MATH4994 — Capstone Projects in Mathematics and Economics

Solution to Homework One

Course instructor: *Prof. Y.K. Kwok*

---

1. We proceed using the backward induction, starting with only 2 persons left.

   (a) When Ed and Fran are left behind, Fran should reject any proposal made by Ed. Hence, Ed gets nothing.

   (b) When Dora, Ed and Fran are left behind, Dora proposes $(5, 1, 0)$ coin allocation to (Dora, Ed, Fran). Ed will vote Yes, otherwise Ed gets nothing in the next round. Hence, Dora and Ed vote Yes. Fran gets nothing.

   (c) When Carl, Dora, Ed and Fran are left behind, Carl proposes $(3, 0, 2, 1)$ allocation to (Carl, Dora, Ed, Fran). Ed and Fran vote Yes, since in the next round Ed and Fran can only get 1 and 0 coin. Hence, Carl, Ed and Fran vote Yes. Dora gets nothing.

   (d) When Bob, Carl, Dora, Ed and Fran are left behind, Bob proposes $(3, 0, 1, 0, 2)$ allocation to (Bob, Carl, Dora, Ed, Fran). Bob, Dora and Fran vote Yes. Carl and Fran get nothing.

   (e) When Ann, Bob, Carl, Dora, Ed and Fran are left behind, Ann proposes $(2, 0, 1, 2, 1, 0)$ allocation to (Ann, Bob, Carl, Dora, Ed, Fran). Ann, Carl, Dora and Ed vote Yes. Ann's proposal is passed. Poor Bob and Fran, they get nothing.

2. (a) Tom cuts equal thirds, so all pieces are acceptable. For each non-cutters, he must think some piece has value at least 1/3.

   (b) Give Tom either X_2 or X_3, and let Dick and Harry play divide-and-choose on the other two pieces. Dick and Harry do not envy each other. Tom is indifferent to the 3 equally valued pieces.

   (c) Give either X_2, X_3, or X_4 to Tom, and let the other three play the three-person game on the other three pieces.

   (d) From part (c), if only Tom likes a particular piece, give it to him and let the others play the three-person game on the rest of the cake. If only two players like a particular piece, say Tom and Dick, give that piece to Dick and let the other three play the three-person game on the other three pieces. Thus we may assume that each column has at least three ones in it. In one case, suppose we assume that the matrix takes the following form (no assumption is made on entries that are left blank):

	X_1	X_2	X_3	X_4
Tom	1	1	1	1
Dick	1	1		
Harry	1	1		
Amy			x	y

If either x or y is one, say $x = 1$, give X_3 to Amy, X_4 to Tom, and let Dick and Harry divide $X_1 \cup X_2$.

Finally, suppose $x = y = 0$ and since there are three ones in each column, we have

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ z & w & 0 & 0 \end{matrix}$$

Either $z = 1$ or $w = 1$. If $z = 1$, then give X_1 to Amy and distribute the others arbitrarily.

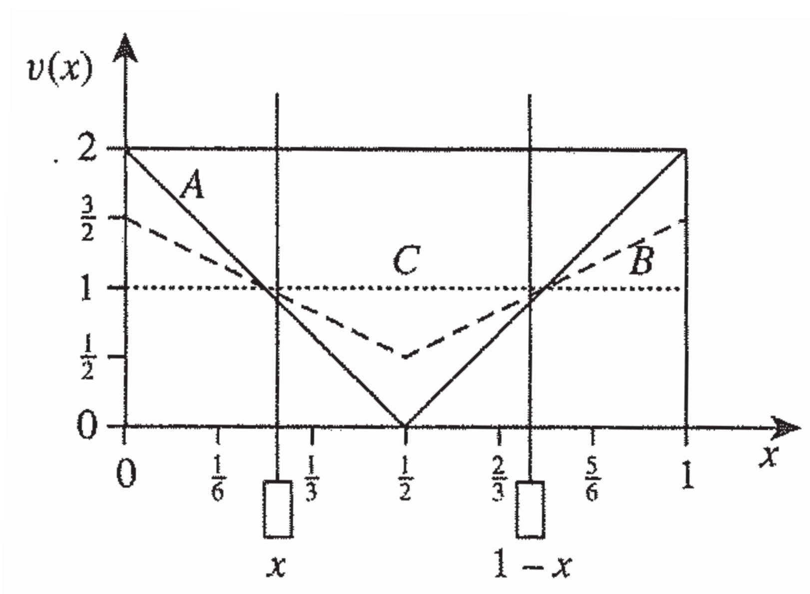
3. Consider 3 players A, B and C, whose value functions are given below:

