtained by solving

$$(Q(\omega_u) \quad Q(w_d)) \begin{pmatrix} 1 & \frac{uS}{R} \\ 1 & \frac{dS}{R} \end{pmatrix} = (1 \quad S).$$

We obtain

$$Q(\omega_u) = 1 - Q(\omega_d) = \frac{R-d}{u-d}.$$

The securities model is complete since there are two states and two securities. Provided that the securities model admits no arbitrage opportunities, we have uniqueness of the risk neutral measure and all contingent claims are attainable.

*Condition on $u$, $d$ and $R$ for absence of arbitrage*

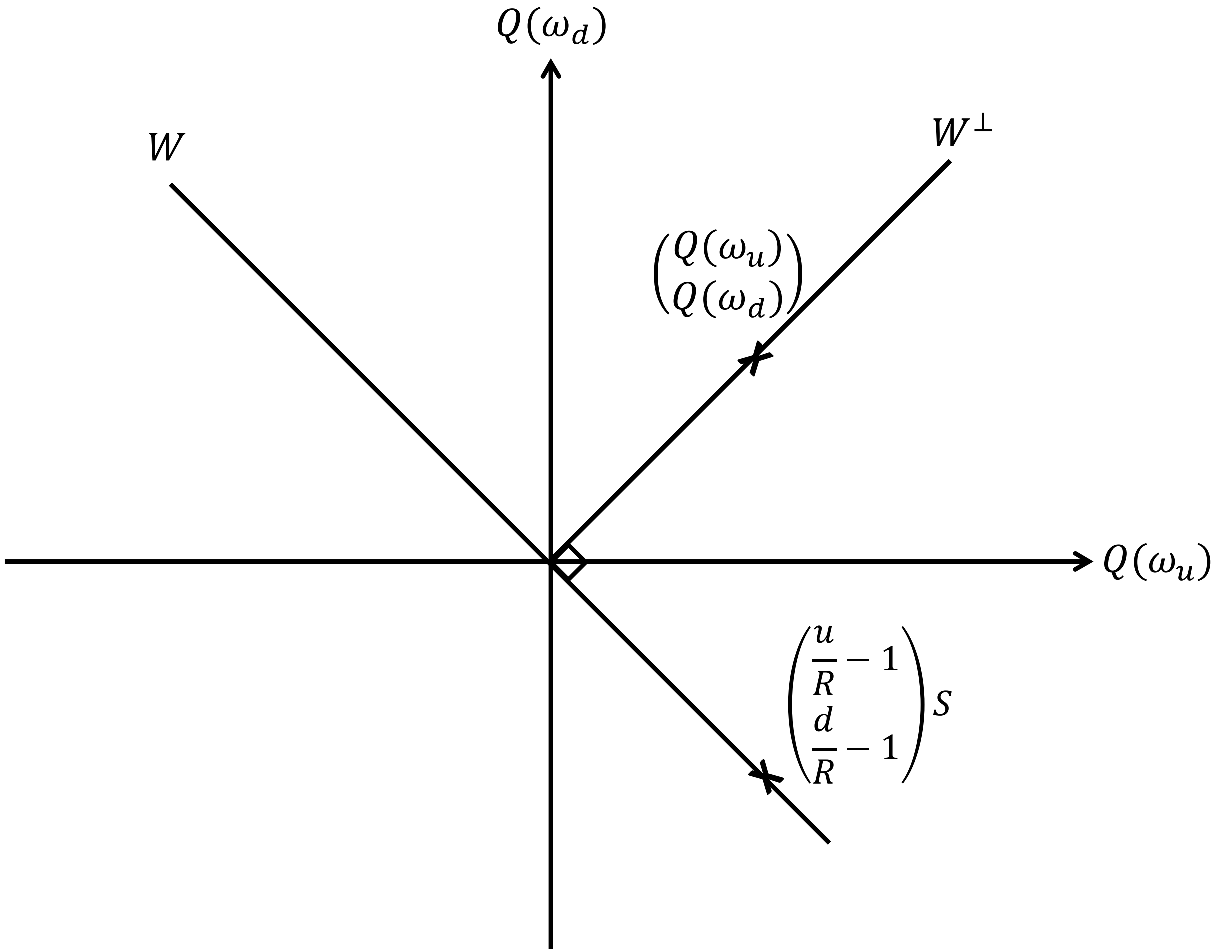
The set of risk neutral measures is given by $= P^+ \cap W^\perp$, where $W$ is the subspace of discounted gains. In the binomial world, $W$ is spanned by the single vector $\begin{pmatrix} \frac{u}{R} - 1 \\ \frac{d}{R} - 1 \end{pmatrix} S$ since there is only one risky asset. Given that $u > d$, we require

$$\frac{u}{R} - 1 > 0 \quad \text{and} \quad \frac{d}{R} - 1 < 0 \quad \Leftrightarrow \quad u > R > d$$

in order that the unique risk neutral measure exists (equivalent to absence of arbitrage). To derive the above "no-arbitrage" condition using geometrical intuition, a vector normal to $\begin{pmatrix} \frac{u}{R} - 1 \\ \frac{d}{R} - 1 \end{pmatrix} S$ lies in the first quadrant of $Q(\omega_u)$-$Q(\omega_d)$ plane if and only if $u > R > d$.

