MATH4994 — Capstone Projects in Mathematics and Economics

Solution to Homework Three

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1. Since A can beat B (two-to-one) in pairwise contest via majority rule, the agenda should be designed such that the other two alternatives are eliminated in the earlier steps. The agenda can be set as follows:

   Round 1: C versus D (D will be eliminated)
   Round 2: B versus C (C will be eliminated)
   Round 3: A versus B (A emerges as the final winner).

2. Both procedures seek for the alternative with majority first-place vote and declare it as the social choice. If such alternative does not exist, then both procedures will delete an alternative with the lowest number of first-place vote and the remaining two alternatives will be voted in the second stage. Thus both procedures are equivalent when there are only 3 alternatives.

3. We allow ties in the output in a social choice procedure but not the input. For (a) and (b), the following simple example

Voter 1	Voter 2
A	B
B	A

   shows that we may have ties for both plurality and Borda count. For the Hare method, we may create an example where after the runoff we obtain the last two alternatives with preferences same as the earlier example. For sequential pairwise voting, we need to create the ties breaking rule in each pairwise contest, otherwise the sequential pairwise voting cannot be continued. With the ties breaking rule in place, we are able to produce a unique social choice (winner). For a dictatorship, since there is only one alternative on the top of the dictator's list, so the social choice is unique.

4. (a) $\text{Net}(x > y) = 1$. $\text{Net}(x > z) = 1$, $B(x) = 2$, $B(z) = -2$

   (b) One can also obtain $B(y) = 0$, $\text{Net}(y > z) = 1$, $\text{Net}(y > x) = -1$, $\text{Net}(z > x) = -1$, $\text{Net}(z > y) = -1$. The three equations can be easily verified.

5. Since the procedure satisfies the monotonicity property, so if D is the social choice and a person who moves D to the higher position will still yield D as the social choice or at least tied for such.

6. If one wants alternative a wins, then choose the agenda "$b \to c \to d \to a$"
   If one wants alternative b wins, then choose the agenda "$a \to c \to b \to d$"
   If one wants alternative c wins, then choose the agenda "$b \to d \to a \to c$"
   If one wants alternative d wins, then choose the agenda "$a \to b \to c \to d$"

7. Suppose we have 2 Condorcet winners x and y. Since x is Condorcet winner, then x wins y in pairwise majority contest. Since y is Condorcet winner, then y wins x in pairwise majority contest which is a contradiction. Thus the Condorcet winner, if exist, must be unique.

8. (a) Plurality voting does not satisfy the Condorcet loser criterion. To see this, we consider the following example:

Voter 1	Voter 2	Voter 3	Voter 4	Voter 5
A	A	B	C	D
B	B	C	D	B
C	C	D	B	C
D	D	A	A	A

It is clear that A is Condorcet loser, but under Plurality voting, A is the social choice.

(b) The Borda count method satisfies the Condorcet Loser Criterion. To see this, let an alternative x be the Condorcet Loser, since it loses to other alternative in pairwise majority contest. The number of vote it receives in each of pairwise majority contest is less than $P/2$, where P is the number of voters. Now Borda Count of $x =$ sum of all votes in $(n-1)$ pairwise majority contests $< \dfrac{(n-1)P}{2}$ and the average Borda Count per alternative is $\dfrac{[0+1+\ldots+(n-1)]P}{n} = \dfrac{(n-1)P}{2}$. Since there always exists an alternative with Borda Count greater than $\dfrac{(n-1)P}{2}$, so the alternative x cannot be the winner.

(c) Hare's procedure does not satisfy the Condorcet Loser criterion by the example in (a).

(d) If an alternative wins in the sequential pairwise voting with fixed agenda, it must win at least 1 alternative. Since a Condorcet loser loses to ALL alternatives, thus it can't win in the sequential pairwise voting and so the sequential pairwise voting satisfies the Condorcet loser property.

(e) The dictatorship does not observe this property by the example in (a) and assume voter 1 is the dictator.

9. Let x, y, z be the three alternatives. Suppose there is no Condorcet loser, then for each alternative (say x), there is another alternative (say y) such that x wins y in pairwise majority contest. Since y is not Condorcet loser also, if y loses to x, then it must win z. Consider the agenda "$y \to z \to x$", then x will be the winner under this agenda.
Conversely, suppose each alternative there is an agenda under which that alternative wins in sequential pairwise voting. Then the alternative must win at least one other alternative in pairwise majority contest, thus it must not be Condorcet loser. So there is no Condorcet loser.

