MATH4994 — Capstone Projects in Mathematics and Economics

Homework One

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1. This problem is related to the design of the rules of a game among 6 contestants for allocating 6 coins among this group of Ann, Bob, Carl, Dora, Ed, and Fran.

   Nature of the game:
   Ann goes first. She is given a bag that everyone knows contains six gold coins. Ann makes a proposal of how to allocate the six coins among the six contestants, including herself. The contestants (including Ann) then vote "yes" or "no" on the proposal. If the proposal gets more than half the votes, then the coins are allocated according to the proposal and everyone leaves the island. If the proposal gets half or fewer than half the votes, then Ann has to leave the island empty-handed and she is out of the game.

   In this case, the bag of six gold coins passes to Bob. He then makes a proposal of how to allocate the coins among the remaining contestants (including Bob but excluding Ann) and these remaining contestants then vote. As before, if the proposal gets more than half the votes then the coins are allocated according to the proposal and everyone leaves the island. If the proposal gets half or fewer than half the votes then Bob has to leave the island empty-handed and is out of the game. In this case, the bag of six gold coins passes to Carl. And so on, the same voting rules apply, with each failed proposal leading to expulsion of the proposer, and with the role of proposer being passed on alphabetically.

   Assumptions made:

   (a) The coins are indivisible, and side contracts to make monetary payments are not allowed.

   (b) There are no abstentions; each surviving voter must vote yes or no.

   (c) The players only care about gold coins. For example, leaving empty handed since your proposal fails is the same as leaving empty-handed since a successful proposal gives you no coins.

   (d) All the contestants are rational.

   It can be deduced that when there are only two contestants left, Fran should reject any proposal made by Ed. This is because Ed would leave the island as the failed proposer. How about the reaction of Ed and Fran to any proposal made by Dora when there are three contestants left? Using backward induction, find the optimal proposal placed by Ann.

2. The Kuhn Algorithm takes one person to be the cutter, who cuts the cake into equal pieces according to his valuations. We start with 3-player division. Tom cuts what he considers equal thirds and asks the other two to identify any of the three pieces they find acceptable. Let us organize this information in a matrix where "1" means "this piece is acceptable" and "0" means "this piece is unacceptable."

	X_1	X_2	X_3
Tom	1	1	1
Dick	1	1	0
Harry	0	1	0

(a) Why all entries in the first row are "1"? Can there be a row with no "1"?

(b) Devise a method of fair division based on envyfreeness if the table looks like:

	X_1	X_2	X_3
Tom	1	1	1
Dick	1	0	0
Harry	1	0	0

Further cuts are allowed.

(c) Considering the problem for four players, how could you proceed if the matrix takes the form given?

	X_1	X_2	X_3	X_4
Tom	1	1	1	1
Dick	1	0	0	0
Harry	1	0	0	0
Amy	1	0	0	0

(d) Assuming that there is always a fair division for three players, consider the problem for four players and show in all cases that a fair division is possible.

3. Consider 3 players A, B and C, whose value functions are given by

$$V_A(t) = \begin{cases} -4t+2, & t \in [0, \frac{1}{2}], \\ 4t-2, & t \in (\frac{1}{2}, 1]; \end{cases}$$
$$V_B(t) = \begin{cases} -2t+\frac{3}{2}, & t \in [0, \frac{1}{2}], \\ 2t-\frac{1}{2}, & t \in (\frac{1}{2}, 1]; \end{cases}$$
$$V_C(t) = 1, \quad t \in [0, 1].$$

Show that it is not always possible to divide a cake among 3 players into envy-free and equitable portions using two cuts.