$$v_A(t) = \begin{cases} -4t + 2 & \text{for } t \in [0, \frac{1}{2}] \\ 4t - 2 & \text{for } t \in (\frac{1}{2}, 1] \end{cases} ;$$

$$v_B(t) = \begin{cases} -2t + \frac{3}{2} & \text{for } t \in [0, \frac{1}{2}] \\ 2t - \frac{1}{2} & \text{for } t \in (\frac{1}{2}, 1] \end{cases} ;$$

$$v_C(t) = 1, \text{ for } t \in [0, 1].$$

Envy-free allocation

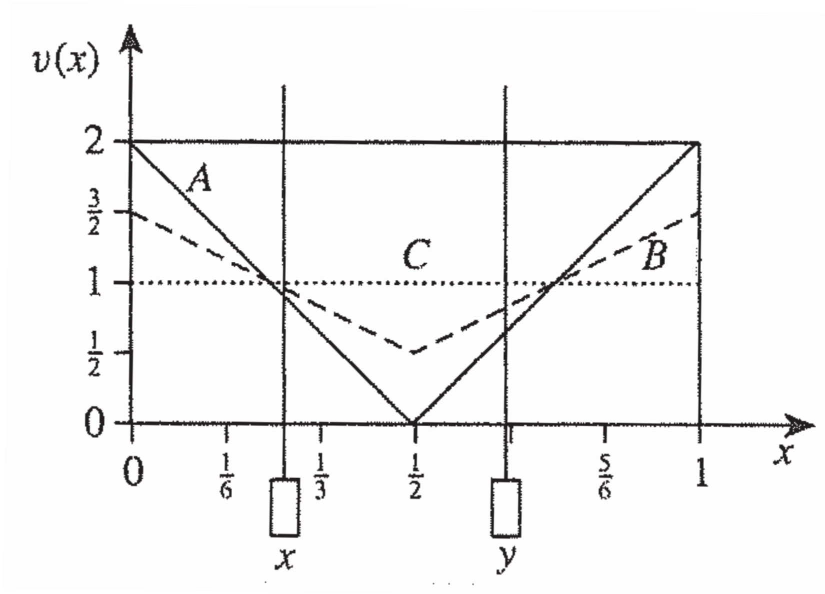


Every envy-free allocation of the cake will be one in which A gets the portion to the left of x, B the portion to the right of $1 - x$ (A and B could be interchanged), and C the portion in the middle (that ensure envy-freeness). If the horizontal lengths of A's and B's portions are not the same (both of length x), the player whose portion is shorter in length will envy the player whose portion is longer.

Failure of equitability

Unfortunately, no such envy-free allocation is equitable. Suppose A and B receive equal-length endpieces (ensuring that each values each piece equally, which precludes envy), A

will receive a larger portion in its eyes than B receives in its eyes, precluding equitability. Thus, such a 2-cut envy-free allocation is not equitable.



Numerical example

We calculate the equitable division in which A gets the left piece defined by the interval $[0, x]$, C gets the middle piece defined by the interval $(x, y]$, where $y < 1 - x$, and B gets the right piece defined by the interval $(y, 1]$. The players' values are equal when

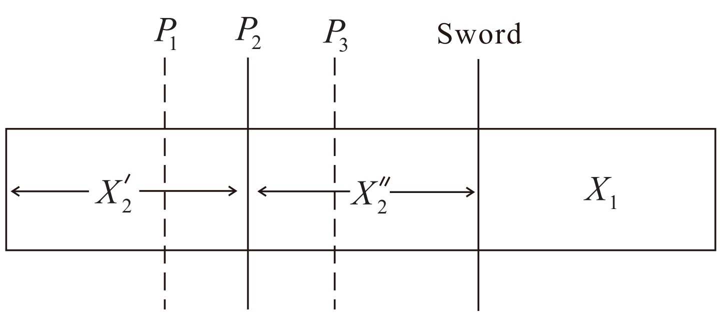
$$\int_0^x (-4t + 2)\,\mathrm{d}t = \int_x^y \mathrm{d}t = \int_y^1 \left(2t - \frac{1}{2}\right)\mathrm{d}t.$$