By the risk neutral valuation formula, we have

$$c = \frac{Q(\omega_u)c_u + Q(\omega_d)c_d}{R} = \frac{1}{R} E^*[c^{\Delta t}|S].$$

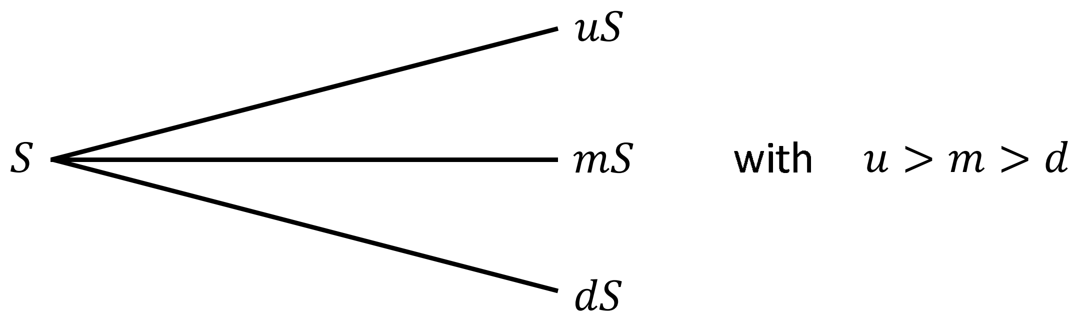



*Two equations for the determination of* $Q(\omega_u)$ *and* $Q(\omega_d)$

$$Q(\omega_u)\left(\frac{u}{R}-1\right)S + Q(\omega_d)\left(\frac{d}{R}-1\right)S = 0$$
$$Q(\omega_u) + Q(\omega_d) = 1.$$

*Extension to the trinomial model with 3 states of the world*

When we extend the two-jump assumption to the three-jump model:




We lose market completeness if we only have the money market account and the underlying risky asset in the securities model. We expect non-uniqueness of risk neutral measures, if they do exist. The system of equations for the determination of the set of risk neutral measures is given by

$$(Q(\omega_u) \quad Q(\omega_m) \quad Q(\omega_d)) \begin{pmatrix} 1 & uS/R \\ 1 & mS/R \\ 1 & dS/R \end{pmatrix} = (1 \quad S).$$

## Summary

- The binomial call value formula can be expressed by the following risk neutral valuation formulation:

$$c = \frac{1}{R} E^*[c^{\Delta t}|S],$$

where $c$ denotes the call value at the current time, and $c^{\Delta t}$ denotes the random variable representing the call value one period later. The call price can be interpreted as the expectation of the payoff of the call option at expiry under the risk neutral probability measure $E^*$ discounted at the riskless interest rate.

- Since there are 3 states of the world in a trinomial model, the application of the principle of replication of claims fails to derive the trinomial option pricing formula. Alternatively, one may use the risk neutral valuation approach for the direct determination of the risk neutral measures.

## Determination of the jump parameters

- For the continuous asset price dynamics of Geometric Brownian motion under the risk neutral measure, we have $d\ln S_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma \; dZ_t$ so that $\ln \frac{S_{t+\triangle t}}{S_t}$ becomes normally distributed with mean $\left(r - \frac{\sigma^2}{2}\right)\triangle t$ and variance $\sigma^2 \triangle t$, where $r$ is the riskless interest rate and $\sigma^2$ is the variance rate.
- The mean and variance of $\frac{S_{t+\triangle t}}{S_t}$ are $R$ and $R^2(e^{\sigma^2 \triangle t} - 1)$, respectively, where $R = e^{r\triangle t}$.
- For the one-period binomial option model under the risk neutral measure, the mean and variance of the asset price ratio $\frac{S_{t+\triangle t}}{S_t}$ are

$$pu + (1-p)d \quad \text{and} \quad pu^2 + (1-p)d^2 - [pu + (1-p)d]^2,$$

respectively.

- By equating the mean and variance of the asset price ratio in both the continuous and discrete models, we obtain

$$pu + (1-p)d = R$$
$$pu^2 + (1-p)d^2 - R^2 = R^2(e^{\sigma^2 \triangle t} - 1).$$

The first equation leads to $p = \dfrac{R-d}{u-d}$, the usual risk neutral probability.