4. To modify Stromquist's Moving Knife Algorithm so that it applies to the dirty work problem, ask a referee to move a sword from left to right dividing X into two pieces, X_1 to the right of the sword and X_2 to the left. Simultaneously, have the three players adjust parallel knives over X_2 so that each of their knives cuts X_2 in exact halves. Instruct the players that when any one of them says "cut", the lawn will be cut in three portions by the referee's sword and the middle knife of the three players, creating partitions $X_2 = X_2' \cup X_2''$ and $X = X_2' \cup X_2'' \cup X_1$. Further instruct them that they should say "cut" whenever they first think X_1 is smaller than or equal to both X_2' and X_2''. Without loss of generality, we can assume that the players' knives from left to right belong to P_1, P_2 and P_3, respectively. Consider the three cases generated by which of the three players says "cut" and show how, in each case, to assign X_2', X_2'', and X_1 to the players in an envy-free way.

5. Consider the following fair division procedure for 3 persons: Amy, Beth, and Colin.

- Amy divides the cake into two pieces of equal value in her opinion.
- Beth takes the larger (in her opinion) of the two pieces, and gives the remaining piece to Amy.
- Amy and Beth each divide their piece of cake into three pieces they consider to be equally valuable. There are now six pieces of cake.
- Colin chooses one piece of cake from Amy's three pieces, and one piece of cake from Beth's three pieces. Amy keeps her remaining two pieces and Beth keeps her remaining two pieces.

(a) Is this procedure proportional? Why or why not?

(b) Is this procedure envy free? Why or why not?

6. Suppose the cake is cut in six pieces and you get the first and last choice.

   (a) How much are you sure to get by your estimation on your first choice? The last choice?

   (b) Show the two combined choices will always guarantee you at least $\frac{1}{5}$ of the cake. Under what conditions will you get only $\frac{1}{5}$ of the cake?

7. The Adjusted Winner procedure can be adapted for unequal entitlements. Suppose that Annie and Ben are getting a divorce, but they signed a pre-nuptial agreement that gives Annie 60% of the joint property and Ben 40%. During the "equitability" adjustment stage of the adjusted winner procedure, Annie's point total should be exactly 1.5 times that of Ben. Determine the allocation dictated by the adjusted winner procedure for the following items.

Annie's Points	*Item*	*Ben's Points*
35	Right to retain lease on apartment	30
20	Entertainment System	15
15	Pool table	25
15	Antique Table	20
15	Washer and Dryer	10
100	Total	100

8. Emma and Kate are planning to open a new restaurant, and have several projects to finish before they will be ready to open. They would rather split up the projects between them so that each person has full control of a few specific issues instead of working together on each of the different projects. Each person has devoted 100 points to the projects listed below.

Emma's Points	*Item*	*Kate's Points*
20	Menu Design	10
25	Interior Design	15
10	Advertising	5
15	Dining Room Layout	20
10	Bar Layout	20
10	Hiring Waitstaff	15
10	Hiring Chefs	15
100	Total	100

Another method for dividing goods or issues between two people is the *Balanced Alternation* procedure, wherein the two parties take turns choosing issues and the party that chooses second is compensated by being able to choose two items during his first turn. For example, if persons A and B are dividing six goods between them, then they might choose in the following order: A,B,B,A,B,A.

(a) If Emma and Kate use Balanced Alternation rather than Adjusted Winner, what is the final allocation of issues?

(b) Is Emma better or worse off with Balanced Alternation than with Adjusted Winner? What about Kate?

(c) Describe a particular example of two sets of goods to be divided where Adjusted Winner is far better than Balanced Alternation.

9. Eighteen cookies are to be divided between three good friends (Michael, Mike, and Peter) after a hard night's work in Athens, Georgia. There are six chocolate chip cookies, 6 peanut butter cookies, and 6 sugar cookies with rainbow sprinkles. Michael is thinking of going vegan (he's already a vegetarian), so the chocolate chip cookies are worthless to him (fortunately, the peanut butter and sugar cookies were made without eggs, butter, or milk). He likes the peanut butter and sugar cookies equally. Mike is allergic to peanuts, so he cannot eat the peanut butter cookies. He likes the chocolate chip and sugar cookies equally. Peter likes the chocolate chip and peanut butter cookies equally but does not like the sugar cookies at all – the sprinkles fall into his mandolin. Give examples of allocations of cookies (all 18 must be accounted for) that are

(a) envy-free but not equitable;

(b) equitable but not envy-free.