10. To show that this procedure does not observe the IIA property, we consider the following set of individual preference lists:

Voter 1	Voter 2	Voter 3
A	B	C
B	C	A
C	A	B

One can easily verify that there is no Condorcet winner. In this case, voter 1 serves as the dictator and A will be the social choice and C is not the social choice.
Suppose voter 2 moves alternative B to the 2^{nd} place without changing the preference of A and C, the new list becomes

Voter 1	Voter 2	Voter 3
A	C	C
B	B	A
C	A	B

Then alternative C will be the Condorcet winner and it is a social choice by definition. Hence it violates the IIA.

To show that this procedure observes the Pareto property, suppose everyone prefers x to y, then y loses to x in pairwise majority contest, thus y cannot be the Condorcet winner. At the same time, it must not be on the top of the voter 1's preference list. Hence y can't be the winner.

11. (a) A plurality voting winner must observe the top condition because if an alternative does not appear on the top of everyone's preference list, then the number of first-place vote of this alternative is 0 and clearly it cannot be a social choice.

    (b) The Borda count method does not observe this property. To see this, we consider the following example

Voter 1	Voter 2	Voter 3
A	C	D
B	B	B
C	D	A
D	A	C

    We see that B gets 6 points and each of A, C, D get 4 points. Therefore, B is the social choice but B does not appear on the top of anyone's list.

    (c) The Hare procedure satisfies the top condition. To see this, suppose an alternative does not appear on the top of any voter's list, then it has fewest first-place vote and will be eliminated. Therefore, it cannot be the social choice.

    (d) Sequential pairwise voting does not observe the top condition property. To see this, we consider the following example:

Voter 1	Voter 2	Voter 3
A	C	D
B	B	B
C	D	A
D	A	C

Consider the agenda $a \to c \to d \to b$, then this agenda yields b as the social choice. Once again, b does not appear at the top of any voter's list.

(e) Dictatorship satisfies the top condition. If x does not appear at the top of any voter's list, then it is not on the top of dictator's list and thus cannot be the social choice.

(f) Suppose everyone prefers x to y, then y is not at the top of ANY voter's preference list. Since the procedure observes the top condition, this implies that y is not the winner. Thus the Pareto condition is verified.

12. (a) Pareto condition: whenever every voter puts a strictly above b, then b cannot be a social choice. Consider the following example:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a \\ d \\ b \\ c \end{pmatrix}.$$

Every voter prefers a to b. However, both alternatives receive zero number of last-place ranking (a ties with b). Hence, b s a social choice. This violates the Pareto condition.

(b) Monotonicity criterion
Let x be a winner under this procedure. Suppose some voters move the ranking of x up without changing the order in which they prefer any other alternatives. Then x receives the same or less number of last-place ranking while all other alternatives receive the same or more number of last-place ranking. Therefore, the winner status of x remains.

(c) Independent of irrelevant alternatives criterion
We quote the following counter example:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} b \\ a \\ c \end{pmatrix}, \begin{pmatrix} b \\ a \\ c \end{pmatrix}, \begin{pmatrix} a \\ c \\ b \end{pmatrix}$$

$$\xrightarrow[\substack{a \text{ with } c \text{ in the } 4^{\text{th}} \\ \text{and } 5^{\text{th}} \text{ lists.}}]{\text{modified by swapping}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} b \\ c \\ a \end{pmatrix}, \begin{pmatrix} b \\ c \\ a \end{pmatrix}, \begin{pmatrix} a \\ c \\ b \end{pmatrix}.$$

Note that the relative positions of the pair a and b remain. Under the original preference list, a is the social choice. However, under the modified preference list, b becomes the social choice.

13. The social preference lists are found to be

(a) $\begin{pmatrix} a \\ b, c, d \\ e \end{pmatrix}$ (b) $\begin{pmatrix} b \\ c \\ d \\ a \\ e \end{pmatrix}$ (c) $\begin{pmatrix} c \\ b \\ d \\ a \\ e \end{pmatrix}$

(d) $\begin{pmatrix} d \\ b \\ c \\ a \\ e \end{pmatrix}$ if the agenda $(a \to b \to c \to d \to e)$ is used

14. (a) Everyone exchanges the positions of a and c in her preference list, then neutrality yields $\begin{pmatrix} c \\ b \\ a \end{pmatrix}$ as the social preference list.

(b) In order to have $\begin{pmatrix} c \\ a \\ b \end{pmatrix}$ as the social preference list, we further interchange the positions of a and b in everyone's list. The new sequence of preference lists become

$$\begin{pmatrix} b \\ a \\ c \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$