We obtain two quadratic equations in two unknowns:

$$-2x^2 + 3x - y = 0 \quad \text{and} \quad -2x + 2y^2 + y - 1 = 0,$$

whose four solutions include one feasible solution: $x \approx 0.269$ and $y \approx 0.662$. Players A, B, and C each value their pieces at 0.393, so each thinks it receives nearly 40 percent of the value of the cake.

4.



The sword held by the referee cuts the cake into $X_1 \cup X_2$. The middle knife held by P_2 cuts X_2 into two equal halves in his valuation: $X_2 = X_2' \cup X_2''$ and $V_2(X_2') = V_2(X_2'')$.

Shouting strategy

Shouts whenever X_1 (the piece on the right side of the sword) becomes equal to the piece to be received if chooses not to shout. How to allocate the pieces to the three players, according to who is the shouter, so that they do not envy on the allocation of dirty works? The shouter always receives X_1. Each non-shouter receives the corresponding divided piece of X_2, X_2' or X_2'', that does not contain his knife.

Allocation of pieces

(i) P_1 shouts when $V_1(X_1) = V_1(X_2'') > V_1(X_2')$.
P_1 receives X_1 (shouter), P_3 receives X_2' (since X_2' does not contain the knife of P_3) and P_2 receives the remaining last piece X_2''.

(ii) P_2 shouts when $V_2(X_1) = V_2(X_2'') = V_2(X_2')$.
P_2 receives X_1 (shouter), P_1 receives X_2'' and P_3 receives X_2'.

(iii) P_3 shouts when $V_3(X_1) = V_3(X_2') > V_3(X_2'')$.
P_3 receives X_1 (shouter), P_1 receives X_2'' (since X_2'' does not contain the knife of P_1) and P_2 receives the remaining last piece X_2'.

5. (a) According to the procedure, the cake is first divided into two pieces, denoted by X_1 and X_2, by Amy. Hence $V_A(X_1) = V_A(X_2) = \frac{1}{2}$. Beth takes the larger of the two pieces, say X_2. Thus, $V_B(X_2) \geq \frac{1}{2}$. Next, X_1 and X_2 are divided into equal three pieces by Amy and Beth respectively. Hence, we have

$$V_A(X_{11}) = V_A(X_{12}) = V_A(X_{13}) = \frac{1}{6},$$

and

$$V_B(X_{21}) = V_B(X_{22}) = V_B(X_{23}) \geq \frac{1}{6}.$$

Say Colin picks X_{11} and X_{21}, then $V_C(X_{11}) \geq \frac{1}{3}V_C(X_1)$ and $V_C(X_{21}) \geq \frac{1}{3}V_C(X_2)$. The above inequality leads to

$$V_C(X_{11}) + V_C(X_{21}) \geq \frac{1}{3}V_C(X_1) + \frac{1}{3}V_C(X_2) = \frac{1}{3}V_C(X_1 + X_2) = \frac{1}{3}. \qquad (1)$$

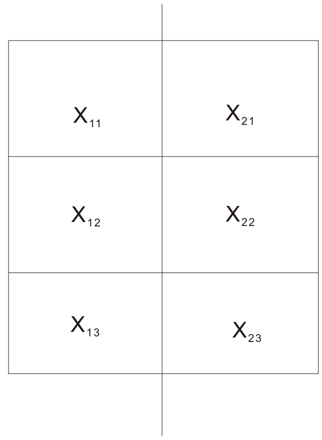
Amy keeps her remaining two pieces (X_{12} and X_{13}) and Beth keeps her remaining two pieces (X_{22} and X_{23}). Hence, Amy's valuation is

$$V_A(X_{12} + X_{13}) = V_A(X_{12}) + V_A(X_{13}) = \frac{1}{3}, \qquad (2)$$

and Beth's valuation is

$$V_B(X_{22} + X_{23}) = V_B(X_{22}) + V_B(X_{23}) \geq \frac{1}{3}. \qquad (3)$$

From (1), (2) and (3) the procedure is proportional.