- A convenient choice of the third condition is the *tree-symmetry condition*

$$u = \frac{1}{d},$$

so that the lattice nodes associated with the binomial tree are symmetrical. Writing $\widetilde{\sigma}^2 = R^2 e^{\sigma^2 \triangle t}$, the solution is found to be

$$u = \frac{1}{d} = \frac{\widetilde{\sigma}^2 + 1 + \sqrt{(\widetilde{\sigma}^2+1)^2 - 4R^2}}{2R}, \qquad p = \frac{R-d}{u-d}.$$

How to obtain a nice approximation to the above daunting expression?

- By expanding $u$ in Taylor series in powers of $\sqrt{\triangle t}$, we obtain

$$u = 1 + \sigma\sqrt{\triangle t} + \frac{\sigma^2}{2}\triangle t + \frac{4r^2 + 4\sigma^2 r + 3\sigma^4}{8\sigma}\triangle t^{\frac{3}{2}} + O(\triangle t^2).$$

- Observe that the first three terms in the above Taylor series agree with those of $e^{\sigma\sqrt{\triangle t}}$ up to $O(\triangle t)$ term.
- This suggests the judicious choice of the following set of parameter values

$$u = e^{\sigma\sqrt{\triangle t}}, \quad d = e^{-\sigma\sqrt{\triangle t}}, \qquad p = \frac{R-d}{u-d}.$$

- With this new set of parameters, the variance of the price ratio $\dfrac{S_{t+\triangle t}}{S_t}$ in the continuous and discrete models agree up to $O(\triangle t)$.

## Continuous limit of the binomial model

We consider the asymptotic limit $\triangle t \to 0$ of the binomial formula

$$c = [pc_u^{\Delta t} + (1-p)c_d^{\Delta t}]\ e^{-r\triangle t}.$$

In the continuous analog, the binomial formula can be written as

$$c(S, t - \triangle t) = [pc(uS, t) + (1-p)c(dS, t)]\ e^{-r\triangle t}.$$

Assuming sufficient continuity of $c(S, t)$, we perform the Taylor expansion of the binomial scheme at $(S, t)$ as follows:

$$
\begin{aligned}
& -c(S, t-\triangle t) + [pc(uS, t) + (1-p)c(dS, t)]e^{-r\triangle t} \\
= \; & \frac{\partial c}{\partial t}(S,t)\triangle t - \frac{1}{2}\frac{\partial^2 c}{\partial t^2}(S,t)\triangle t^2 + \cdots - (1 - e^{-r\triangle t})c(S,t) \\
& + \; e^{-r\triangle t}\left\{[p(u-1) + (1-p)(d-1)]S\frac{\partial c}{\partial S}(S,t)\right. \\
& + \; \frac{1}{2}[p(u-1)^2 + (1-p)(d-1)^2]S^2\frac{\partial^2 c}{\partial S^2}(S,t) \\
& + \; \left.\frac{1}{6}[p(u-1)^3 + (1-p)(d-1)^3]S^3\frac{\partial^3 c}{\partial S^3}(S,t) + \cdots\right\}.
\end{aligned}
$$

First, we observe that

$$1 - e^{-r\triangle t} = r\triangle t + O(\triangle t^2),$$

and it can be shown that

$$
\begin{aligned}
e^{-r\triangle t}\,[p(u-1) + (1-p)(d-1)] &= \; r\triangle t + O(\triangle t^2), \\
e^{-r\triangle t}\,[p(u-1)^2 + (1-p)(d-1)^2] &= \sigma^2\triangle t + O(\triangle t^2), \\
e^{-r\triangle t}\,[p(u-1)^3 + (1-p)(d-1)^3] &= O(\triangle t^2).
\end{aligned}
$$

Combining the results, we obtain

$$
\begin{aligned}
& -c(S, t-\triangle t) + [pc(uS, t) + (1-p)c(dS, t)]\, e^{-r\triangle t} \\
= \quad & \left[\frac{\partial c}{\partial t}(S,t) + rS\frac{\partial c}{\partial S}(S,t) + \frac{\sigma^2}{2}S^2\, \frac{\partial^2 c}{\partial S^2}(S,t) - rc(S,t)\right]\triangle t + O(\triangle t^2).
\end{aligned}
$$

Since $c(S,t)$ satisfies the binomial formula, so we obtain

$$
0 = \frac{\partial c}{\partial t}(S,t) + rS\frac{\partial c}{\partial S}(S,t) + \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2}(S,t) - rc(S,t) + O(\triangle t).
$$

In the limit $\Delta t \to 0$, the binomial call value $c(S,t)$ satisfies the Black-Scholes equation.