(b) Suppose the preferences for Amy, Beth and Colin are

A: Cherry > no icing > Apple
B and C: indifferent on cherry, apple and no icing

Also suppose X_{21} has cherry, X_{23} has apple and other pieces have no icing. Then

$$V_A(X_{21}) > V_A(X_{22}) = V_A(X_{11}) = V_A(X_{12}) = V_A(x_{13}) > V_A(X_{23}),$$
$$V_B(X_{11}) = V_B(X_{12}) = V_B(X_{13}) = V_B(X_{21}) = V_B(x_{22}) = V_B(x_{23}),$$

and

$$V_C(X_{11}) = V_C(X_{12}) = V_C(X_{13}) = V_C(X_{21}) = V_C(X_{22}) = V_C(X_{23}).$$

Suppose Colin picks X_{11} and X_{21}, then

$$V_A(X_{11}) + V_A(X_{21}) > \frac{1}{3},$$

greater than what Amy gets. The procedure is not envy free.

6. (a) Order the 6 pieces in value to the chooser: $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq 0$. Since $6a_1 \geq a_1 + a_2 + \cdots + a_6 = 1$, so $a_1 \geq \frac{1}{6}$. It is possible that $a_6 = 0$.

(b) Write $a_6 = \frac{1}{6} - \epsilon$, where $0 \leq \epsilon \leq \frac{1}{6}$. Note that

$$5a_1 \geq 1 - a_6 = 1 - \left(\frac{1}{6} - \epsilon\right) = \frac{5}{6} + \epsilon \Rightarrow a_1 \geq \frac{1}{6} + \frac{\epsilon}{5}.$$

Therefore, $a_1 + a_6 \geq \frac{1}{3} - \frac{4}{5}\epsilon$.

(i) When $\epsilon = 0$, the chooser achieves the maximum value of $a_1 + a_6$, which is $\frac{1}{3}$. This corresponds to $a_6 = \frac{1}{6}$, which implies all pieces have the same value. It is not surprising that the chooser can achieve $\frac{1}{3}$ by taking 2 pieces out of 6 pieces.

(ii) When $\epsilon = \frac{1}{6}$, $a_6 = 0$. In this case,

$$a_1 = a_1 + a_6 \geq \frac{1}{3} - \frac{4}{5} \times \frac{1}{6} = \frac{1}{5}.$$

When the 6^{th} piece is worthless, there are essentially 5 pieces to be divided. The largest piece a_1 is guaranteed to have at least $\frac{1}{5}$. The chooser receives the worst amount when $a_1 = a_2 = \cdots = a_5 = \frac{1}{5}$ and $a_6 = 0$.

7. In the first phase, Annie and Ben are assigned the more valuable items as valued by them. Therefore, Annie is assigned the lease, entertainment and washer, while Ben is assigned the pool table and antique table. The initial point received by Annie is 70, which is more than 3/2 times that of Ben's initial point (which is 45 points).

   In the second phase, portion of an item is transferred from Annie (initial winner) to Ben. According to the adjusted winner procedure, the one with the smallest ratio of values is transferred. Since the ratio of values is smallest for lease ($35/30 = 1.167$), we transfer x portion of lease from Annie to Ben. The equation for solving x is given by

   $$70 - 35x = (45 + 30x) \times 1.5.$$

   This gives $x = 0.03125$. As a check, the final point received by Annie is $70 - 35 \times 0.03125 = 68.90625$, while that of Ben is $45 + 30 \times 0.03125 = 45.9375$. It is easy to verify that the ratio of the final points is 3/2, where

   $$68.90625 = 3/2 \times 45.9375.$$

8. (a) Suppose Emma picks first: (Allocation 1)

   Emma: Interior Design (25 points)
   Kate: Dining Room Layout (20 points), Bar Layout (20 points)
   Emma: Menu Design (20 points)
   Kate: Hiring Waitstaff (15 points)
   Emma: Advertising (10 points)
   Kate: Hiring Chefs (15 points)

   Total points for Emma and Katie in Allocation 1

   Emma: 55 Kate: 70

   There is another allocation when Emma picks first (Allocation 2)

   Emma: Interior Design (25 points)
   Kate: Dining Room Layout (20 points), Bar Layout (20 points)
   Emma: Menu Design (20 points)
   Kate: Hiring Waitstaff (15 points)
   Emma: Hiring Chefs (10 points)
   Kate: Advertising (5 points)

   Total points for Emma and Katie in Allocation 2:

   Emma: 55 Kate: 60

   Suppose Kate picks first: (Allocation 3)

   Kate: Bar Layout (20 points)
   Emma: Interior Design (25 points), Menu Design (20 points)
   Kate: Dining Room Layout (20 points)
   Emma: Advertising (10 points)
   Kate: Hiring Waitstaff (15 points)
   Emma: Hiring Chefs (10 points)

Total points for Emma and Katie in Allocation 3:

Emma: 65 Kate: 55

Using the Adjusted Winner procedure, Emma gets Menu Design, Interior Design, Advertising and 60% of Hiring Chefs while Kate gets Dining Room Layout, Bar Layout, Hiring Waitstaff and 40% of Hiring Chefs. Both receives 61 points.

(b) For Allocation 1: Emma is worse off and Kate is better off in the Balanced Alternation procedure.
For Allocation 2: Both Emma and Kate are worse off in the Balanced Alternation procedure.
For Allocation 3: Emma is better off and Kate is worse off in the Balanced Alternation procedure. The person which chooses first is worse off in the Balanced Alternation procedure.

(c) Consider the following example:

A's points	*Item*	*B's points*
25	Dog	85
25	Cat	5
25	Bird	5
25	Tiger	5
100	Total	100

In the Adjusted Winner procedure. A gets Cat, Bird, and Tiger and shares x portion of Dog with B. Then solving the following equation for x

$$85x = 75 + 25(1 - x),$$

we have $x = 0.9091$. Hence the total points for A and B are both equal to 77.27. In the Balanced Alternation procedure, suppose A picks Dog in the first move. The total points for A and B are 50 and 10, respectively. In this example the Adjusted Winner procedure is far better than the Balanced Alternation procedure for both parties.

9. (a) Consider the allocation

	Michael	Mike	Peter
Chocolate (C)	1	2	3
Peanut (P)	3	1	2
Sugar (S)	3	3	0

The individual value function counts the number of desirable pieces received by the corresponding player. Let V_1, V_2, V_3 be the value function of Michael, Mike and Peter, respectively. We observe

$$\begin{aligned} V_1(1C + 3P + 3S) &= V_1(3P + 3S) \quad \text{(Chocolate chips are worthless to Michael)} \\ &> V_2(2C + 3S) \quad \text{(Peanuts are worthless to Mike)} \\ &= V_3(3C + 2P), \quad \text{(Both Mike and Peter receive 5 desirable pieces)} \end{aligned}$$

which means that this allocation is not equitable. It is envy-free because

$$V_1(1C + 3P + 3S) > V_1(2C + P + 3S), \quad V_1(1C + 3P + 3S) > V_1(3C + 2P),$$
$$V_2(2C + P + 3S) > V_2(C + 3P + 3S), \quad V_2(2C + P + 3S) > V_2(3C + 2P),$$
$$V_3(3C + 2P) > V_3(C + 3P + 3S), \quad V_3(3C + 2P) > V_3(2C + P + 3S).$$

(b) Consider the allocation

	Michael	Mike	Peter
Chocolate (C)	0	2	4
Peanut (P)	2	3	1
Sugar (S)	3	3	0

It is equitable because Michael and Peter receive 5 desirable pieces while the 3 peanuts are worthless to Mike, so

$$V_1(2P + 3S) = V_2(2C + 3P + 3S) = V_3(4C + P).$$

It is not envy-free since

$$V_1(2C + 3P + 3S) > V_1(2P + 3S).$$